Shortest Paths

CSE 373

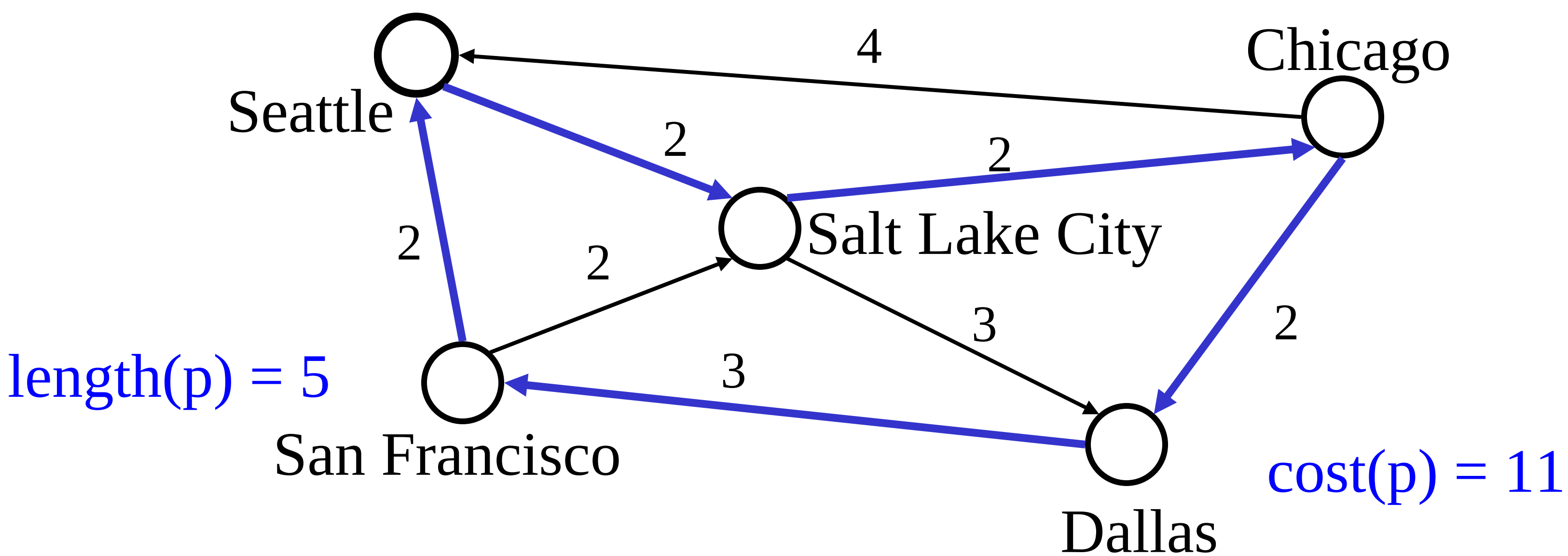
Data Structures

Readings

- Reading Chapter 13
  - Sections 13.5 to 13.7

Recall Path cost ,Path length

- Path cost: the sum of the costs of each edge
- Path length: the number of edges in the path
  - › Path length is the unweighted path cost



Shortest Path Problems

- Given a graph G = (*V*, *E*) and a "source" vertex *s* in *V*, find the minimum cost paths from *s* to every vertex in *V*
- Many variations:
  - › unweighted vs. weighted
  - › cyclic vs. acyclic
  - › pos. weights only vs. pos. and neg. weights
  - › etc

Why study shortest path problems?

- Traveling on a budget: What is the cheapest airline schedule from Seattle to city X?
- Optimizing routing of packets on the internet:
  - › Vertices are routers and edges are network links with different delays. What is the routing path with smallest total delay?
- Shipping: Find which highways and roads to take to minimize total delay due to traffic
- etc.

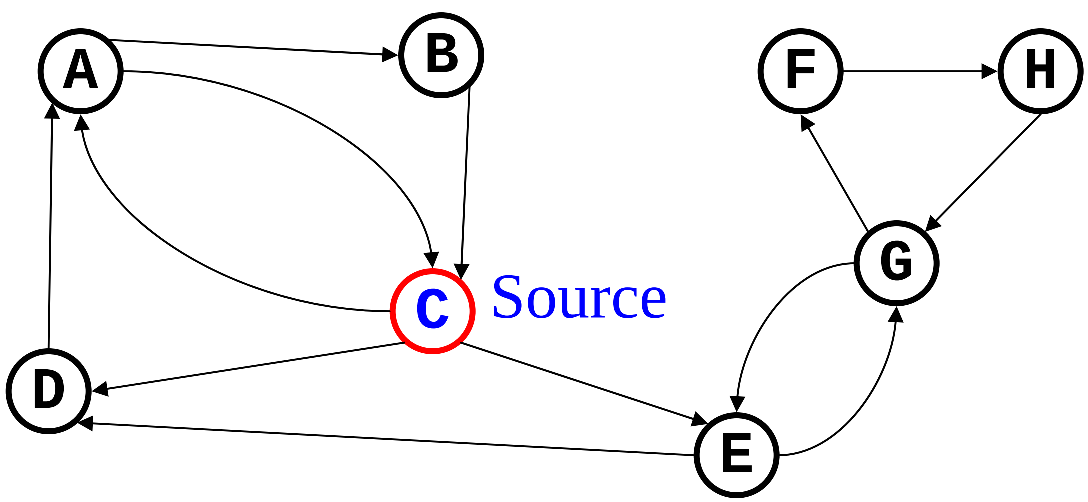
Unweighted Shortest Path

Problem: Given a “source” vertex *s* in an unweighted directed graph

G = (*V*,*E*), find the shortest path from *s* to all vertices in G

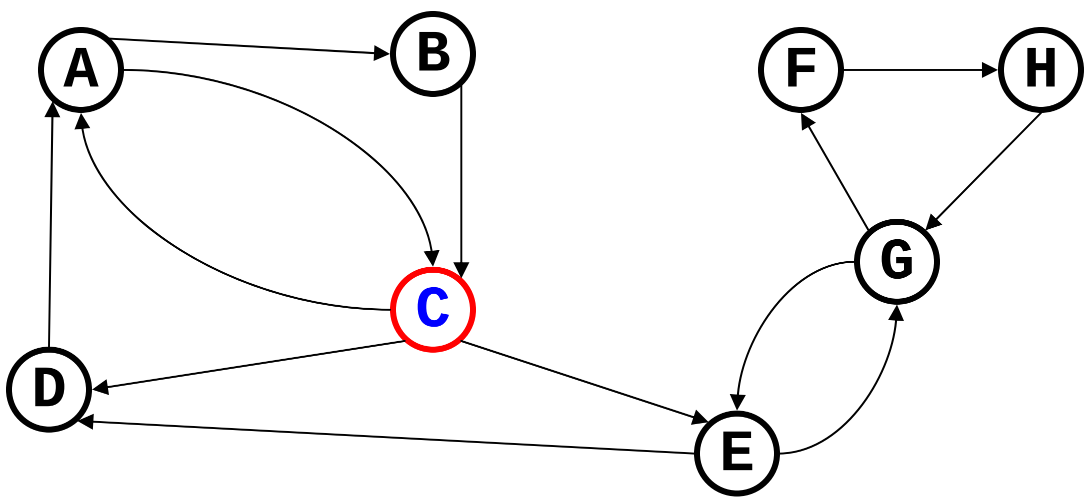
Only interested

in path lengths



Breadth-First Search Solution

- Basic Idea: Starting at node s, find vertices that can be reached using 0, 1, 2, 3, …, N-1 edges (works even for cyclic graphs!)



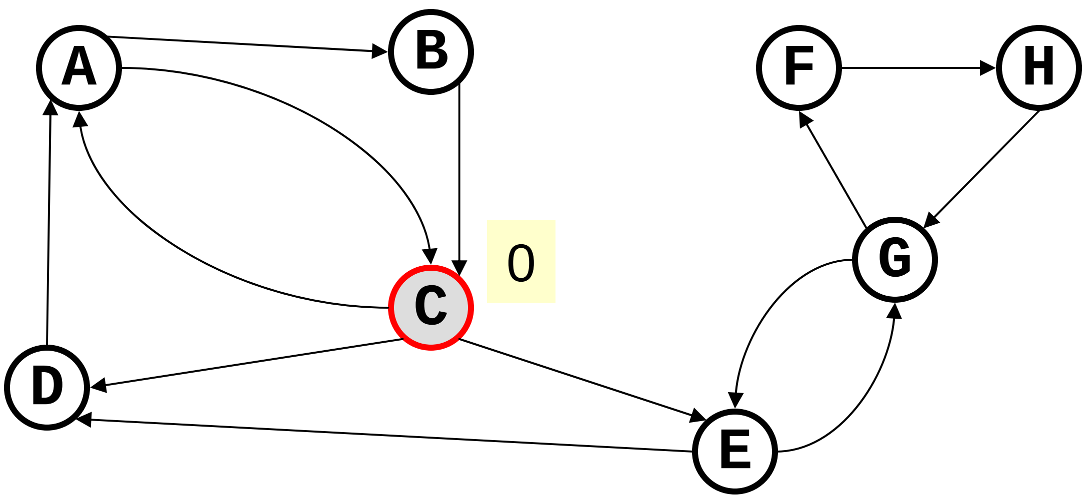
Breadth-First Search Alg.

- Uses a queue to track vertices that are “nearby”
- source vertex is **s**

```
Distance[s] := 0
Enqueue(Q,s); Mark(s)//After a vertex is marked once
                    // it won’t be enqueued again
while queue is not empty do
    X := Dequeue(Q);
    for each vertex Y adjacent to X do
        if Y is unmarked then
            Distance[Y] := Distance[X] + 1;
            Previous[Y] := X;//if we want to record paths
            Enqueue(Q,Y); Mark(Y);
```

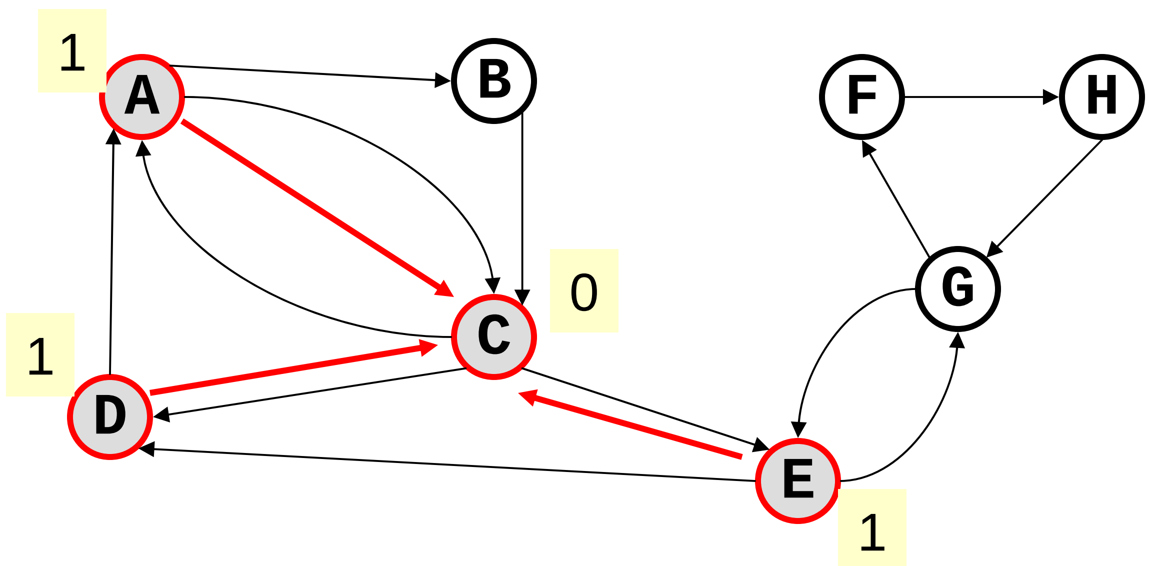
- Running time = $O(|V| + |E|)$

Example: Shortest Path length



Queue Q = C

Example (ct'd)

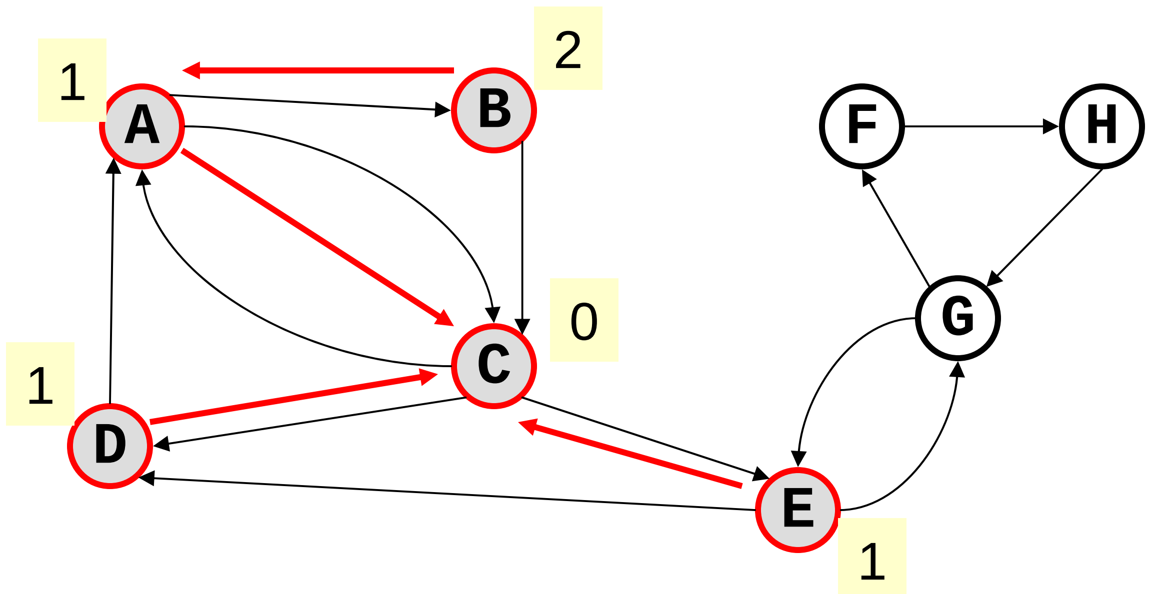


Queue Q = A D E

Previous pointer

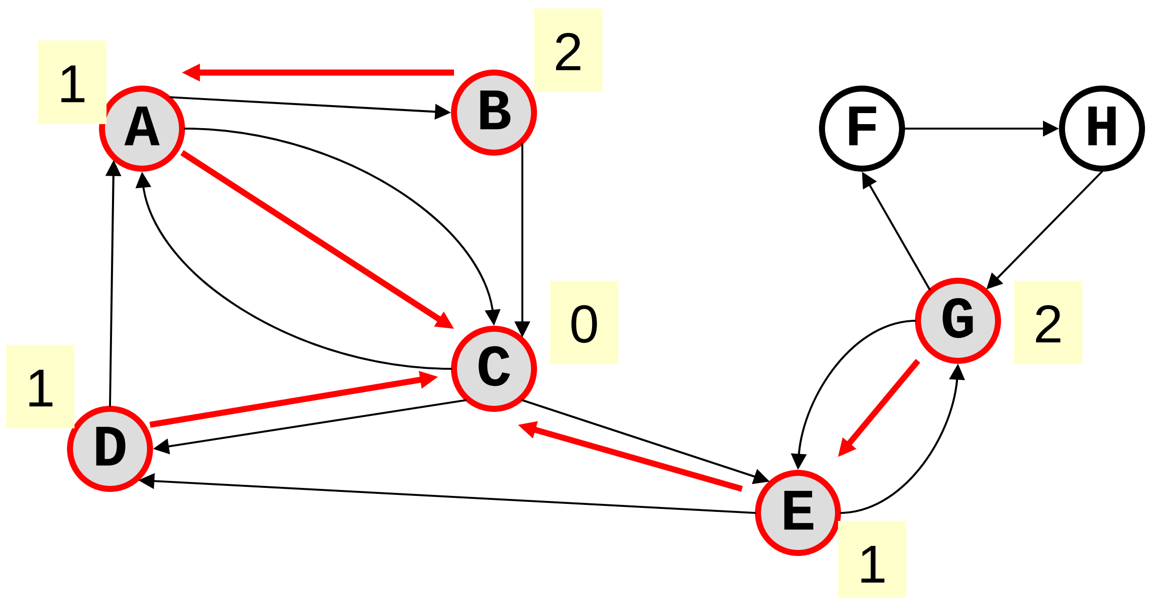
Indicates the vertex is marked

Example (ct'd)



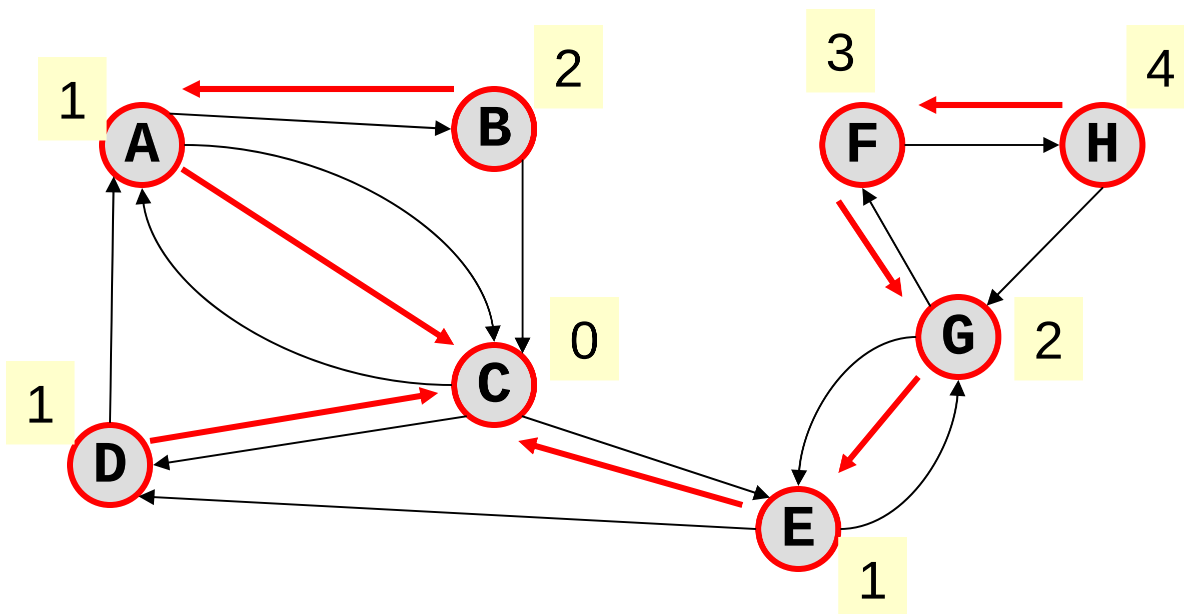
Q = D E B

Example (ct'd)



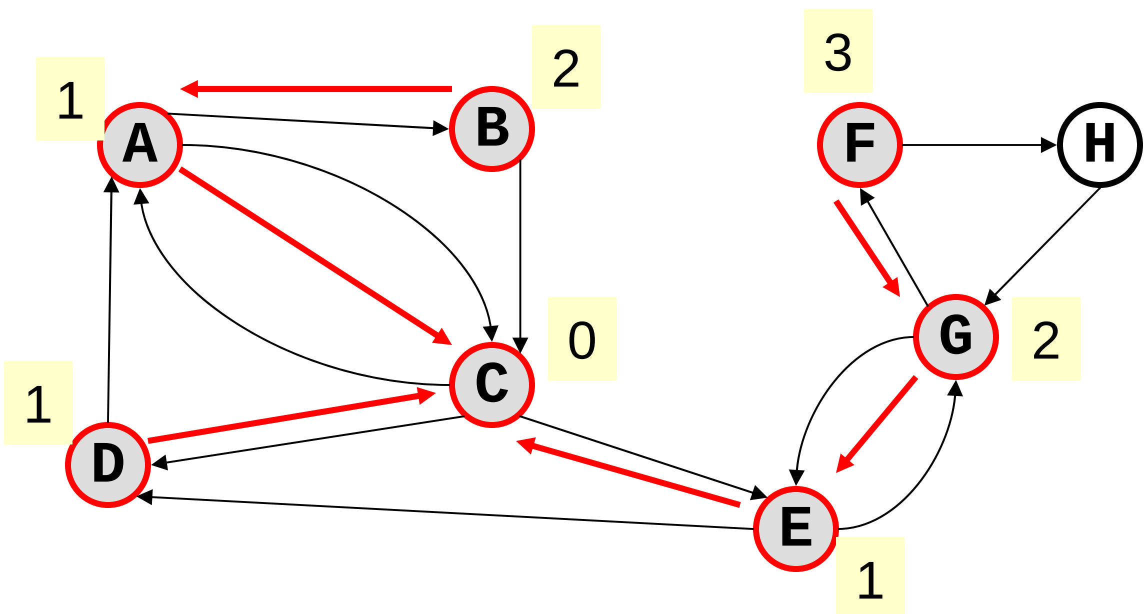
Q = B G

Example (ct'd)



Q = F

Example (ct'd)



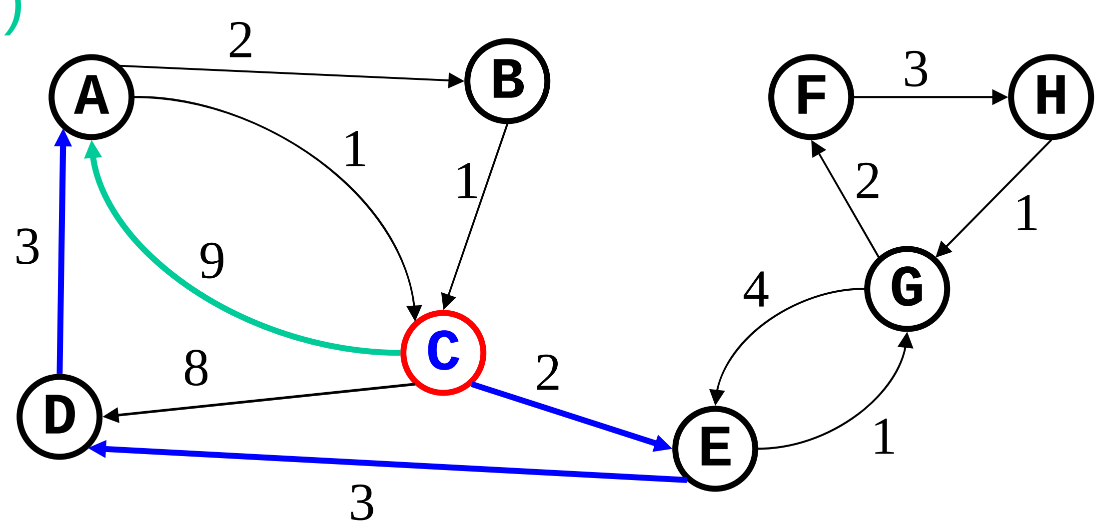
Q = H

What if edges have weights?

- Breadth First Search does not work anymore
  - minimum *cost* path may have more edges than minimum *length* path

Shortest path (length) from C to A:
C→A (cost = 9)

Minimum Cost Path = C→E→D→A
(cost = 8)



Dijkstra’s Algorithm for Weighted Shortest Path

- Classic algorithm for solving shortest path in weighted graphs (without negative weights)
- A greedy algorithm (irrevocably makes decisions without considering future consequences)
- Each vertex has a cost for path from initial vertex

Dijkstra's Algorithm

- Edsger Dijkstra (1930-2002)

Basic Idea of Dijkstra's Algorithm (1959)

- Find the vertex with smallest cost that has not been "marked" yet.
- Mark it and compute the cost of its neighbors.
- Do this until all vertices are marked.
- Note that each step of the algorithm we are marking one vertex and we won't change our decision: hence the term "greedy" algorithm
- Works for directed and undirected graphs

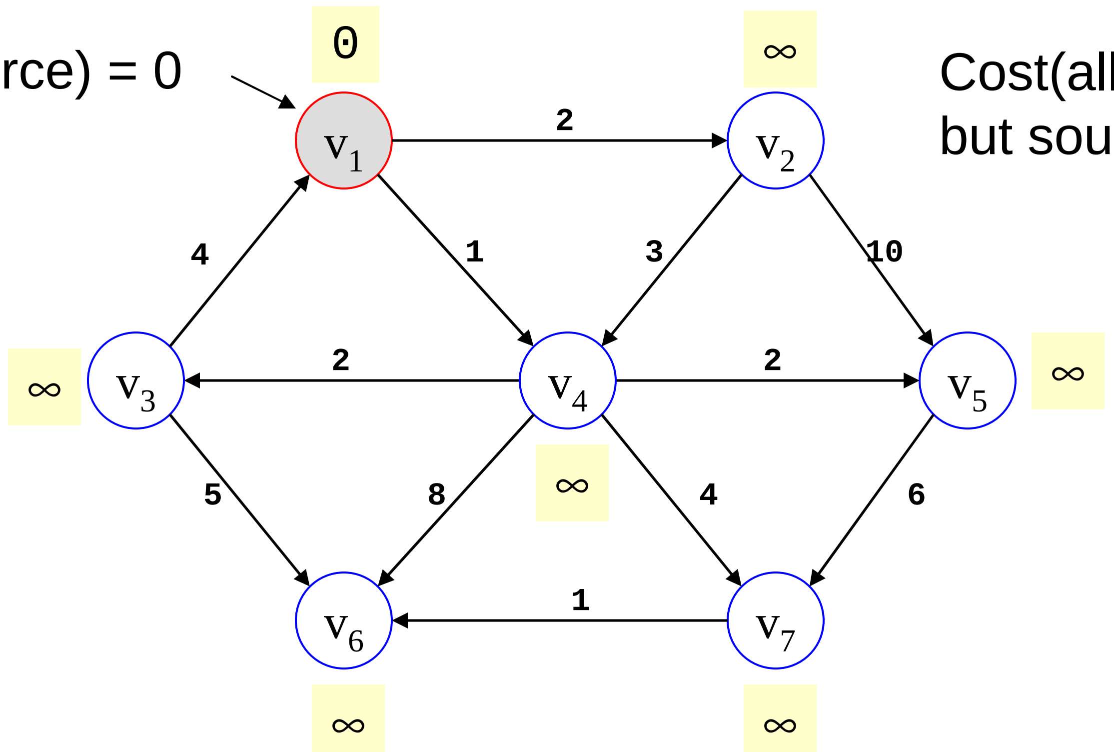
Dijkstra's Shortest Path Algorithm

- Initialize the cost of s to 0, and all the rest of the nodes to ∞
- Initialize set S to be ∅
  - › S is the set of nodes to which we have a shortest path
- While S is not all vertices
  - › Select the node A with the lowest cost that is not in S and identify the node as now being in S
  - › for each node B adjacent to A
    - if cost(A)+cost(A,B) < B's currently known cost
      - set cost(B) = cost(A)+cost(A,B)
      - set previous(B) = A so that we can remember the path

Example: Initialization

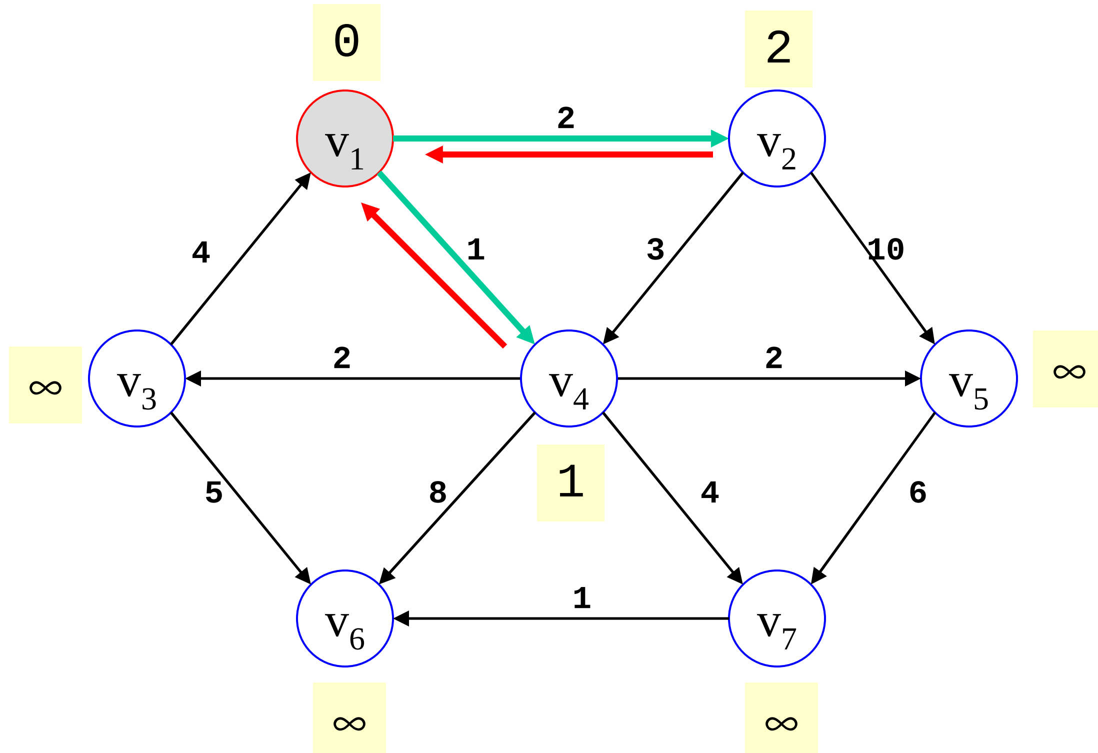
Cost(source) = 0

Cost(all vertices but source) = ∞



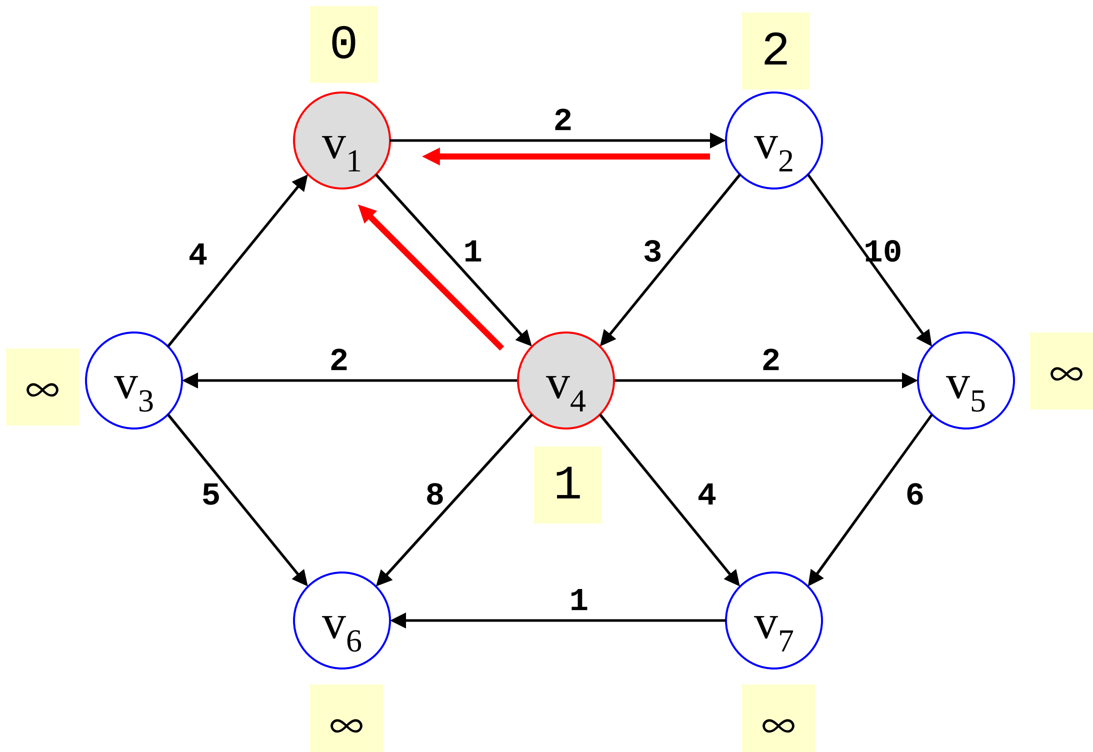
Pick vertex not in S with lowest cost.

Example: Update Cost neighbors



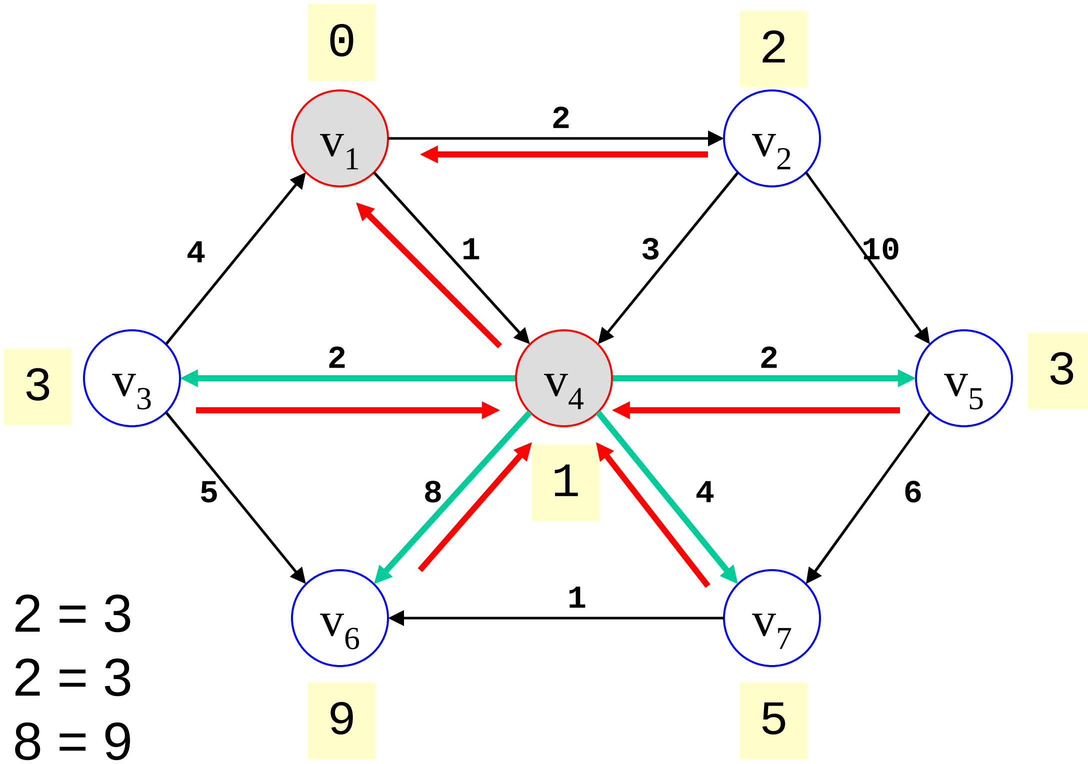
Cost(v_2) = 2
Cost(v_4) = 1

Example: pick vertex with lowest cost and add it to S



Pick vertex not in S with lowest cost, i.e., v_4

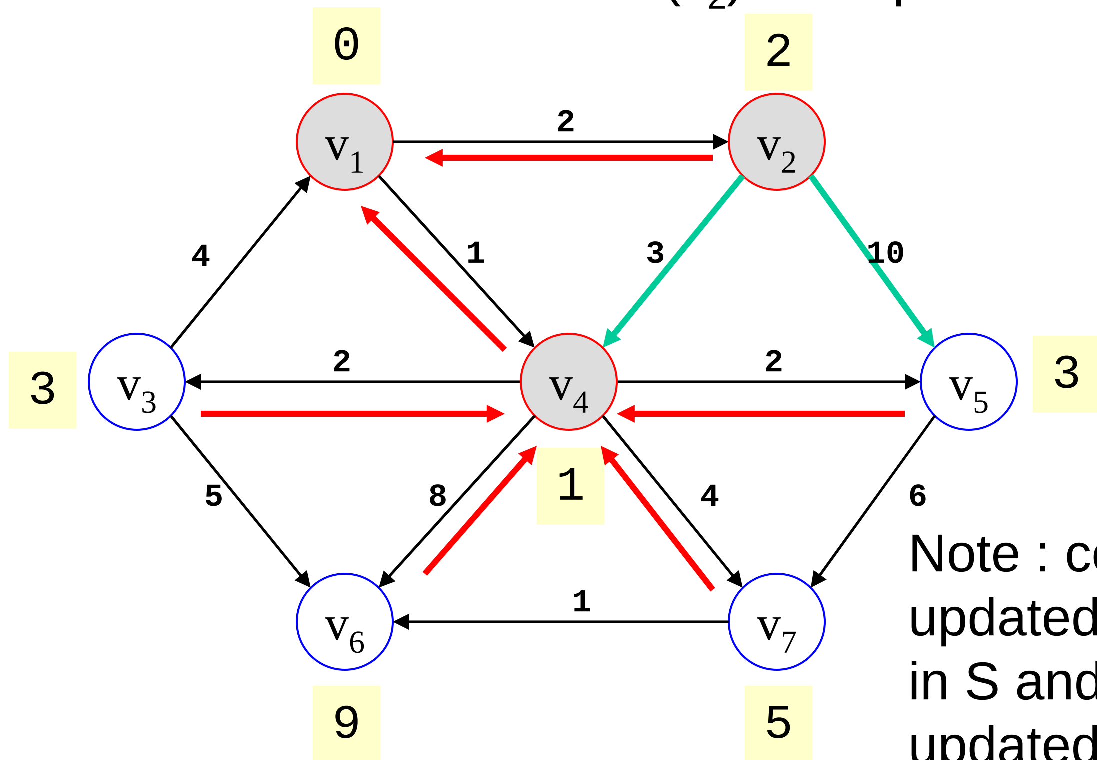
Example: update neighbors



Cost(v_3) = 1 + 2 = 3
Cost(v_5) = 1 + 2 = 3
Cost(v_6) = 1 + 8 = 9
Cost(v_7) = 1 + 4 = 5

Example (Ct'd)

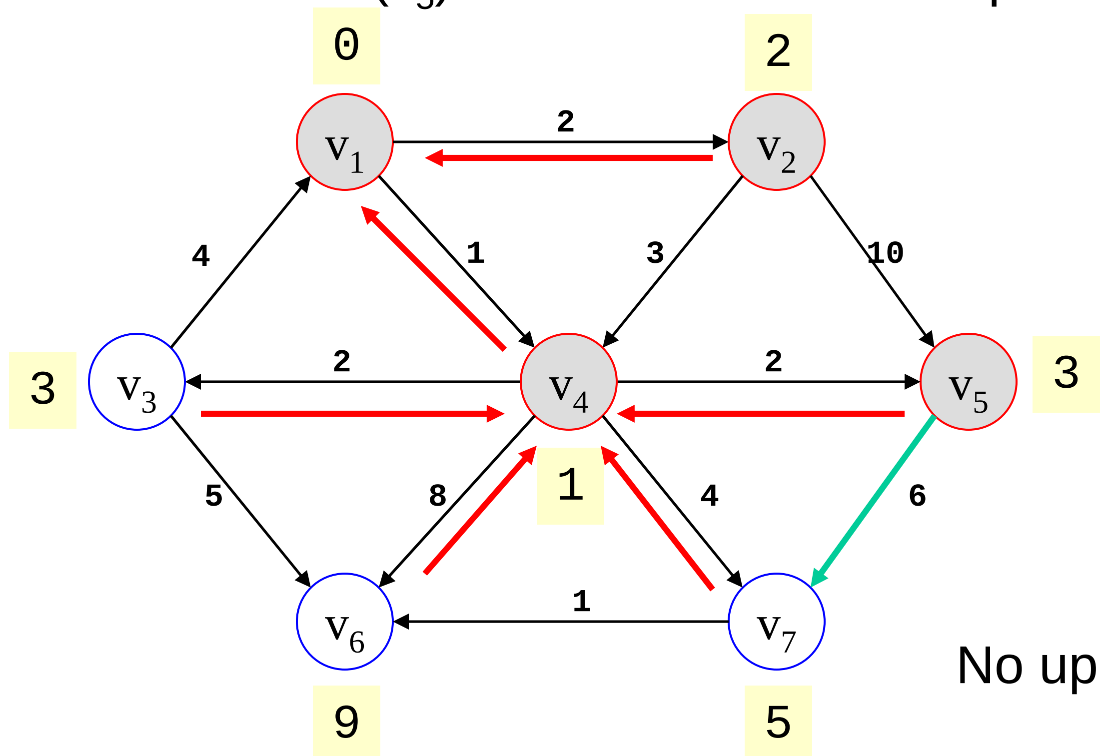
Pick vertex not in S with lowest cost (v_2) and update neighbors



Note : cost(v_4) not updated since already in S and cost(v_5) not updated since it is larger than previously computed

Example: (ct'd)

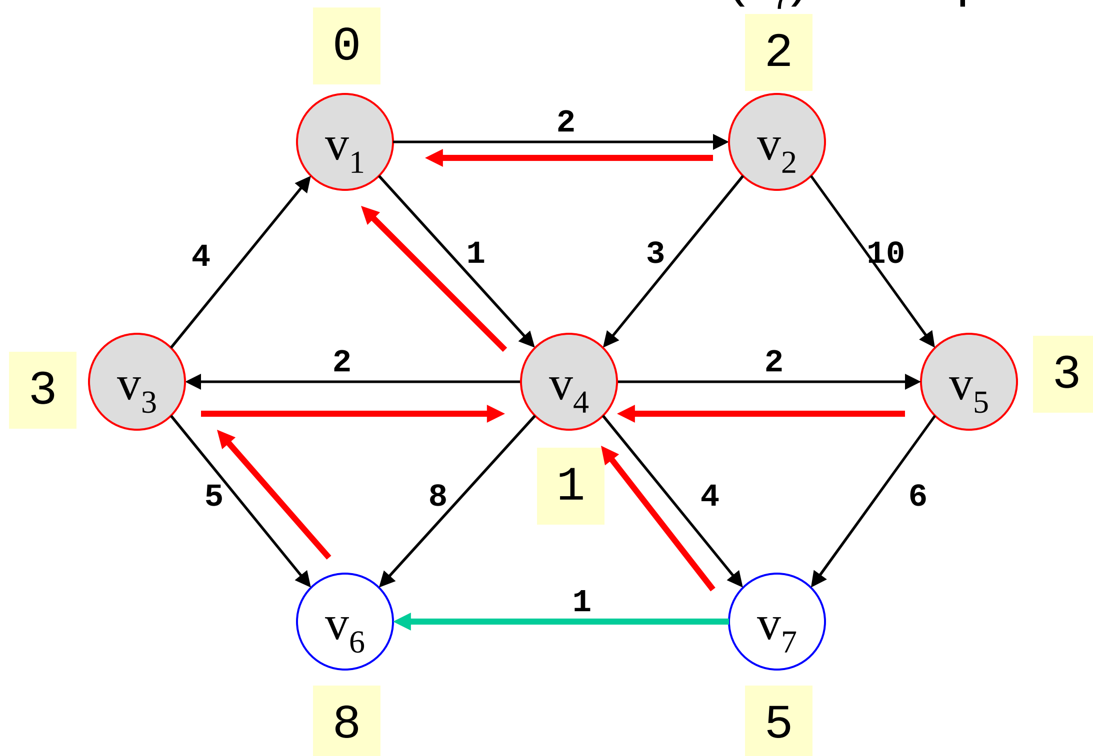
Pick vertex not in S (v_5) with lowest cost and update neighbors



No updating

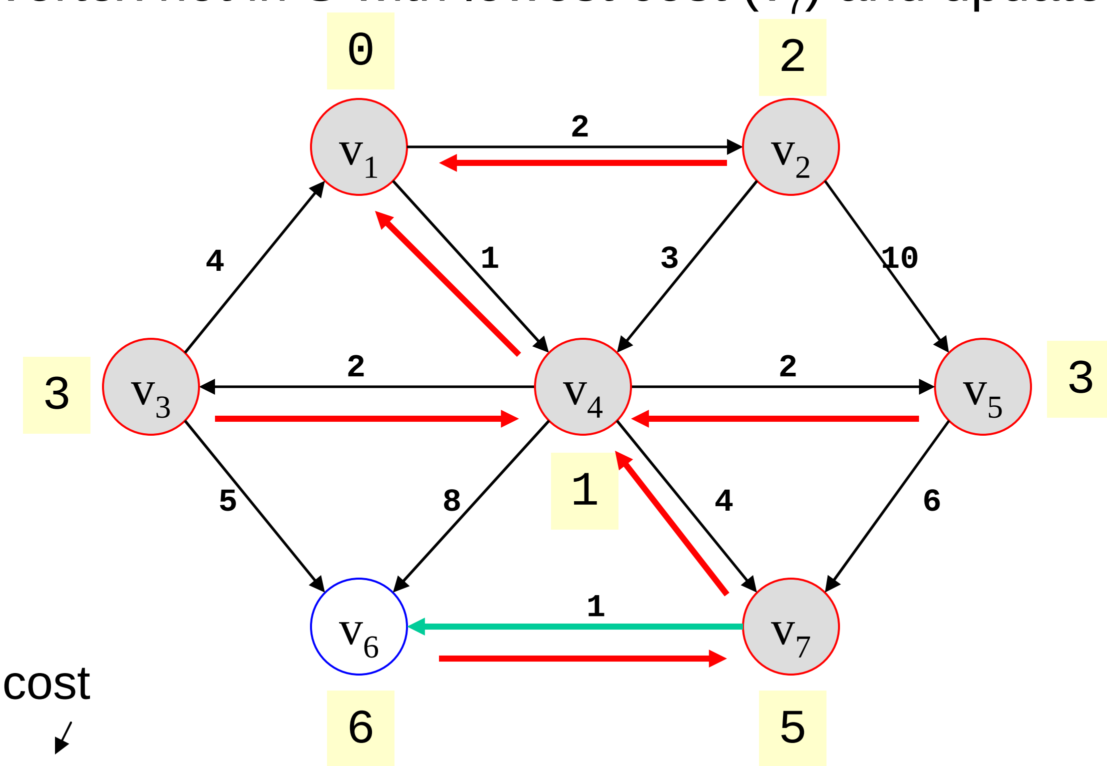
Example: (ct'd)

Pick vertex not in S with lowest cost (v_7) and update neighbors



Example: (ct'd)

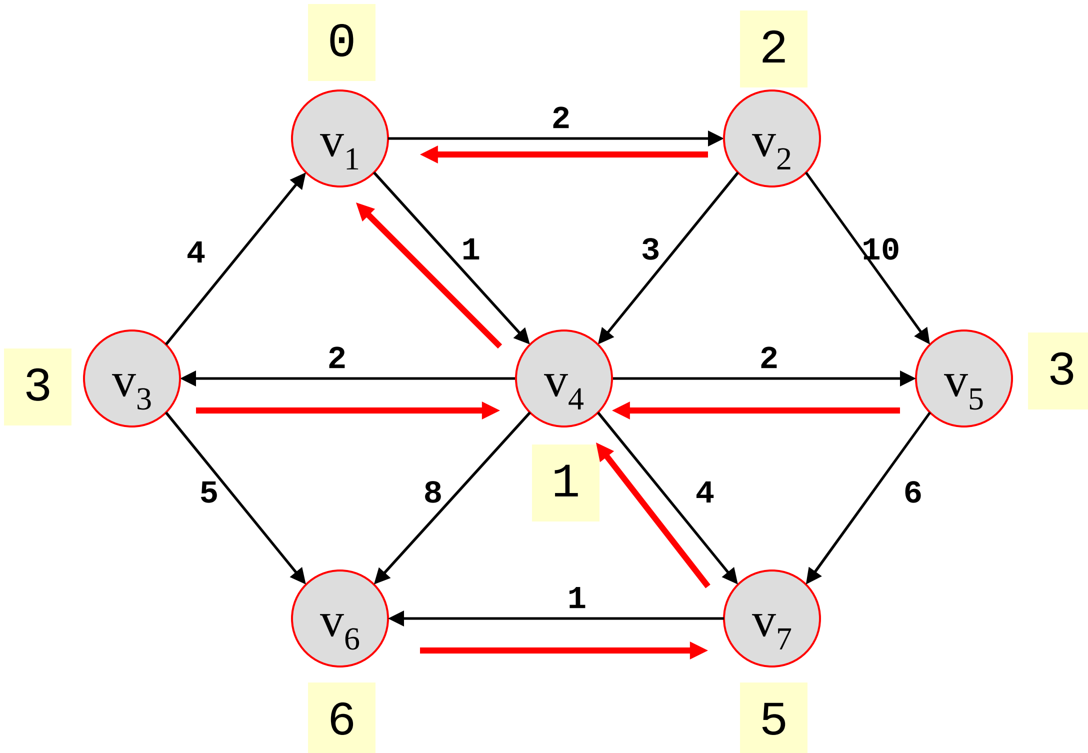
Pick vertex not in S with lowest cost (v_7) and update neighbors



Previous cost

$\text{Cost}(v_6) = \min(8, 5+1) = 6$

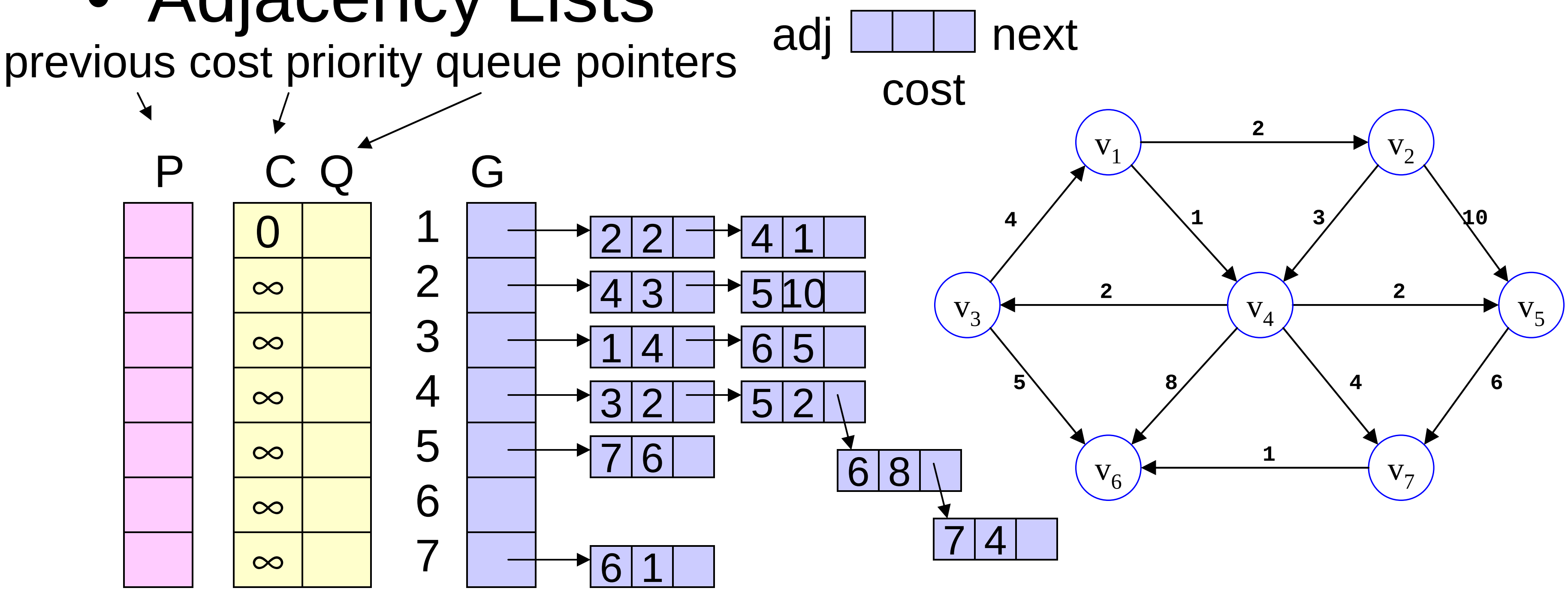
Example (end)



Pick vertex not in S with lowest cost (v_6) and update neighbors

Data Structures

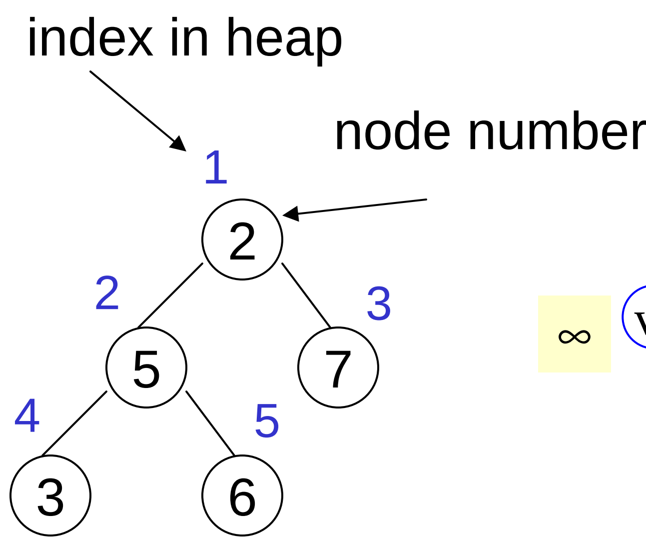
- Adjacency Lists



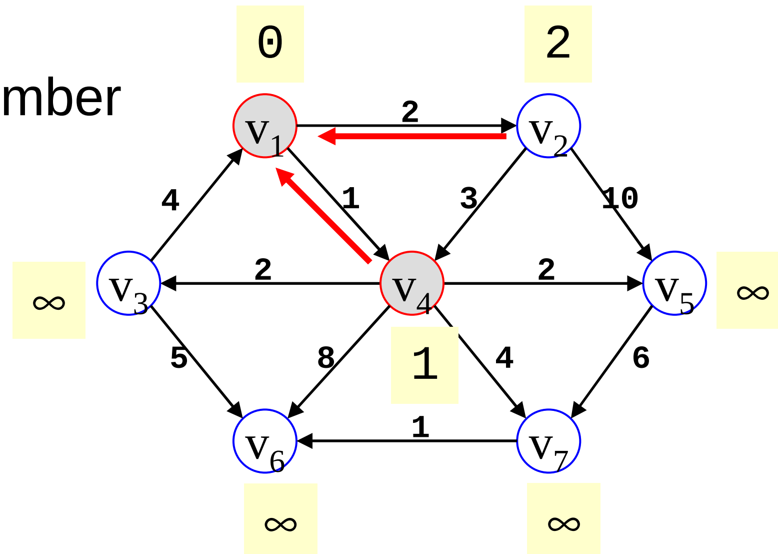
Priority queue for finding and deleting lowest cost vertex and for decreasing costs (Binary Heap works)

Priority Queue

		C	Q
1		0	
2	1	2	1
3		∞	4
4	1	1	
5		∞	2
6		∞	5
7		∞	3



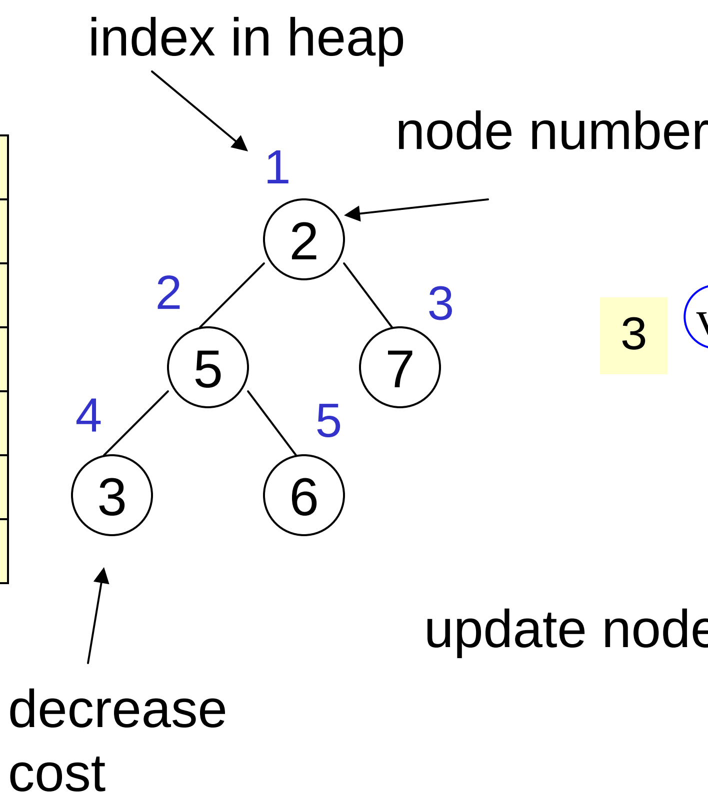
This is somewhat arbitrary and depends when the heap was first built

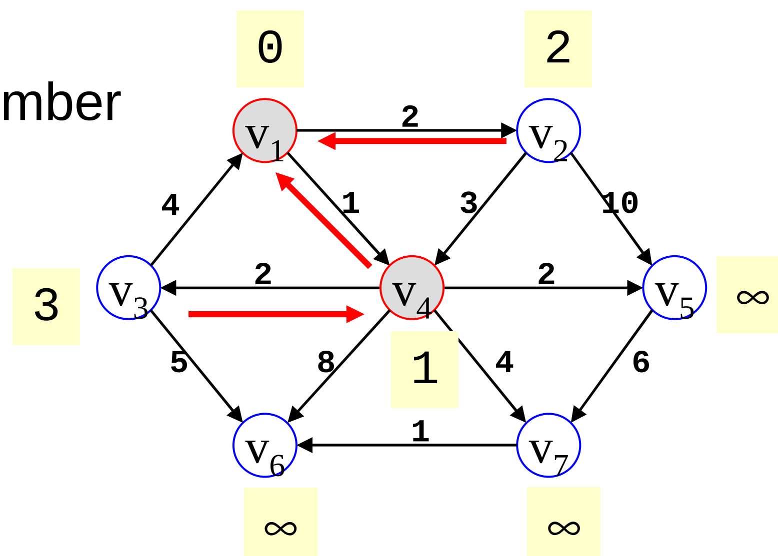


Before the update, but after find min.,i.e., v1 and v4 have been "deletemin"

Priority Queue

		C	Q
1		0	
2	1	2	1
3	<u>4</u>	<u>3</u>	4
4	1	1	
5		∞	2
6		∞	5
7		∞	3

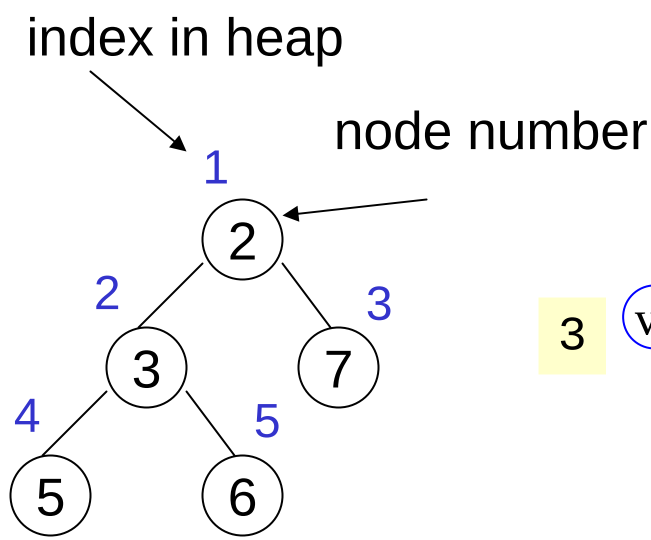




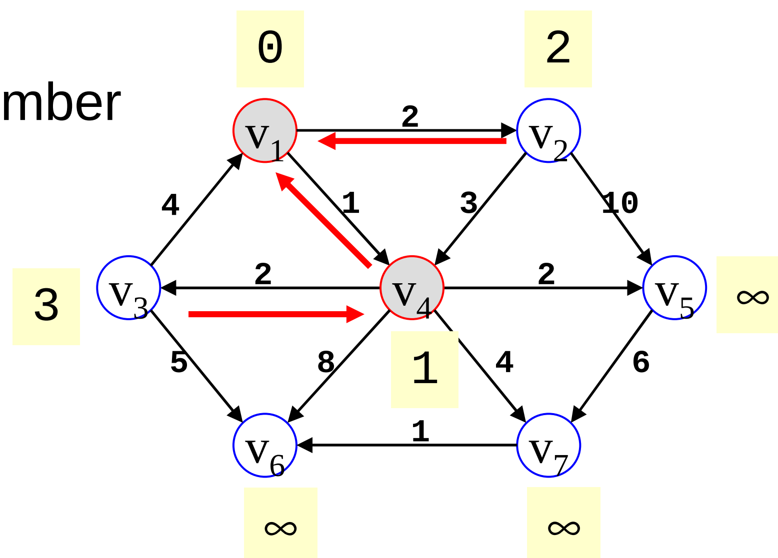
update node 3

Priority Queue

		C	Q
1		0	
2	1	2	1
3	4	3	2
4	1	1	
5		∞	4
6		∞	5
7		∞	3



percolate up



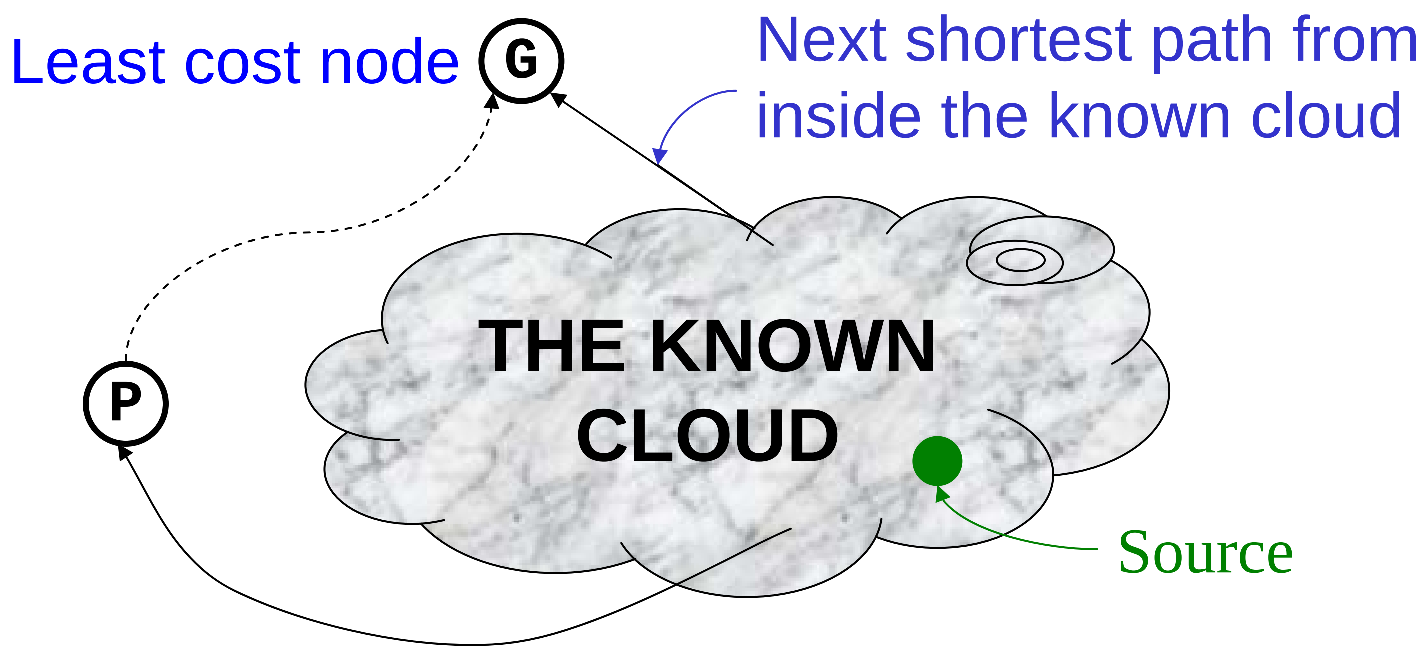
Time Complexity

- n vertices and m edges
- Initialize data structures O(n+m)
- Find min cost vertices O(n log n)
  - n delete mins
- Update costs O(m log n)
  - Potentially m updates
- Update previous pointers O(m)
  - Potentially m updates
- Total time O((n + m) log n) - very fast.

Correctness

- Dijkstra’s algorithm is an example of a greedy algorithm
- Greedy algorithms always make choices that currently seem the best
  - › Short-sighted – no consideration of long-term or global issues
  - › Locally optimal does not always mean globally optimal
- In Dijkstra’s case – choose the least cost node, but what if there is another path through other vertices that is cheaper?

“Cloudy” Proof



- If the path to G is the next shortest path, the path to P must be at least as long. Therefore, any path through P to G cannot be shorter!

Inside the Cloud (Proof)

- Everything inside the cloud has the correct shortest path
- Proof is by induction on the number of nodes in the cloud:
  - › Base case: Initial cloud is just the source with shortest path 0
  - › Inductive hypothesis: cloud of k-1 nodes all have shortest paths
  - › Inductive step: choose the least cost node G → has to be the shortest path to G (previous slide). Add k-th node G to the cloud