

HW 2 Pen & Paper Solutions:

R-3.2: total of 2 points possible

The number of operations executed by algorithms A and B are $8n \log n$ and $2n^2$, respectively. Determine n_0 such that A is better than B for all $n > n_0$.

Solution

The graphs describing the behavior of these algorithms start out with A higher (slower) than B, and eventually cross. After the point where they cross, B is always higher than A. Therefore we need to find the point where they cross, that is the value where

$8n \log n = 2n^2$. Applying algebra we get:

$$8n \log n = 2n^2$$

$$4n \log n = n^2$$

$$4 \log n = n$$

$$4 = n / \log n$$

Solving for n we get $n=16$, since $4 = 16/\log_2 16 = 16/4 = 4$

Thus $n_0 = 17$, since for all $n \geq 17$, A will be faster than B (at 16 they're equal.)

Another acceptable solution was to show the number of operations for each algorithm for all n up to the point where they cross.

Common problems

Many people used logs with bases other than 2. Common ones were \log_{10} and \ln . It's common in algorithmic analysis to use log base 2, as many algorithms use a divide and conquer strategy that naturally produces log base 2 behavior. Some people had the right idea but made an algebraic mistake. Partial credit was given. No credit was lost if you put 16 instead of 17 (i.e. if you confused $>$ with \geq .)

R-3.3: total of 2 points possible

The number of operations executed by algorithms A and B are $40n^2$ and $2n^3$, respectively. Determine n_0 such that A is better than B for all $n > n_0$.

Solution

The approach is the same as in 3.2. Equate the formulae and solve for n .

$$40n^2 = 2n^3$$

$$20n^2 = n^3$$

$$20 = n$$

Thus $n_0 = 21$, since for all $n \geq 21$, A will be faster than B (at 20, they're equal.)

R-3.6: total of 2 points possible

What is the sum of all the even numbers from 0 to $2n$ for any positive integer n ? Use induction to prove your answer.

Solution

The question asks what the sum is for:

$$0 + 2 + 4 + 6 + 8 + \dots + 2n$$

For any value of n .

The answer is $n(n+1)$.

Here is the inductive proof

Base case: $n = 1$

Sum(2^*i) for $i = 0, 1$ is $0 + 2 = 2$.

$$n(n+1) = 1(1+1) = 1(2) = 2$$

So the base case works.

Inductive hypothesis:

Assume: Sum(2^*i) for $i=0..n = n(n+1)$

Show: same holds for $n+1$

$$\text{Sum}(2^*i) \text{ for } i=0..n+1 = (n+1)((n+1)+1) = (n+1)(n+2) = n^2 + 3n + 2$$

Sum(2^*i) for $i=0..n+1 =$

Sum(2^*i) for $i=0..n$ + $2^*(n+1)$ *Replace underlined portion using inductive assumption*

$n(n+1)$ + $2^*(n+1)$ *Multiply out $2^*(n+1)$ term*

$n(n+1) + 2n+2$ *Multiple out $n(n+1)$ term*

$n^2 + n + 2n + 2$ *Gather terms*

$$n^2 + 3n + 2$$

This is equal to the **show** line above, so we're done.

Common problems

Many people simply didn't do any inductive proof.

R-3.8: total of 2 points possible

Order the following functions by asymptotic growth rate.

$$4n \log n + 2n \quad 2^{10} \quad 2^{\log n} \quad 3n + 100 \log n \quad 4n \quad 2^n \quad n^2 + 10n \quad n^3 \quad n \log n$$

Solution

Ordered from least to greatest:

$$2^{10} \quad 2^{\log n} \quad 3n + 100 \log n \quad 4n \quad n \log n \quad 4n \log n + 2n \quad n^2 + 10n \quad n^3 \quad 2^n$$

Explanation:

2^{10} is constant time

$2^{\log n}$ is linear $O(n)$ by the definition of log

$3n + 100 \log n$ is $O(n)$

$4n$ is also $O(n)$ and larger than $3n+100\log n$ because the $4n$ term is larger than the $3n$ term
 $n\log n$ is $O(n\log n)$

$4n\log n+2n$ is also $O(n\log n)$ because the $n\log n$ term dominates the $2n$ term

n^2+10n is $O(n^2)$

n^3 is $O(n^3)$

2^n is $O(2^n)$ - exponential

Common mistakes

Many people missed the fact that $2^{\log n}$ is linear. If that was your only mistake, I didn't take off. I also didn't penalize as heavily in comparing within an order (e.g. $3n+100\log n$ and $4n$ are both $O(n)$ so I didn't count off as much as you flipped them but kept them in the same place relative to all the other entries.)

Relational properties: total of 4 points, one for each part

Give an example of each of these situations or explain why it is impossible. The answer should be non-trivial, i.e., not the empty set.

- A relation which is reflexive, not transitive, and not symmetric
 - Answer: $\{(a,a), (b,b), (c,c), (a,b), (b,c)\}$ over the set $\{a, b, c\}$
This is reflexive, $R(x,x)$ is true for all x . It is not transitive because $R(a,b)$ and $R(b,c)$ but it's not true that $R(a,c)$. It is not symmetric because $R(a,b)$ but it's not true that $R(b,a)$
 - Common mistakes: Some people put something like $\{(a,a), (b,b)\}$. However this is both transitive and symmetric, though trivially so. In all cases where (x,y) and (y,z) , it's also the case that (x,z) . The only cases where this is true is where $x = y = z = a$ OR $x = y = z = b$, but nonetheless it is true in those cases. A similar argument holds for symmetry.
- A relation which is transitive and not reflexive
 - Answer: $\{(a,b), (b,c), (a,c)\}$
 - Another answer, $<$. (The $<$ relation over the integers is transitive but not reflexive)
- A function of the integers which is reflexive
 - Answer: The $=$ (equality) function is reflexive.
 - Common mistake: Some people thought that because the problem statement pointed out that a function of the integers has all integers as its domain and (a subset of) the integers as its range, this was impossible because $=$ requires that the range and domain are the same. However, this isn't a problem since it may not be a proper subset. Another common mistake was to give the function in terms of a domain $\{1, 2, 3\}$ rather than all integers. While I accepted this (and added "...") to indicate that it should have been extended), this type of thinking caused problems for the next question. Similarly many people completely ignore the part about it being a function of the integers and wrote functions whose domains (and ranges) were things like $\{a,b,c\}$ or $\{x,y,z\}$.
- A function of the integers which is transitive and not reflexive.

- Answer: this is impossible because to be transitive, it must be the case that if (x,y) and (y,z) are in the function then so must be (x, z) , however that would imply two different values for (x) . But functions (as opposed to relations) must have a unique value for any input. Put another way, you can't have $f(x) = y$, $f(y) = z$ and then $f(x)=z$, since $f(x)$ already equals y . The only way for a function to be transitive is if it's trivially so, as in the case of $=$. But in that case, it's also reflexive.
- Common problems: Though a number of people pointed out the problem with having repeated x values with different y, z , etc. Very few mentioned the case of the trivially transitive function being reflexive. However as long as you got the first part about repeated x values, full credit was given. A number of people did things like $\{(1,2), (2,3), (1,3)\}$ which is not a function, or worse, $\{(a,b), (b,c), (a,c)\}$ which is neither a function, nor does it have the integers as its domain.