Directed Graphs (Part II)

CSE 373

Data Structures

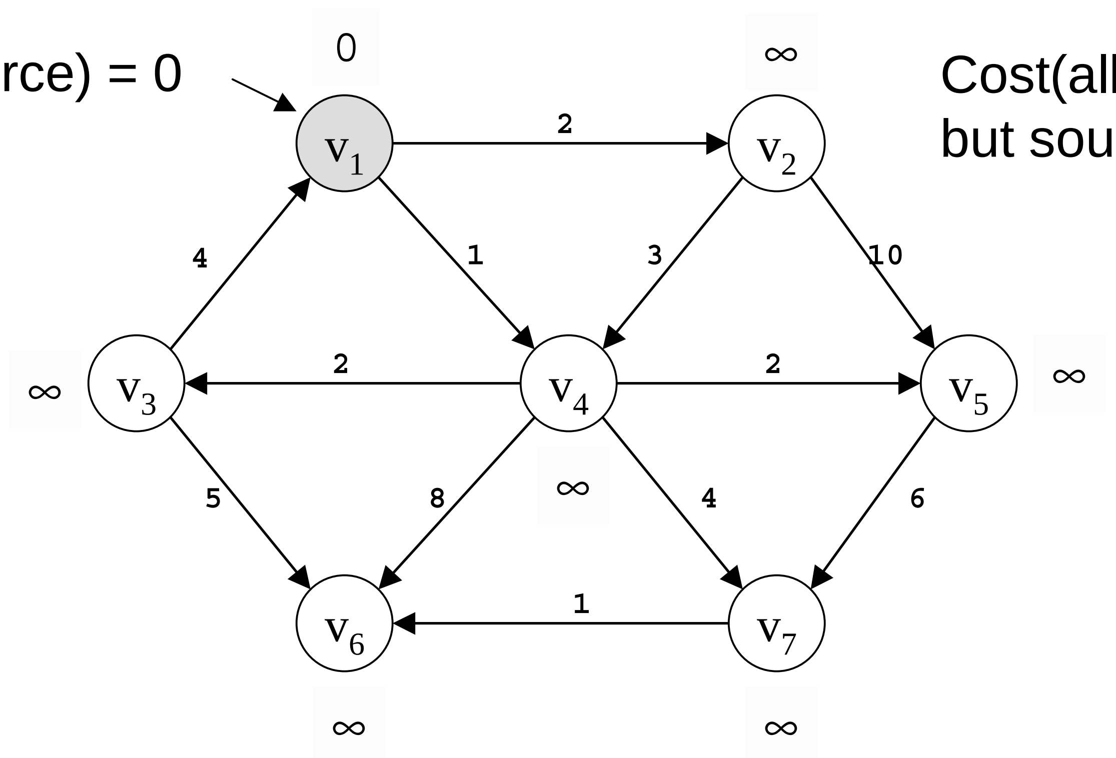
Lecture 19

Dijkstra’s Shortest Path Algorithm

- Initialize the cost of s to 0, and all the rest of the nodes to ∞
- Initialize set S to be ∅
  - › S is the set of nodes to which we have a shortest path
- While S is not all vertices
  - › Select the node A with the lowest cost that is not in S and identify the node as now being in S
  - › for each node B adjacent to A
    - if cost(A)+cost(A,B) < B’s currently known cost
      - – set cost(B) = cost(A)+cost(A,B)
      - – set previous(B) = A so that we can remember the path

Example: Initialization

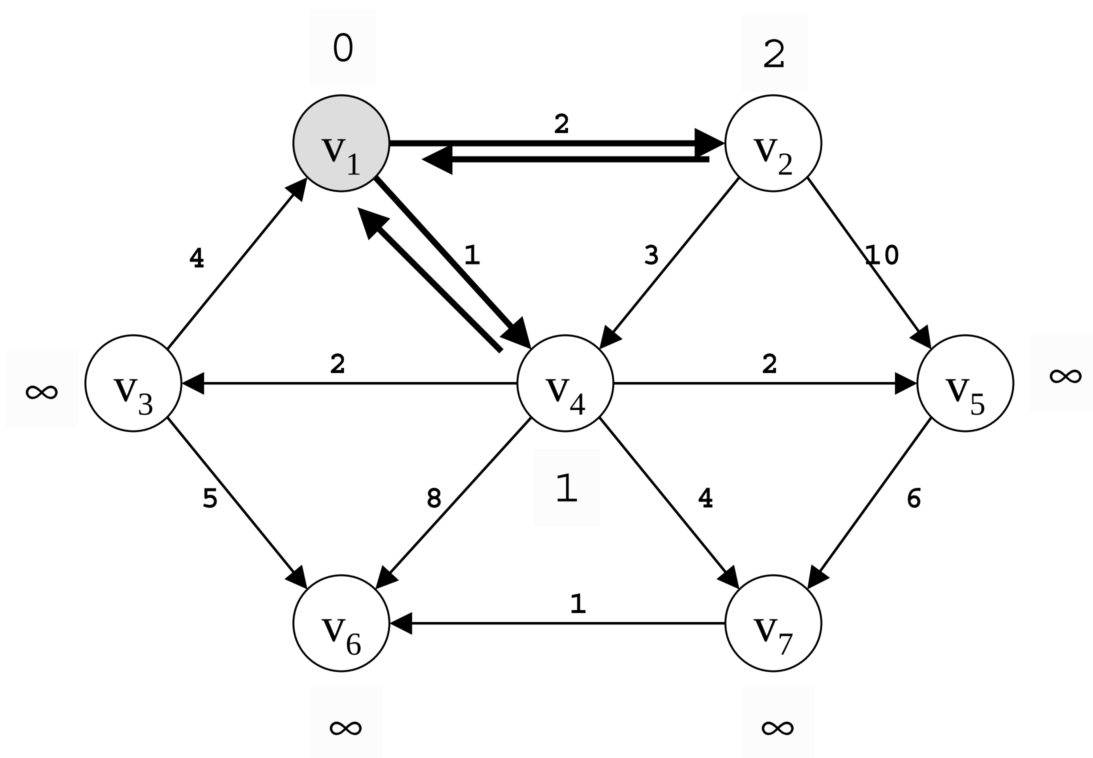
Cost(source) = 0



Cost(all vertices but source) = ∞

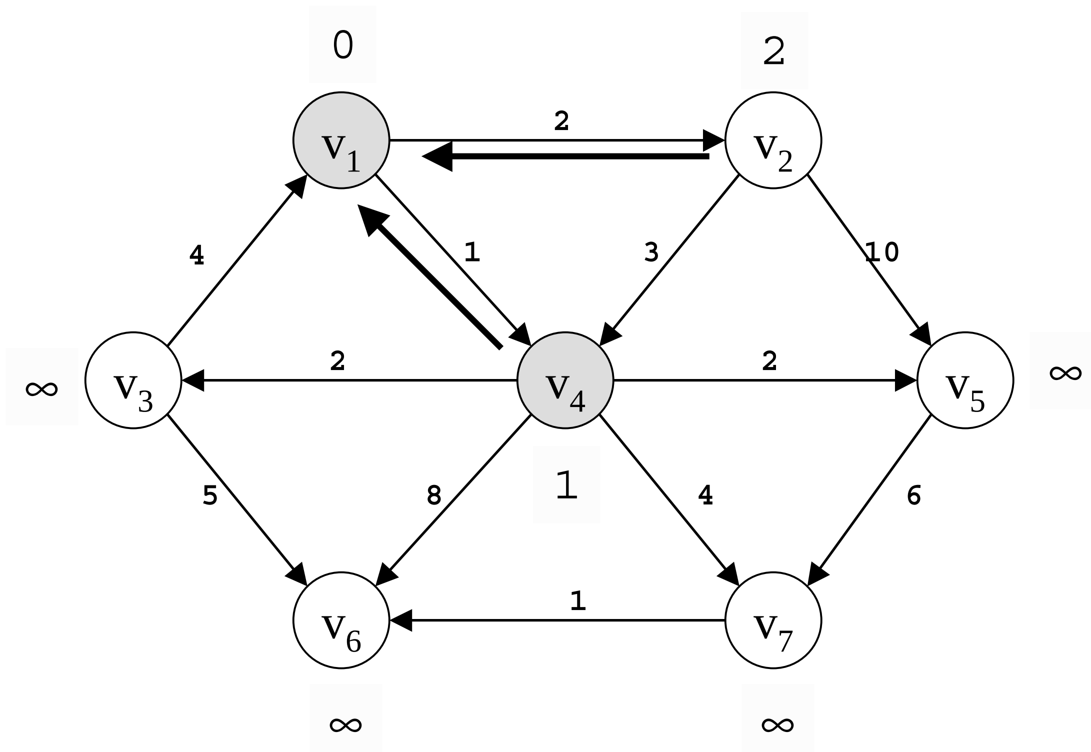
Pick vertex not in S with lowest cost.

Example: Update Cost neighbors



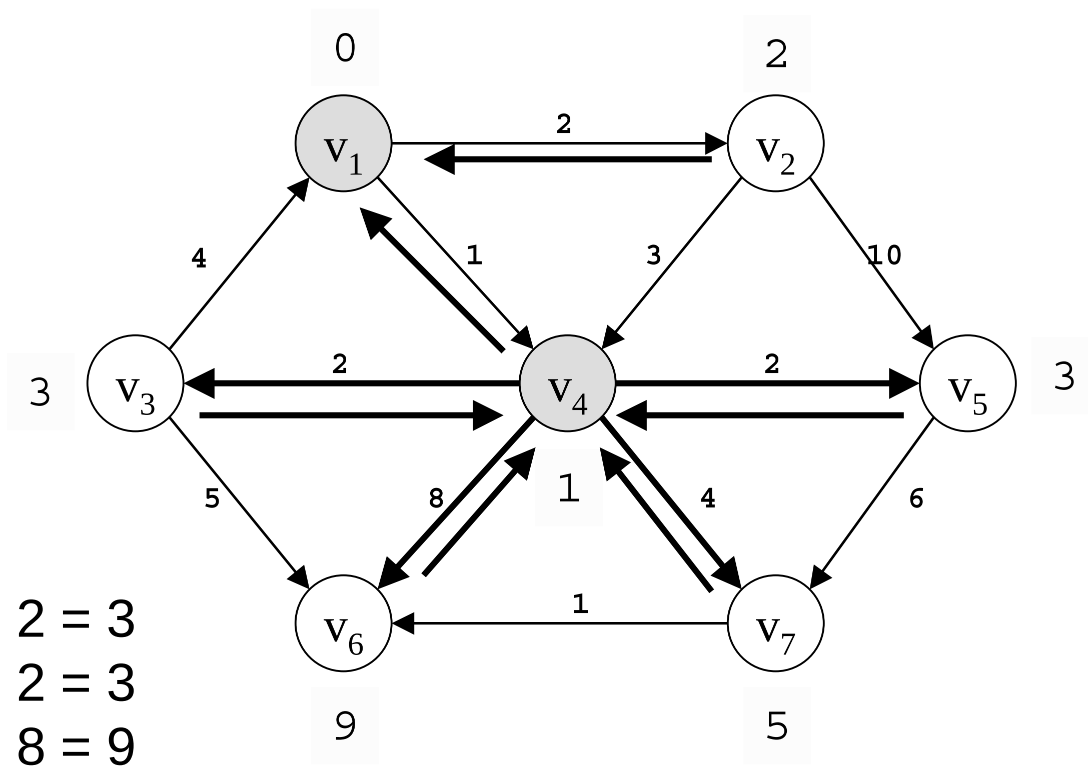
Cost(v_2) = 2
Cost(v_4) = 1

Example: pick vertex with lowest cost and add it to S



Pick vertex not in S with lowest cost, i.e., v_4

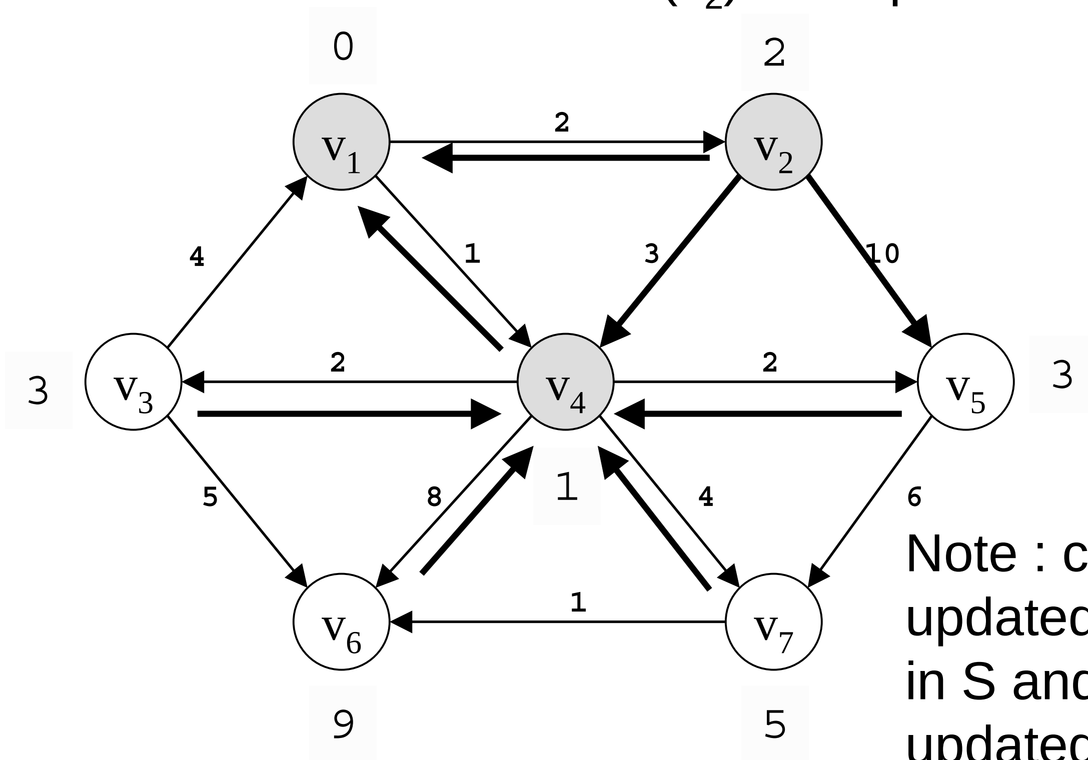
Example: update neighbors



$Cost(v_3) = 1 + 2 = 3$
$Cost(v_5) = 1 + 2 = 3$
$Cost(v_6) = 1 + 8 = 9$
$Cost(v_7) = 1 + 4 = 5$

Example (Ct'd)

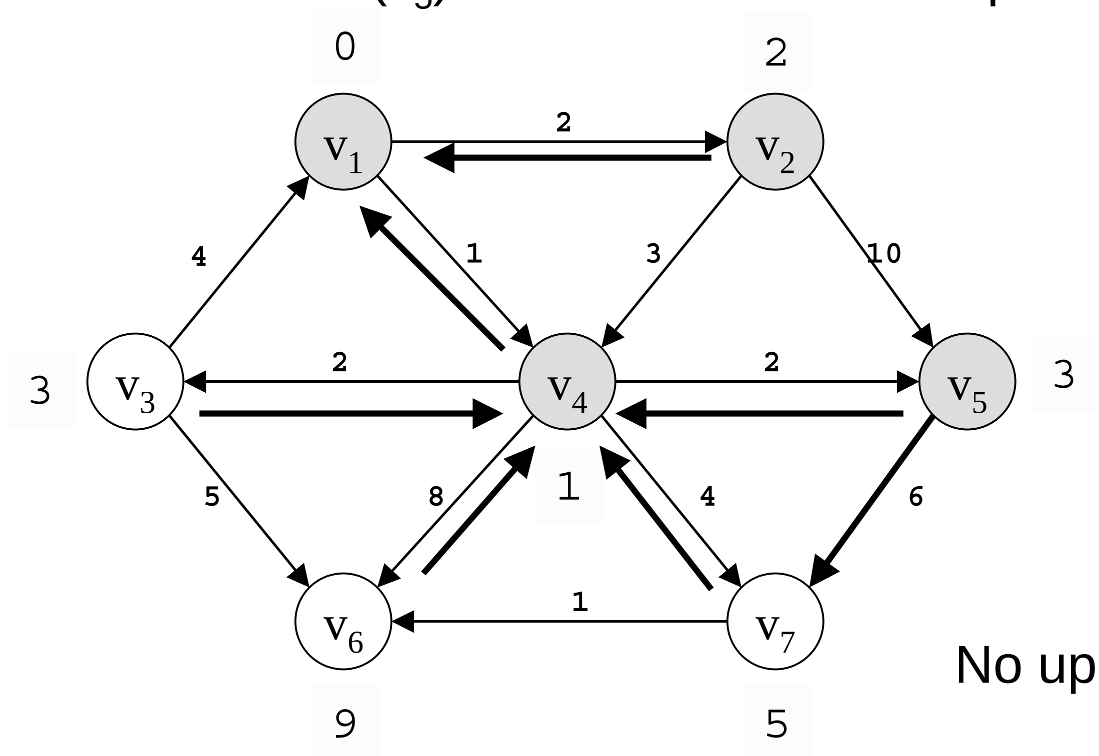
Pick vertex not in S with lowest cost (v_2) and update neighbors



Note : cost(v_4) not updated since already in S and cost(v_5) not updated since it is larger than previously computed

Example: (ct'd)

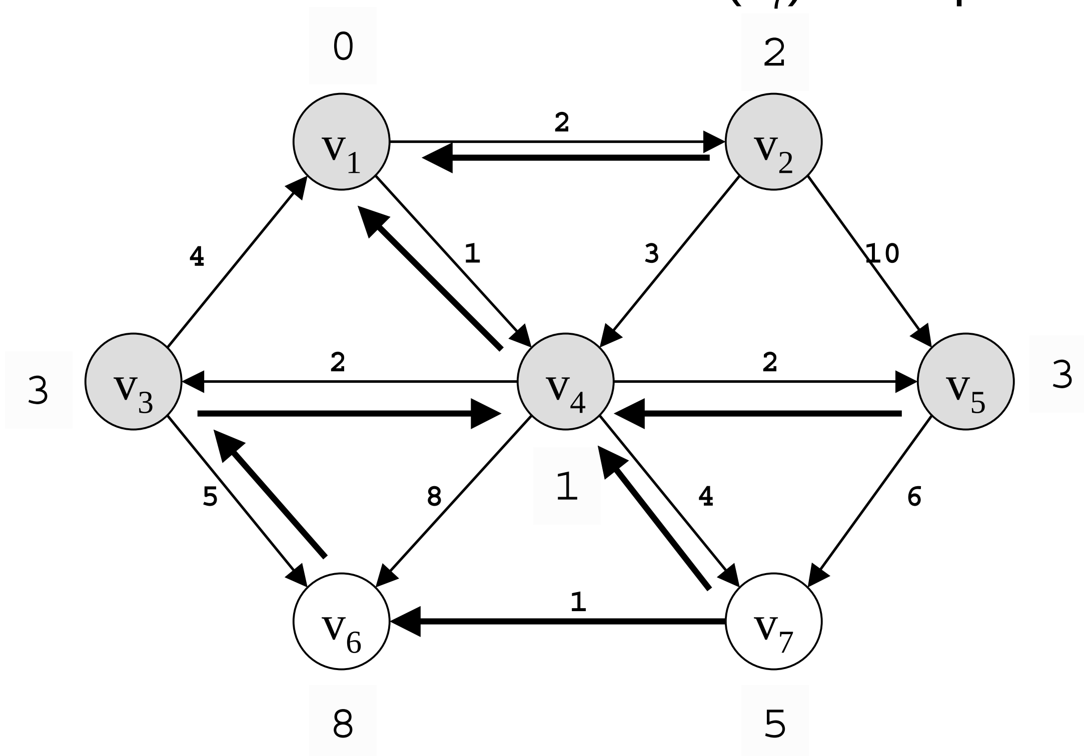
Pick vertex not in S (v_5) with lowest cost and update neighbors



No updating

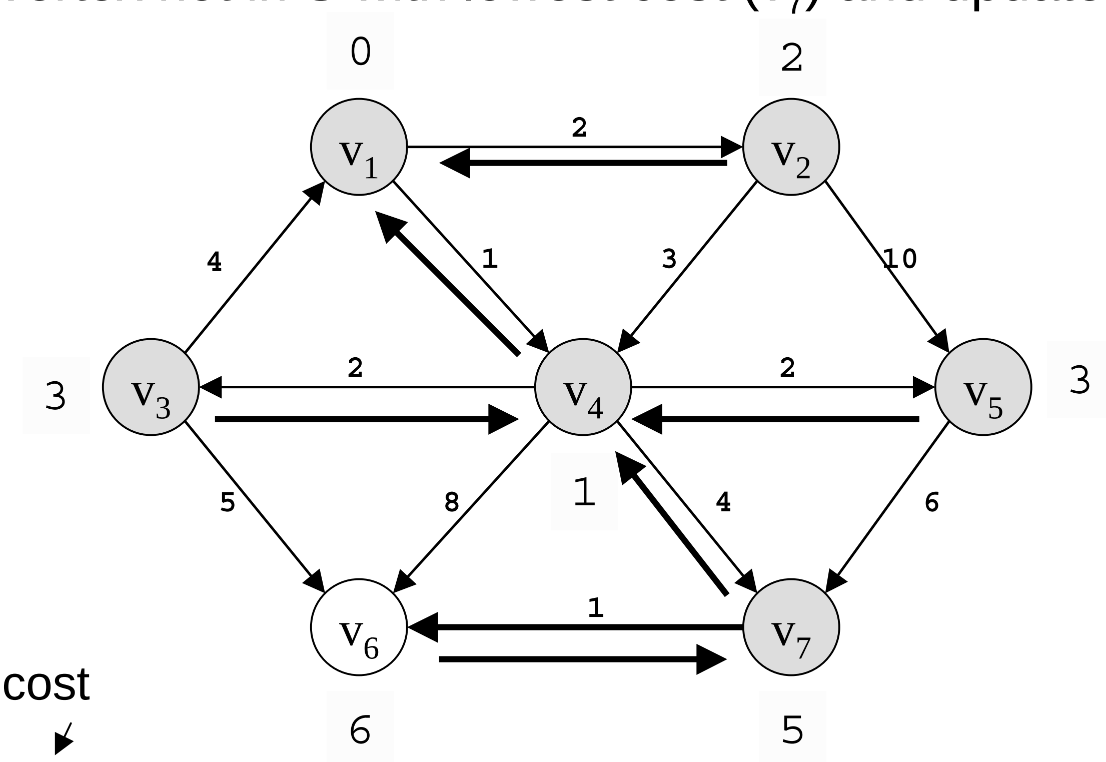
Example: (ct'd)

Pick vertex not in S with lowest cost (v_7) and update neighbors



Example: (ct'd)

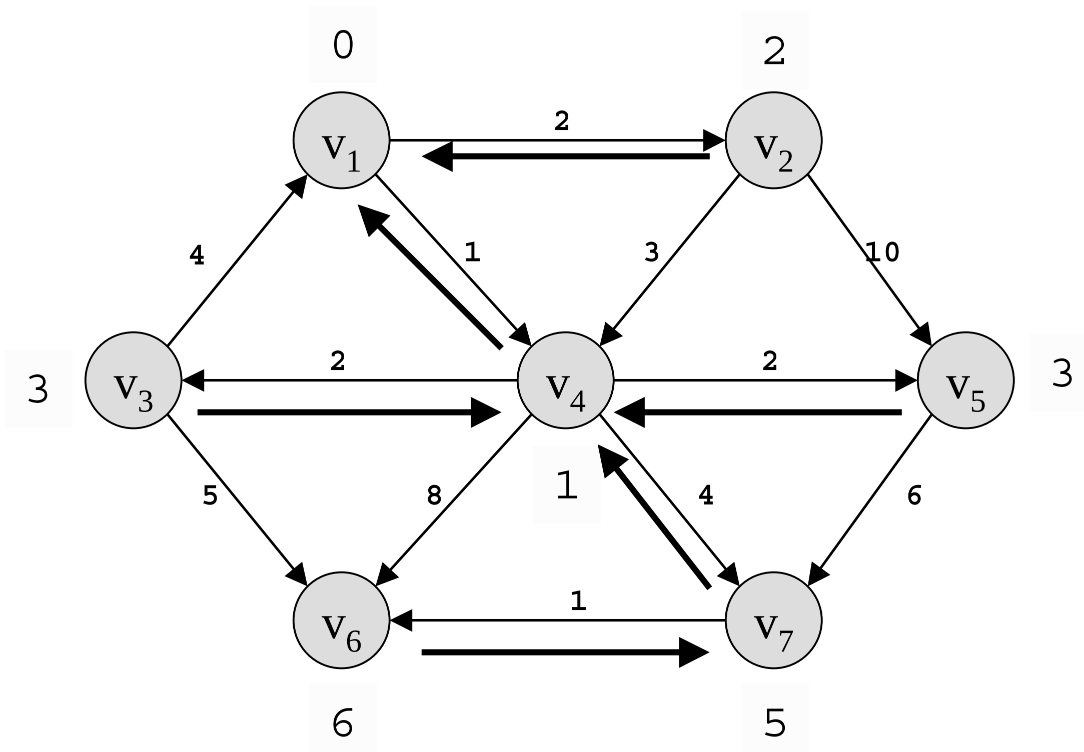
Pick vertex not in S with lowest cost (v_7) and update neighbors



Previous cost

Cost(v_6) = min (8, 5+1) = 6

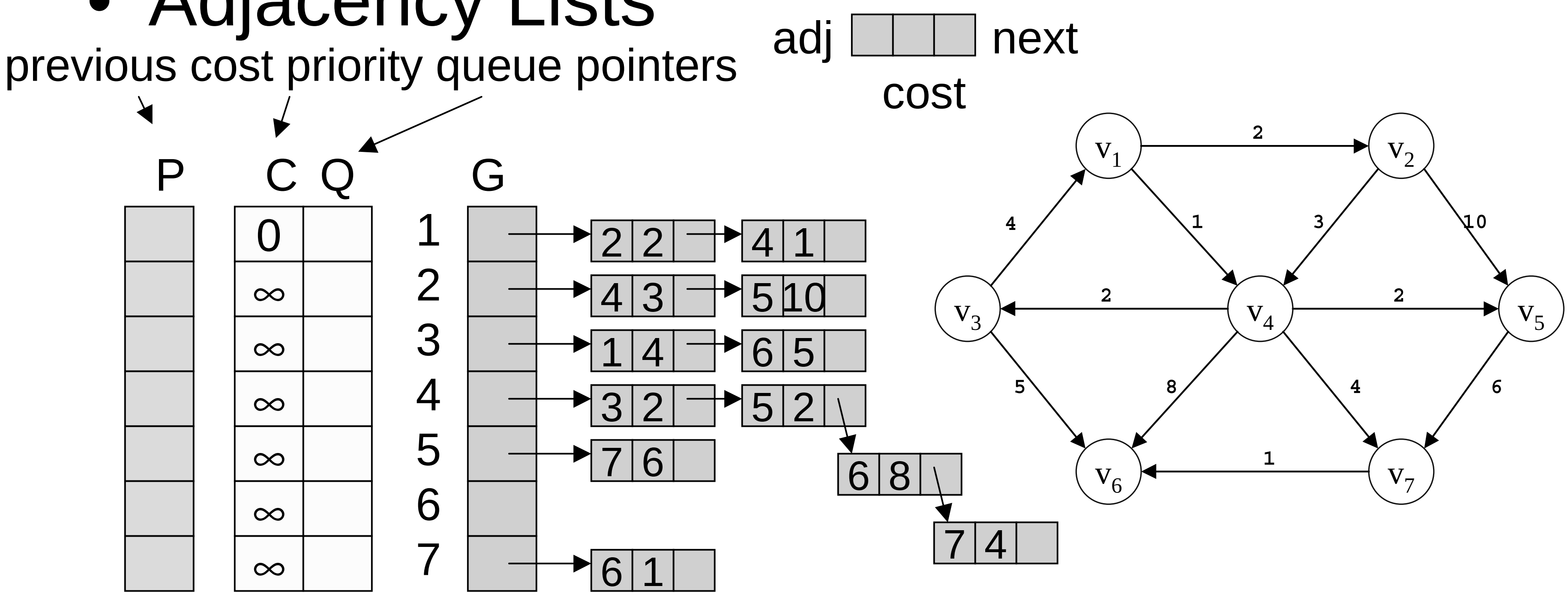
Example (end)



Pick vertex not in S with lowest cost (v_6) and update neighbors

Data Structures

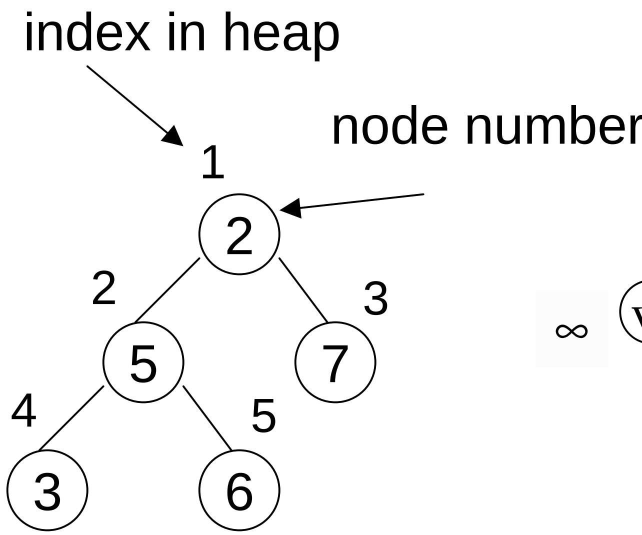
- Adjacency Lists



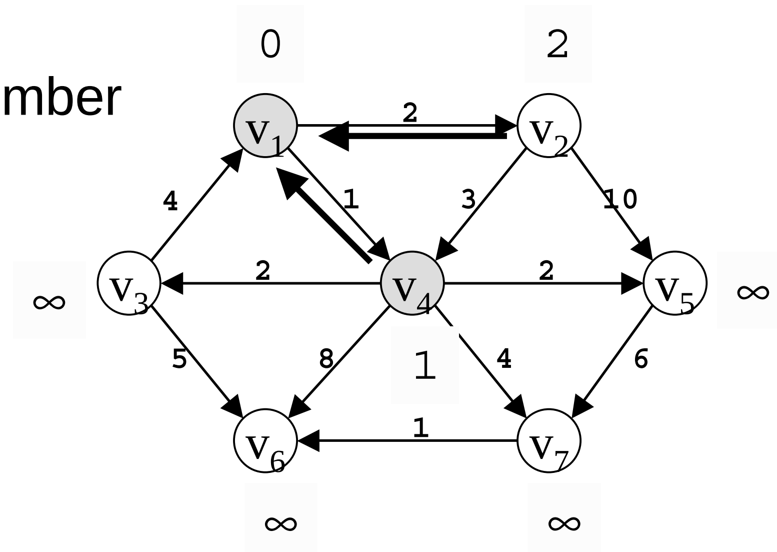
Priority queue for finding and deleting lowest cost vertex and for decreasing costs (Binary Heap works)

Priority Queue

		C	Q
1		0	
2	1	2	1
3		∞	4
4	1	1	
5		∞	2
6		∞	5
7		∞	3



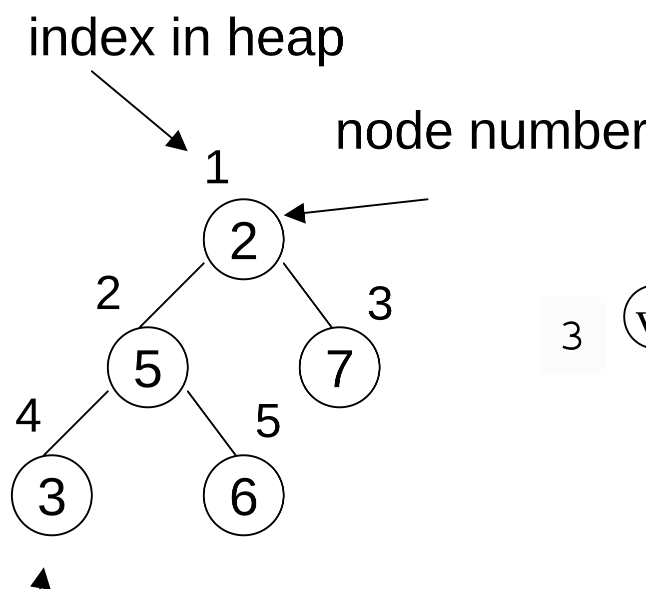
This is somewhat arbitrary and depends when the heap was first built



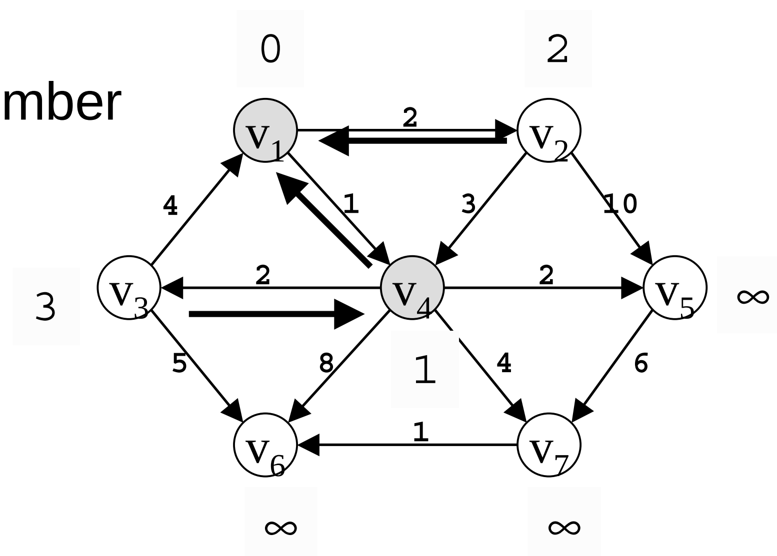
Before the update, but after find min.

Priority Queue

		C	Q
1		0	
2	1	2	1
3	<u>4</u>	<u>3</u>	4
4	1	1	
5		∞	2
6		∞	5
7		∞	3



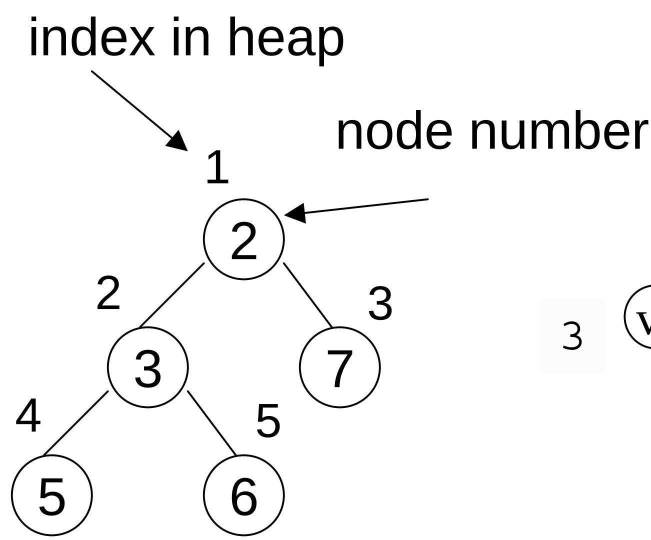
decrease
cost

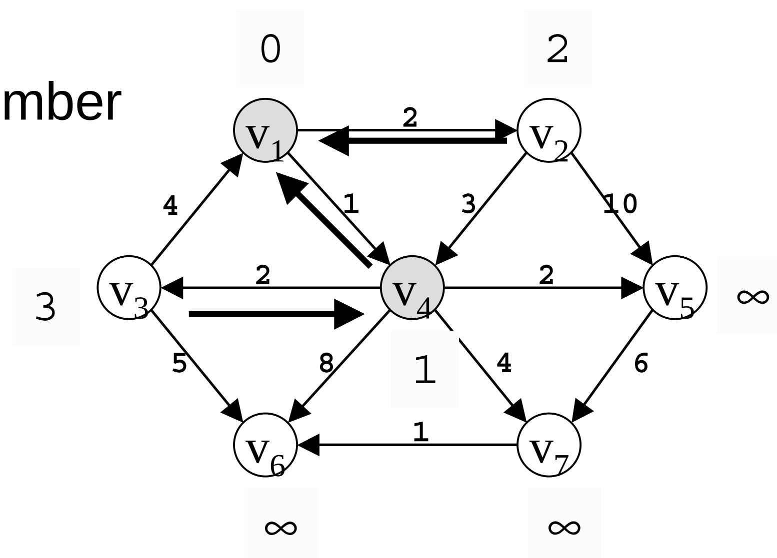


update node 3

Priority Queue

		C	Q
1		0	
2	1	2	1
3	4	3	2
4	1	1	
5		∞	4
6		∞	5
7		∞	3





percolate up

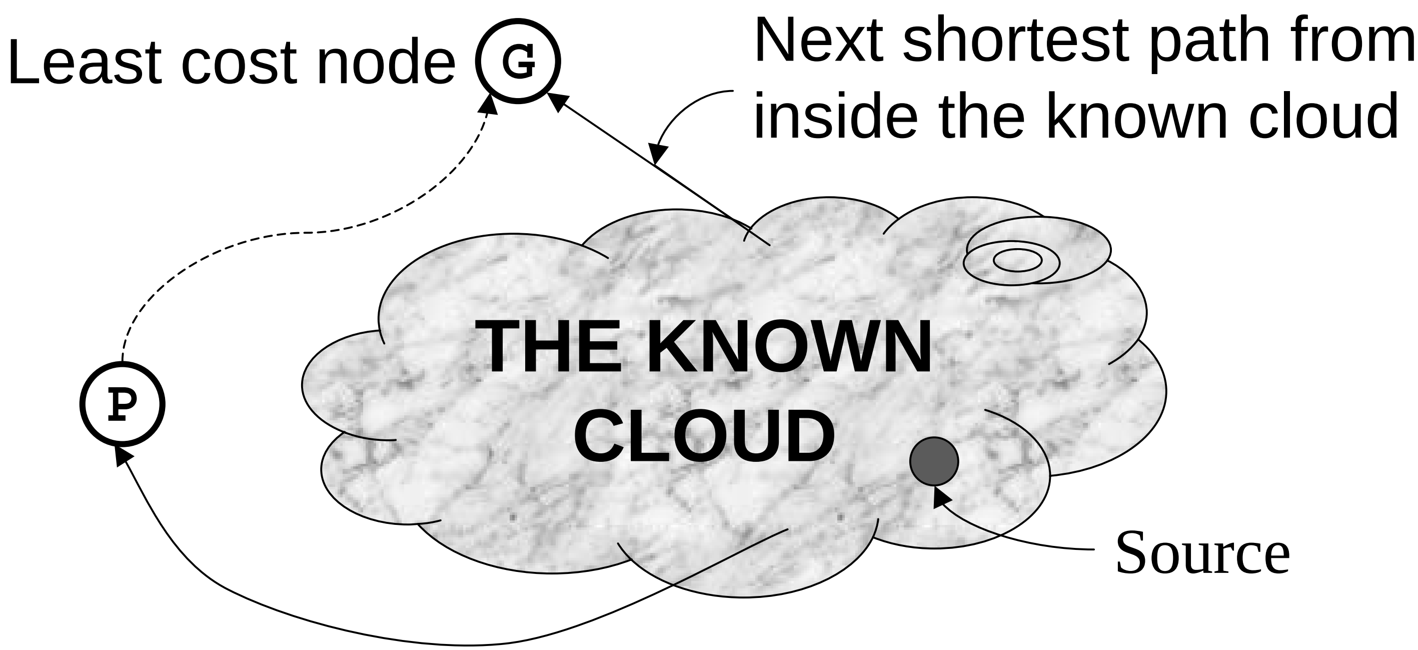
Time Complexity

- n vertices and m edges
- Initialize data structures O(n+m)
- Find min cost vertices O(n log n)
  - n delete mins
- Update costs O(m log n)
  - Potentially m updates
- Update previous pointers O(m)
  - Potentially m updates
- Total time O((n + m) log n) - very fast.

Correctness

- Dijkstra’s algorithm is an example of a greedy algorithm
- Greedy algorithms always make choices that currently seem the best
  - › Short-sighted – no consideration of long-term or global issues
  - › Locally optimal does not always mean globally optimal
- In Dijkstra’s case – choose the least cost node, but what if there is another path through other vertices that is cheaper?

“Cloudy” Proof



- If the path to G is the next shortest path, the path to P must be at least as long. Therefore, any path through P to G cannot be shorter!

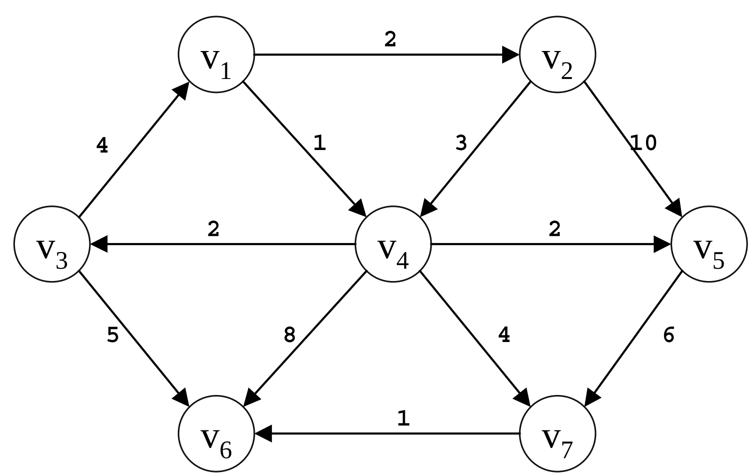
Inside the Cloud (Proof)

- Everything inside the cloud has the correct shortest path
- Proof is by induction on the number of nodes in the cloud:
  - › Base case: Initial cloud is just the source with shortest path 0
  - › Inductive hypothesis: cloud of k-1 nodes all have shortest paths
  - › Inductive step: choose the least cost node G → has to be the shortest path to G (previous slide). Add k-th node G to the cloud

All Pairs Shortest Path

- Given a edge weighted directed graph G = (V,E) find for all u,v in V the length of the shortest path from u to v. Use matrix representation.

$$
\begin{array}{cc}
C & \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array} &
\begin{pmatrix}
0 & 2 & : & 1 & : & : & : \\
: & 0 & : & 3 & 10 & : & : \\
4 & : & 0 & : & : & 5 & : \\
: & : & 2 & 0 & 2 & 8 & 4 \\
: & : & : & : & 0 & : & 6 \\
: & : & : & : & : & 0 & : \\
: & : & : & : & : & 1 & 0
\end{pmatrix}
\end{array}
$$



: = infinity

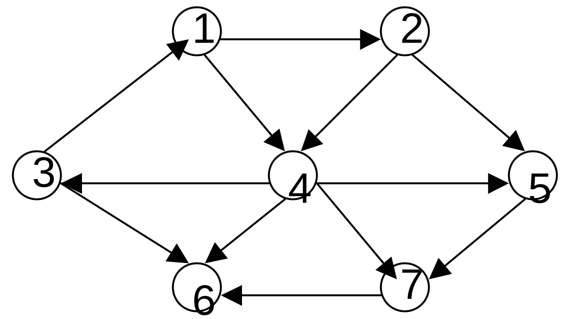
A (simpler) Related Problem: Transitive Closure

- Given a digraph $G(V,E)$ the transitive closure is a digraph $G'(V',E')$ such that
  - › $V' = V$ (same set of vertices)
  - › If $(v_i, v_{i+1},\ldots,v_k)$ is a path in G, then (v_i, v_k) is an edge of E'

Unweighted Digraph Boolean Matrix Representation

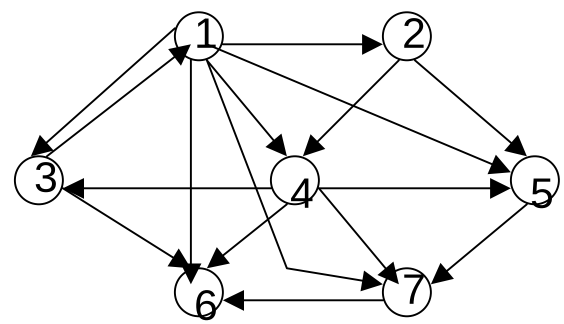
- C is called the connectivity matrix

$$
C \quad
\begin{array}{c}
\\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7
\end{array}
\begin{array}{c}
\begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\
\begin{pmatrix}
0 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0
\end{pmatrix}
\end{array}
$$



Transitive Closure

$$
\begin{array}{cc}
C & \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array} &
\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0
\end{pmatrix}
\end{array}
$$



On the graph, we show only the edges added with 1 as origin. The matrix represents the full transitive closure.

Finding Paths of Length 2

```
Length2 {          //Initialization of C2[i,j]
for k = 1 to n     // to all 0's not shown
 for i = 1 to n do
  for j = 1 to n do
     C2[i,j] := C2[i,j] ∪ (C[i,k] ∩ C[k,j]);
}
```

where ∩ is Boolean And (&&) and ∪ is Boolean OR (||)
This means if there is an edge from i to k
AND an edge from k to j, then there is a path
of length 2 between i and j.
Column k (C[i,k]) represents the predecessors of k
Row k (C[k,j]) represents the successors of k

Paths of Length 2

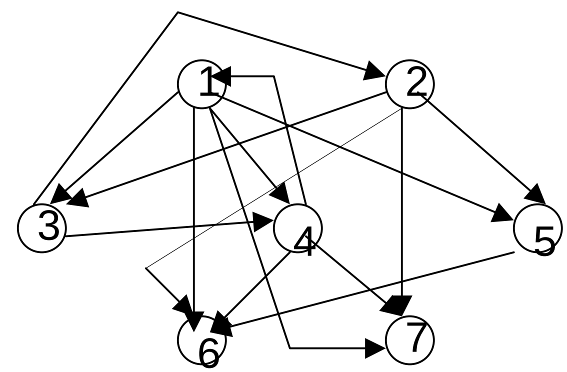
C

$$
\begin{array}{c|ccccccc}
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
\hline
1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
2 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
3 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
4 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\
5 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
7 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{array}
$$

C2

$$
\begin{array}{c|ccccccc}
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
\hline
1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\
2 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\
3 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
4 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\
5 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
7 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}
$$

Time $O(n^3)$



Transitive Closure

- Union of paths of length 0, length 1, length 2, …, length n-1.
  - Time complexity n * $O(n^3) = O(n^4)$
- There exists a better ($O(n^3)$) algorithm: Warshall's algorithm

Warshall Algorithm

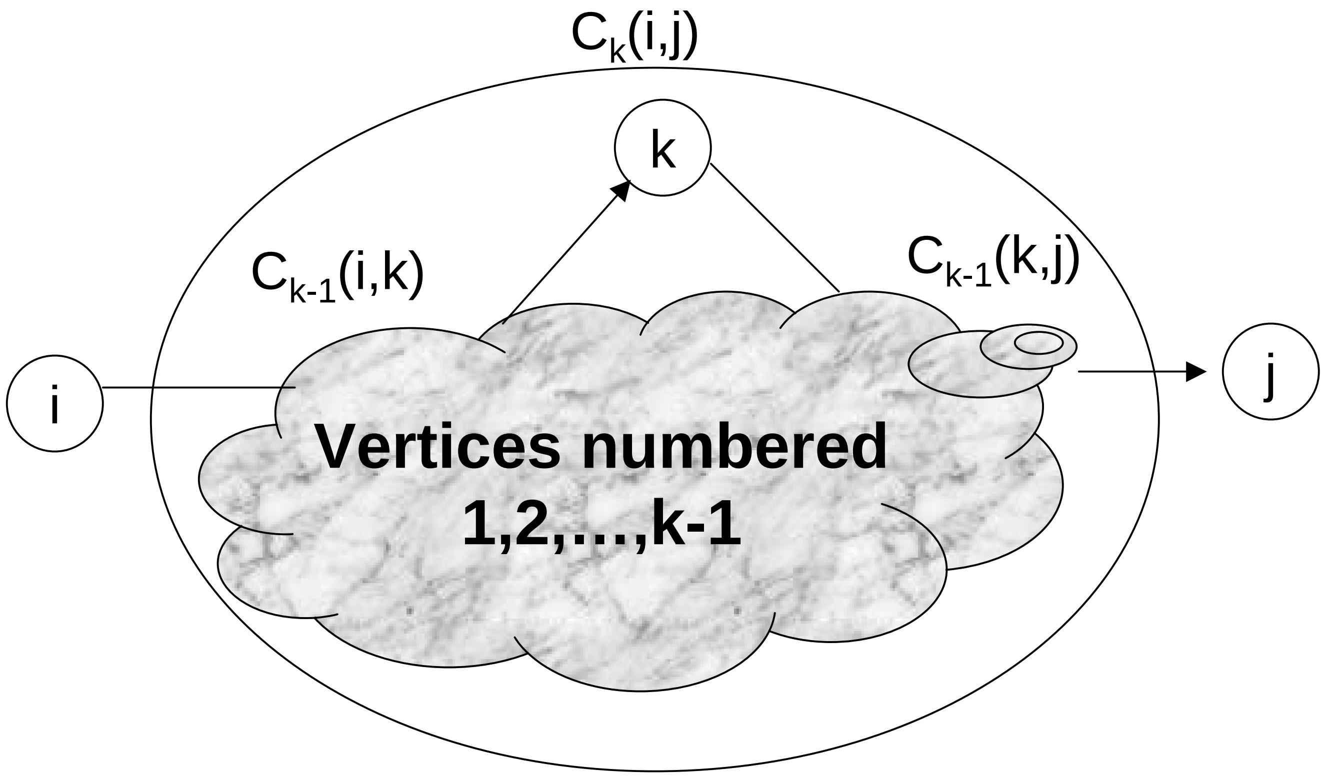
```
TransitiveClosure {
for k = 1 to n do
  for i = 1 to n do
    for j = 1 to n do
      C [i,j] := C[i,j] ∪ (C[i,k] ∩ C[k,j]);
}
```

where C[i,j] is the original connectivity matrix

Proof of Correctness

- After the k-th time through the loop, C[i,j] =1 if there is a path from i to j that only passes through vertices numbered 1,2,…,k (except for the initial edges)
- Base case: k = 1. C [i,j] = 1 for the initial connectivity matrix (path of length 0) and C [i,j] = 1 if there is a path (i,1,j)

Cloud Argument



Inductive Step

- Assume true for k-1.
  - › All paths from i to j that only go through vertices 1,2, …, k do not go through vertex k at all.
    - $C_k[i,j] = C_{k-1}[i,j]$ ($C_k[i,j]$ is result after k passes)
  - › A path from i to j that goes through (vertices 1,2, …, k must go through vertex k.
    - $C_k[i,j] = C_{k-1}[i,k] + C_{k-1}[k,j]$

Back to Weighted graphs: Matrix Representation

- C[i,j] = the cost of the edge (i,j)
  - › C[i,i] = 0 because no cost to stay where you are
  - › C[i,j] = infinity (:) if no edge from i to j.

$$
\begin{array}{c c}
C & \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array} &
\begin{pmatrix}
0 & 2 & : & 1 & : & : & : \\
: & 0 & : & 3 & 10 & : & : \\
4 & : & 0 & : & : & 5 & : \\
: & : & 2 & 0 & 2 & 8 & 4 \\
: & : & : & : & 0 & : & 6 \\
: & : & : & : & : & 0 & : \\
: & : & : & : & : & 1 & 0
\end{pmatrix}
\end{array}
$$

Floyd – Warshall Algorithm

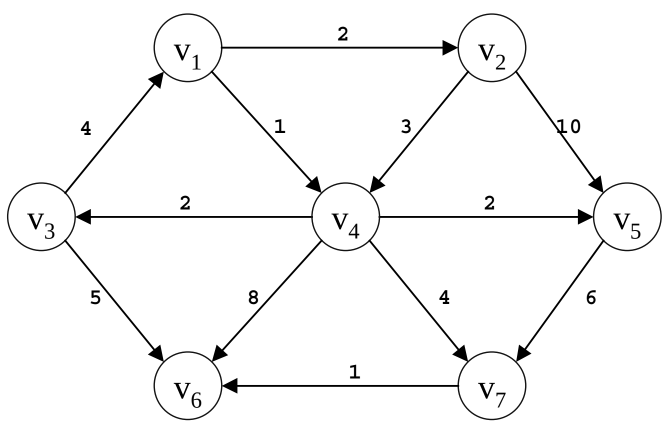
```
All_Pairs_Shortest_Path {
for k = 1 to n do
  for i = 1 to n do
    for j = 1 to n do
      C[i,j] := min(C[i,j], C[i,k] + C[k,j]);
}

Note x + : = : by definition
```

On termination C[i,j] is the length of the shortest path from i to j.

The Computation

$$\begin{array}{cc} C & \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array} & \begin{pmatrix} 0 & 2 & : & 1 & : & : & : \\ : & 0 & : & 3 & 10 & : & : \\ 4 & : & 0 & : & : & 5 & : \\ : & : & 2 & 0 & 2 & 8 & 4 \\ : & : & : & : & 0 & : & 6 \\ : & : & : & : & : & 0 & : \\ : & : & : & : & : & 1 & 0 \end{pmatrix} \end{array} \Rightarrow \begin{array}{cc} C & \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array} & \begin{pmatrix} 0 & 2 & 3 & 1 & 3 & 6 & 5 \\ 9 & 0 & 5 & 3 & 5 & 8 & 7 \\ 4 & 6 & 0 & 5 & 4 & 5 & 6 \\ 6 & 8 & 2 & 0 & 2 & 5 & 4 \\ : & : & : & : & 0 & 7 & 6 \\ : & : & : & : & : & 0 & : \\ : & : & : & : & : & 1 & 0 \end{pmatrix} \end{array}$$



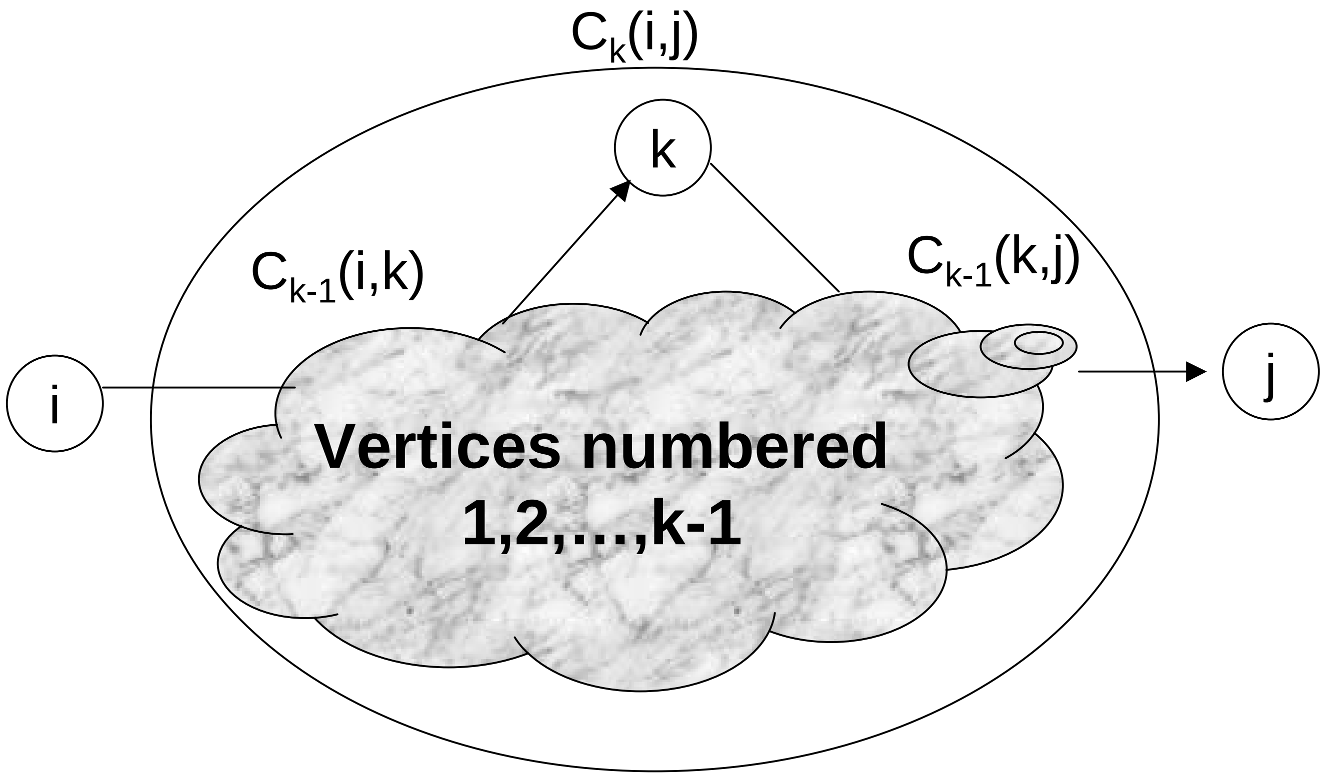
Proof of Correctness

- After the k-th time through the loop C[i,j] is the length of the shortest path that only passes through vertices numbered 1,2,…,k.
  - › Let C_k[i,j] be C[i,j] after k time through the loop.
- Base case: k = 0. C_0[i,j] is the cost of an edge that does not pass through any vertices.

Inductive Step

- Assume true for k-1.
  - › A shortest path from i to j that only goes through vertices 1,2, …, k does not go through vertex k at all.
    - $C_k[i,j] = C_{k-1}[i,j]$
  - › All shortest paths from i to j that only go through vertices 1,2, …, k must go through vertex k.
    - $C_k[i,j] = C_{k-1}[i,k] + C_{k-1}[k,j]$

Cloud Argument



Time Complexity of All Pairs Shortest Path

- n is the number of vertices
- Three nested loops. $O(n^3)$
  - › Shortest paths can be found too (see the book).
- Repeated Dijkstra's algorithm
  - › $O(n(n+m)\log n)$ (= $O(n^3 \log n)$ for dense graphs).
  - › Run Dijkstra starting at each vertex.
  - › Dijkstra also gives the shortest paths not just their lengths.