Sorting

CSE 373

Data Structures

Lecture 19

Reading

- Reading
  - Sections 7.1-7.3 and 7.5
  - Section 7.6, Mergesort
  - Section 7.7, Quicksort

Sorting

- Input
  - › an array A of data records (Note: we have seen how to sort when elements are in linked lists: Mergesort)
  - › a key value in each data record
  - › a comparison function which imposes a consistent ordering on the keys (e.g., integers)
- Output
  - › reorganize the elements of A such that
    - For any i and j, if $i < j$ then $A[i] \leq A[j]$

Space

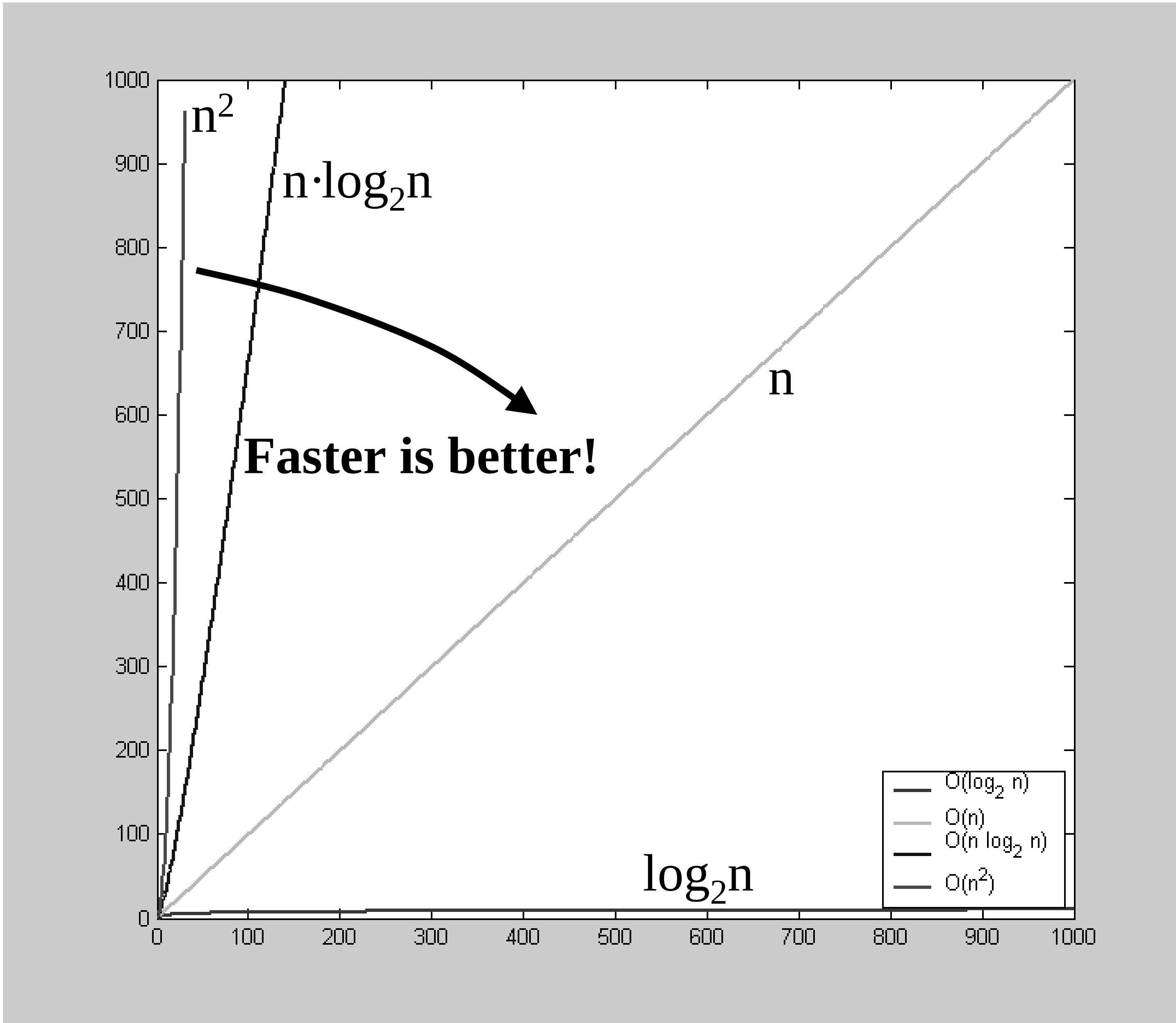
- How much space does the sorting algorithm require in order to sort the collection of items?
  - › Is copying needed? O(n) additional space
  - › In-place sorting – no copying – O(1) additional space
  - › Somewhere in between for “temporary”, e.g. O(logn) space
  - › External memory sorting – data so large that does not fit in memory

Time

- How fast is the algorithm?
  - The definition of a sorted array A says that for any i<j, A[i] < A[j]
  - This means that you need to at least check on each element at the very minimum, I.e., at least O(N)
  - And you could end up checking each element against every other element, which is $O(N^2)$
  - The big question is: How close to O(N) can you get?

Stability

- Stability: Does it rearrange the order of input data records which have the same key value (duplicates)?
  - E.g. Phone book sorted by name. Now sort by county – is the list still sorted by name within each county?
  - Extremely important property for databases
  - A stable sorting algorithm is one which does not rearrange the order of duplicate keys



n^2
n·log2n
n
log2n
Faster is better!
O(log2 n)
O(n)
O(n log2 n)
O(n^2)
0
100
200
300
400
500
600
700
800
900
1000

Bubble Sort

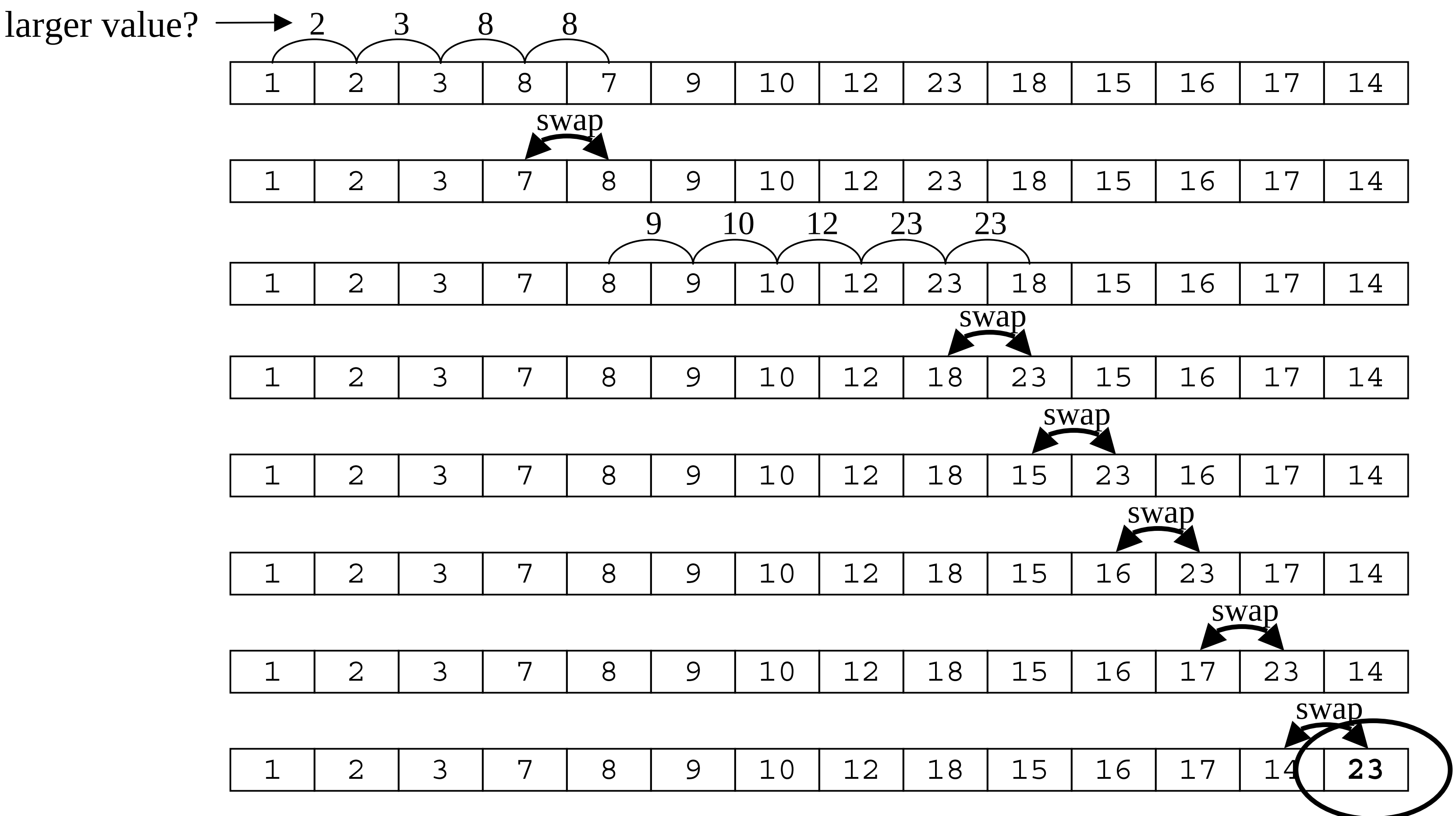
- "Bubble" elements to to their proper place in the array by comparing elements i and i+1, and swapping if A[i] > A[i+1]
  - › Bubble every element towards its correct position
    - last position has the largest element
    - then bubble every element except the last one towards its correct position
    - then repeat until done or until the end of the quarter, whichever comes first ...

Bubblesort

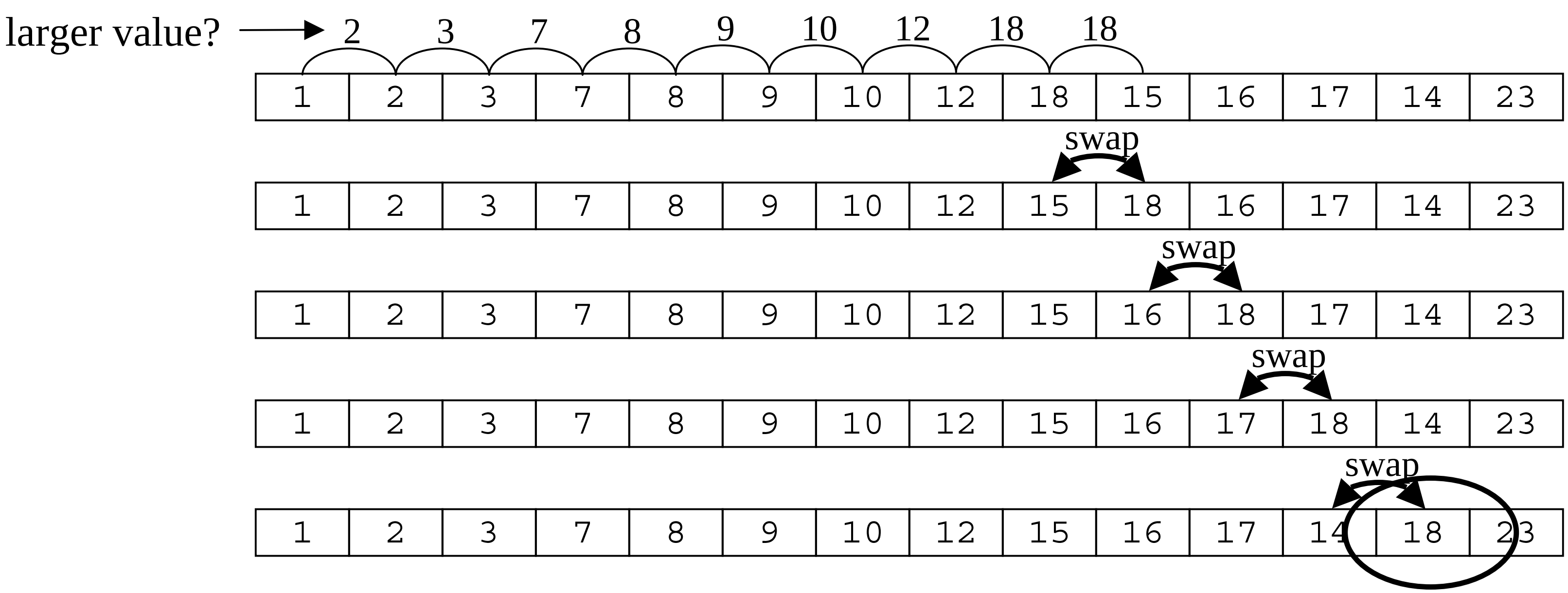
```
bubble(A[1..n]: integer array, n : integer): {
  i, j : integer;
  for i = 1 to n-1 do
    for j = 2 to n-i+1 do
       if A[j-1] > A[j] then SWAP(A[j-1],A[j]);
  }

SWAP(a,b) :  {
  t :integer;
  t:=a; a:=b; b:=t;
}
```

Put the largest element in its place



Put 2nd largest element in its place



Two elements done, only n-2 more to go ...

Bubble Sort: **Just Say No**

- "Bubble" elements to to their proper place in the array by comparing elements i and i+1, and swapping if A[i] > A[i+1]
- We bubblize for i=1 to n (i.e, n times)
- Each bubblization is a loop that makes n-i comparisons
- This is $O(n^2)$

Insertion Sort

- What if first *k* elements of array are already sorted?
  - › 4, 7, 12, 5, 19, 16
- We can shift the tail of the sorted elements list down and then *insert* next element into proper position and we get k+1 sorted elements
  - › 4, 5, 7, 12, 19, 16

Insertion Sort

```
InsertionSort(A[1..N]: integer array, N: integer) {
   i, j, temp: integer ;
   for i = 2 to N {
     temp := A[i];
     j := i-1;
     while j > 1 and A[j-1] > temp {
           A[j] := A[j-1]; j := j–1;}
     A[j] = temp;
   }
}
```

- Is Insertion sort in place?
- Running time = ?

Example

1	2	3	8	7	9	10	12	23	18	15	16	17	14
1	2	3	7	8	9	10	12	23	18	15	16	17	14
1	2	3	7	8	9	10	12	18	23	15	16	17	14
1	2	3	7	8	9	10	12	18	15	23	16	17	14
1	2	3	7	8	9	10	12	15	18	23	16	17	14
1	2	3	7	8	9	10	12	15	18	16	23	17	14
1	2	3	7	8	9	10	12	15	16	18	23	17	14

Example

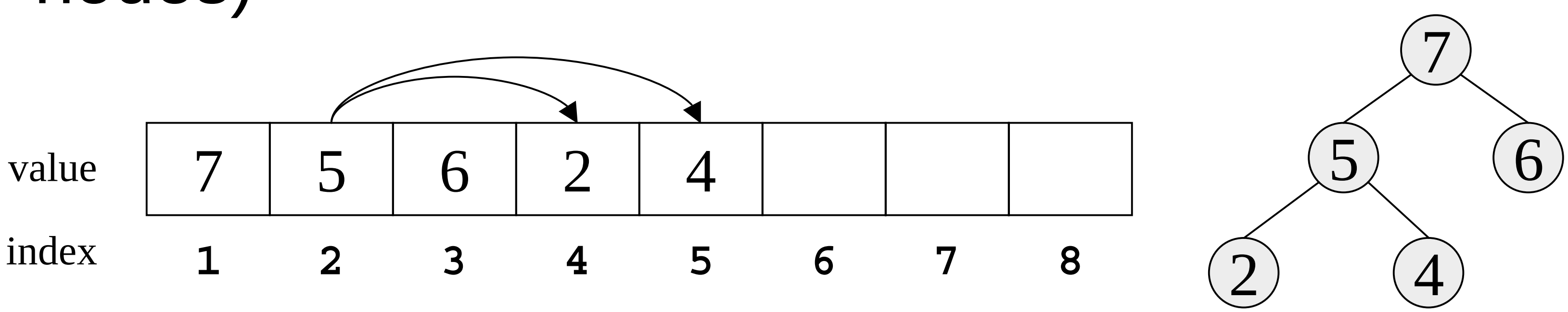
1	2	3	7	8	9	10	12	15	16	18	17	23	14
1	2	3	7	8	9	10	12	15	16	17	18	23	14
1	2	3	7	8	9	10	12	15	16	17	18	14	23
1	2	3	7	8	9	10	12	15	16	17	14	18	23
1	2	3	7	8	9	10	12	15	16	14	17	18	23
1	2	3	7	8	9	10	12	15	14	16	17	18	23
1	2	3	7	8	9	10	12	14	15	16	17	18	23

Insertion Sort Characteristics

- In place and Stable
- Running time
  - › Worst case is $O(N^2)$
    - reverse order input
    - must copy every element every time
- Good sorting algorithm for almost sorted data
  - › Each item is close to where it belongs in sorted order.

Heap Sort

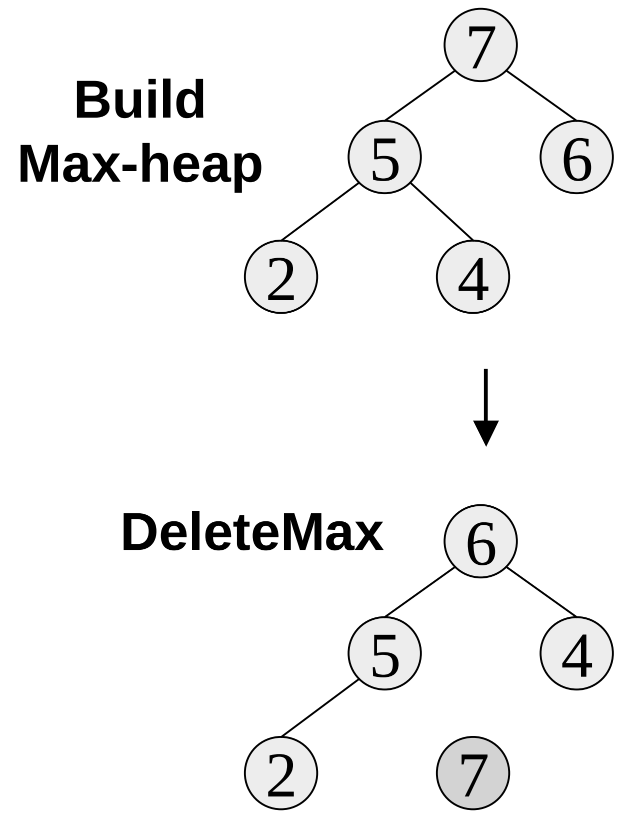
- We use a Max-Heap
- Root node = A[1]
- Children of A[i] = A[2i], A[2i+1]
- Keep track of current size N (number of nodes)



N = 5

Using Binary Heaps for Sorting

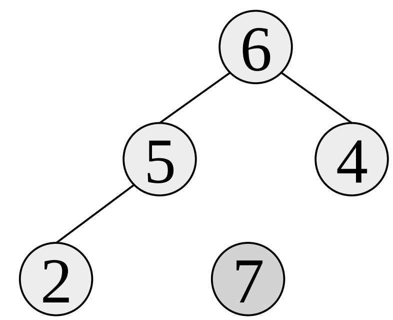
- Build a max-heap
- Do N DeleteMax operations and store each Max element as it comes out of the heap
- Data comes out in largest to smallest order
- Where can we put the elements as they are removed from the heap?



1 Removal = 1 Addition

- Every time we do a DeleteMax, the heap gets smaller by one node, and we have one more node to store
  - › Store the data at the end of the heap array
  - › Not "in the heap" but it is in the heap array

value	6	5	4	2	7			
index	1	2	3	4	5	6	7	8

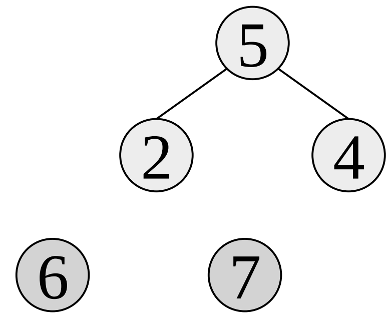


N = 4

Repeated DeleteMax

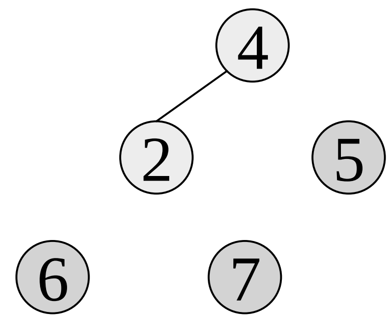
5	2	4	6	7			
1	2	3	4	5	6	7	8

N = 3



4	2	5	6	7			
1	2	3	4	5	6	7	8

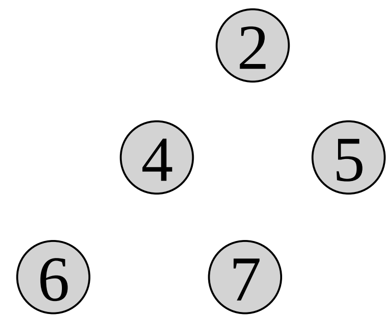
N = 2



Heap Sort is In-place

- After all the DeleteMaxs, the heap is gone but the array is full and is in sorted order

value	2	4	5	6	7			
index	1	2	3	4	5	6	7	8



N = 0

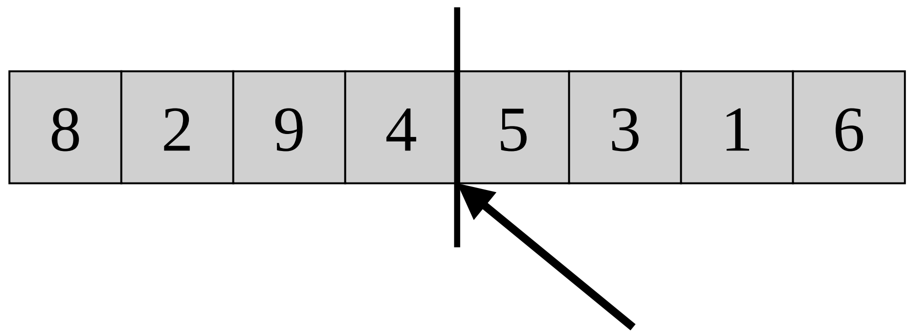
Heapsort: Analysis

- Running time
  - › time to build max-heap is O(N)
  - › time for N DeleteMax operations is N O(log N)
  - › total time is **O(N log N)**
- Can also show that running time is Ω(N log N) for some inputs,
  - › so *worst case* is Θ**(N log N)**
  - › *Average case* running time is also O(N log N)
- Heapsort is in-place but not stable (why?)

“Divide and Conquer”

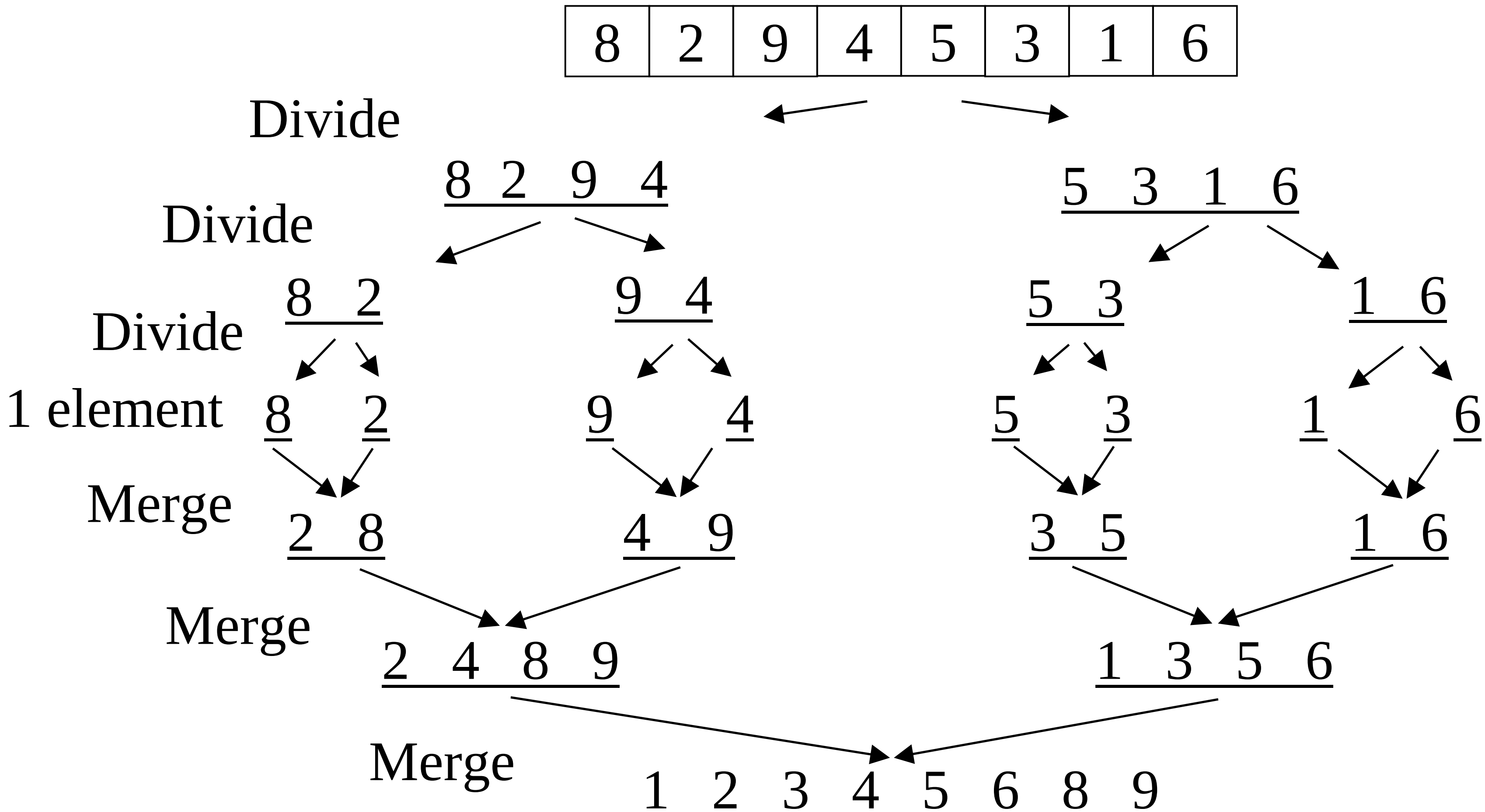
- Very important strategy in computer science:
  - › Divide problem into smaller parts
  - › Independently solve the parts
  - › Combine these solutions to get overall solution
- **Idea 1**: Divide array into two halves, *recursively* sort left and right halves, then *merge* two halves → Mergesort
- **Idea 2 :** Partition array into items that are “small” and items that are “large”, then recursively sort the two sets → Quicksort

Mergesort



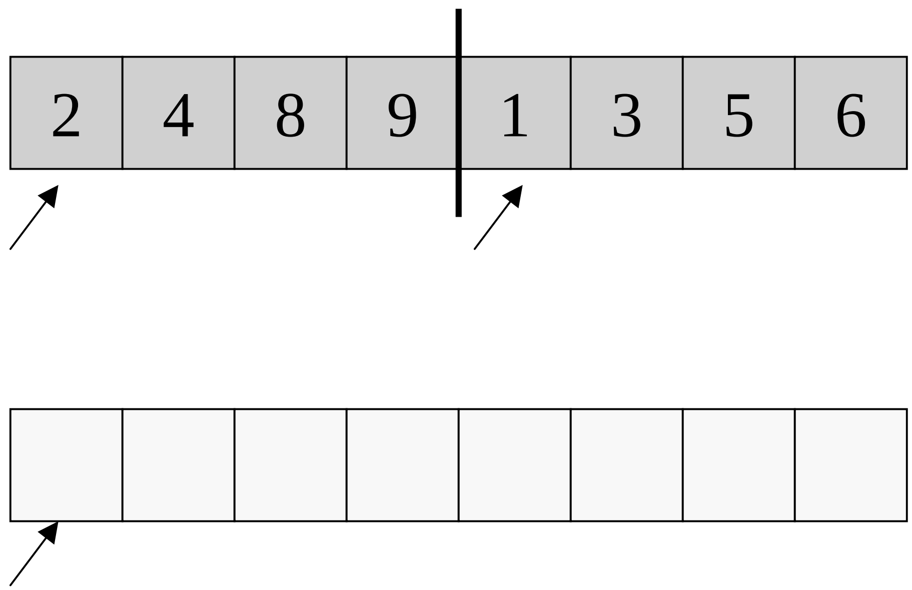
- Divide it in two at the midpoint
- Conquer each side in turn (by recursively sorting)
- Merge two halves together

Mergesort Example



Auxiliary Array

- The merging requires an auxiliary array.



Auxiliary array

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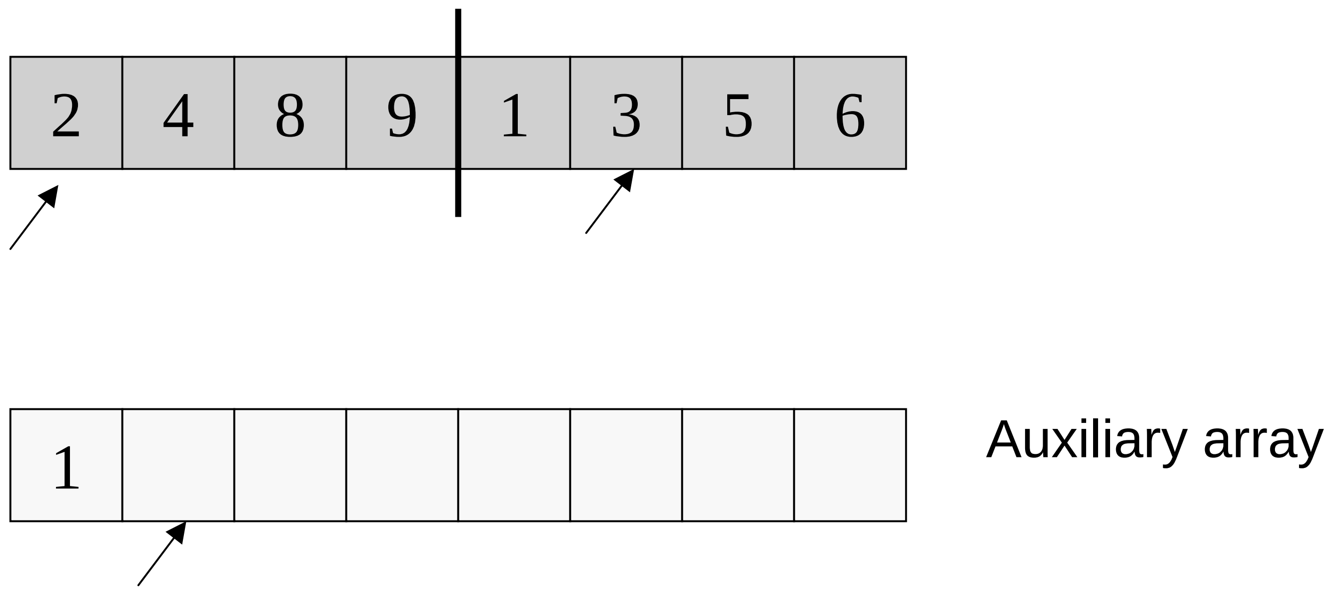

Auxiliary Array

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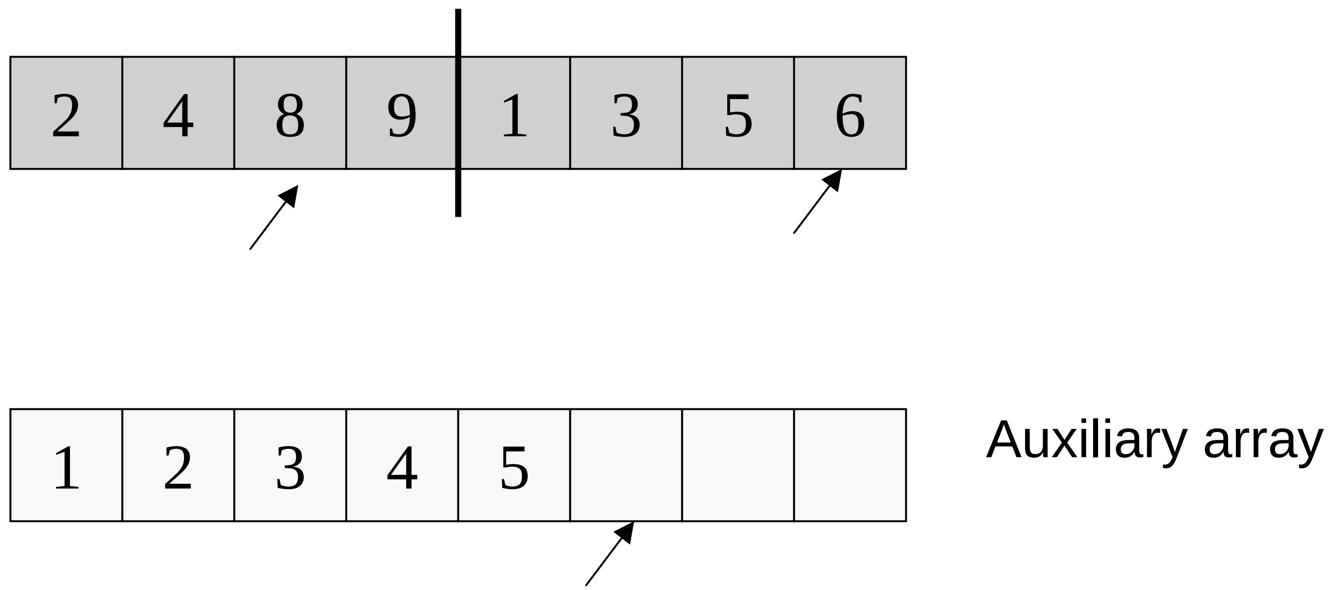
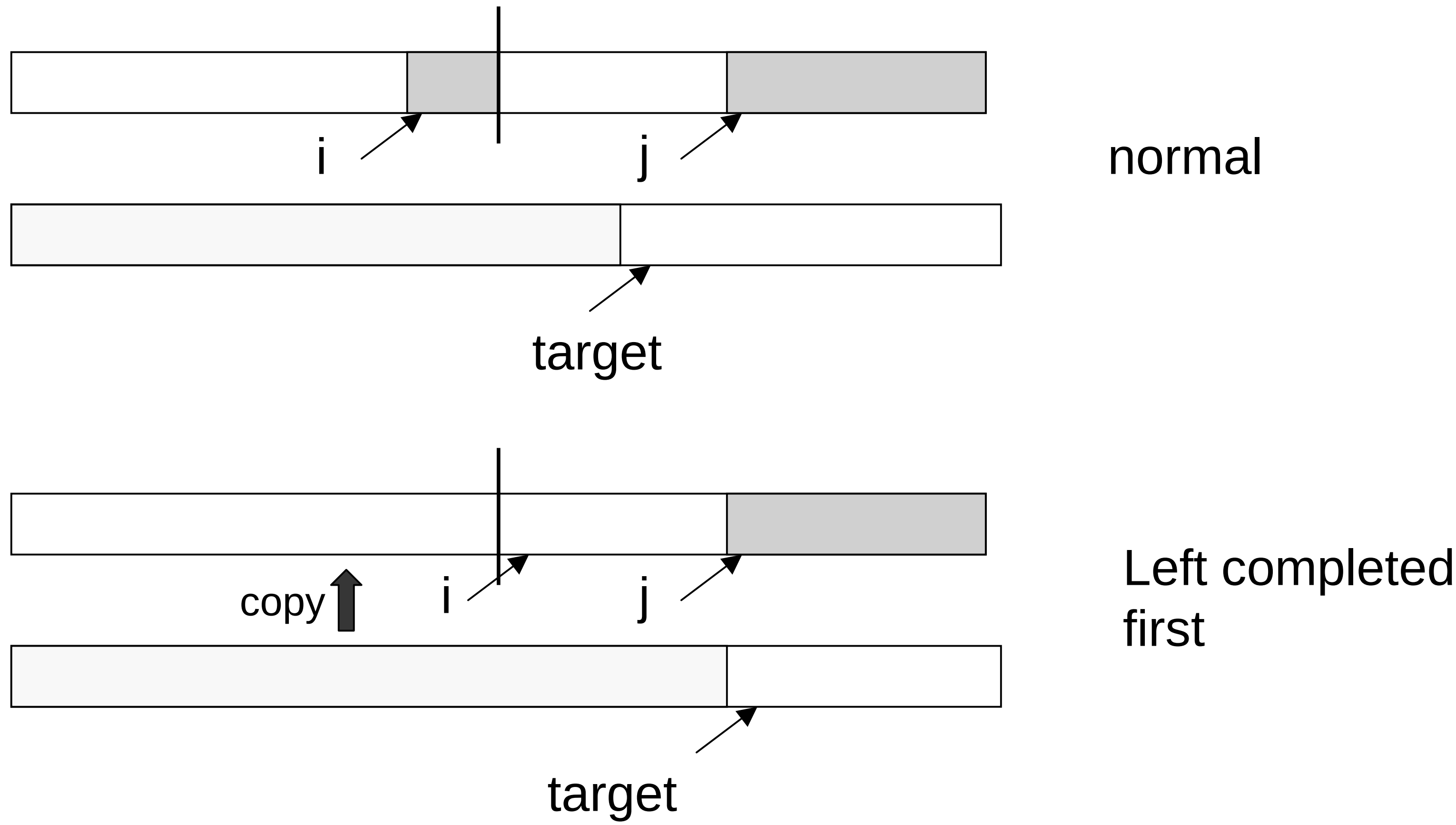
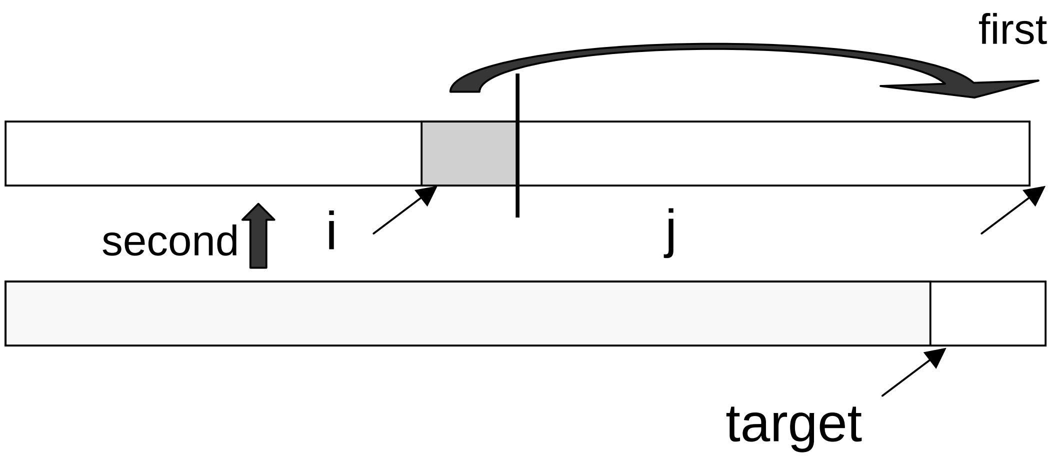


Merging



Merging



Right completed first

Merging Algorithm

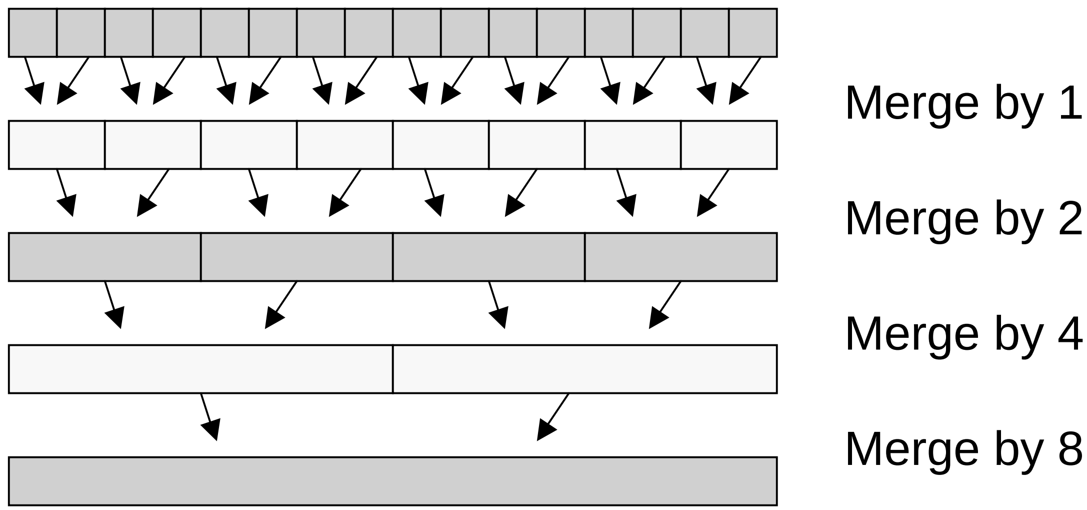
```
Merge(A[], T[] : integer array, left, right : integer) : {
  mid, i, j, k, l, target : integer;
  mid := (right + left)/2;
  i := left; j := mid + 1; target := left;
  while i ≤ mid and j ≤ right do
    if A[i] ≤ A[j] then T[target] := A[i] ; i:= i + 1;
      else T[target] := A[j]; j := j + 1;
    target := target + 1;
  if i > mid then //left completed//
    for k := left to target-1 do A[k] := T[k];
  if j > right then //right completed//
    k : = mid; l := right;
    while k ≥ i do A[l] := A[k]; k := k-1; l := l-1;
    for k := left to target-1 do A[k] := T[k];
}
```

Recursive Mergesort

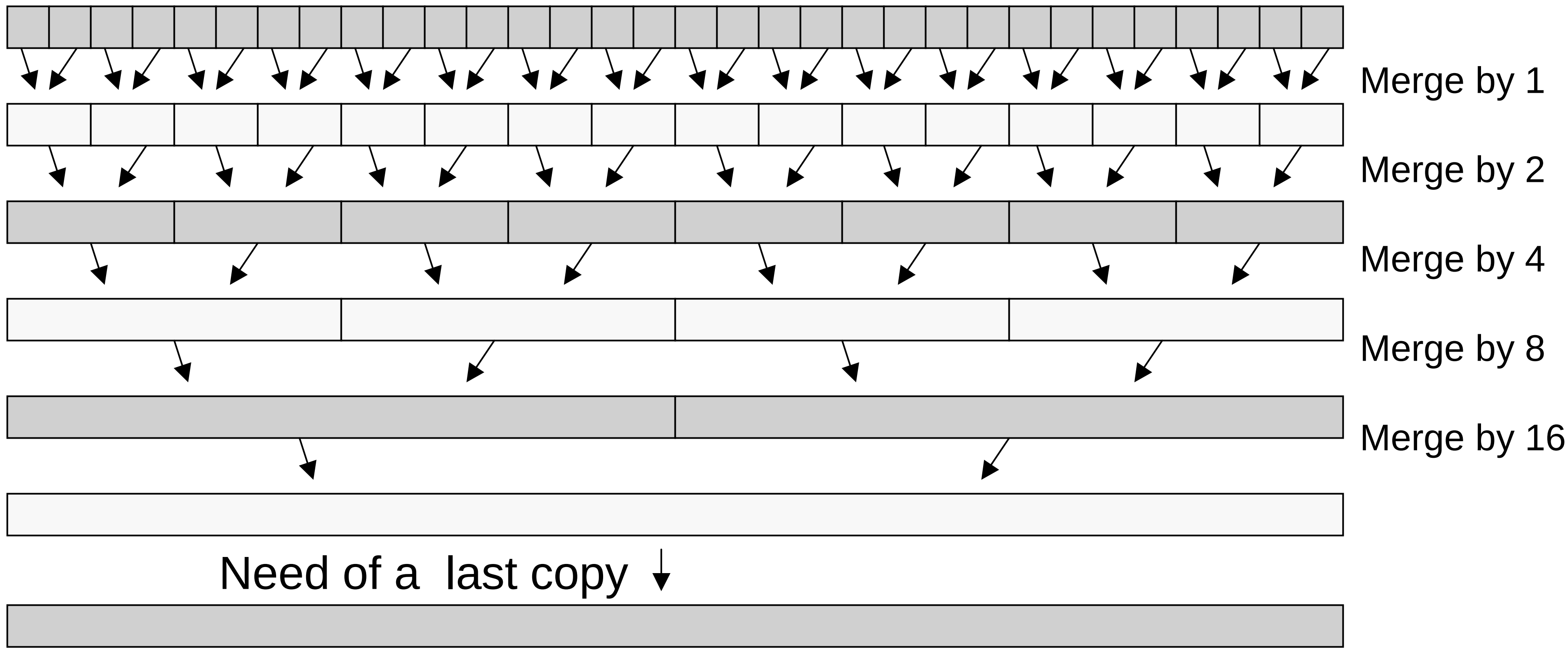
```
Mergesort(A[], T[] : integer array, left, right : integer) : {
  if left < right then
    mid := (left + right)/2;
    Mergesort(A,T,left,mid);
    Mergesort(A,T,mid+1,right);
    Merge(A,T,left,right);
}

MainMergesort(A[1..n]: integer array, n : integer) : {
  T[1..n]: integer array;
  Mergesort[A,T,1,n];
}
```

Iterative Mergesort



Iterative Mergesort



Iterative Mergesort

```
IterativeMergesort(A[1..n]: integer array, n : integer) : {
//precondition: n is a power of 2//
  i, m, parity : integer;
  T[1..n]: integer array;
  m := 2; parity := 0;
  while m ≤ n do
    for i = 1 to n - m + 1 by m do
       if parity = 0 then Merge(A,T,i,i+m-1);
         else Merge(T,A,i,i+m-1);
    parity := 1 - parity;
    m := 2*m;
  if parity = 1 then
    for i = 1 to n do A[i] := T[i];
}
```

How do you handle non-powers of 2?
How can the final copy be avoided?

Mergesort Analysis

- Let T(N) be the running time for an array of N elements
- Mergesort divides array in half and calls itself on the two halves. After returning, it merges both halves using a temporary array
- Each recursive call takes T(N/2) and merging takes O(N)

Mergesort Recurrence Relation

- The recurrence relation for T(N) is:
  - › $T(1) \le a$
    - base case: 1 element array → constant time
  - › $T(N) \le 2T(N/2) + bN$
    - Sorting N elements takes
      - – the time to sort the left half
      - – plus the time to sort the right half
      - – plus an O(N) time to merge the two halves
- $T(N) = O(n \log n)$

Properties of Mergesort

- Not in-place
  - › Requires an auxiliary array (O(n) extra space)
- Stable
  - › Make sure that left is sent to target on equal values.
- Iterative Mergesort reduces copying.

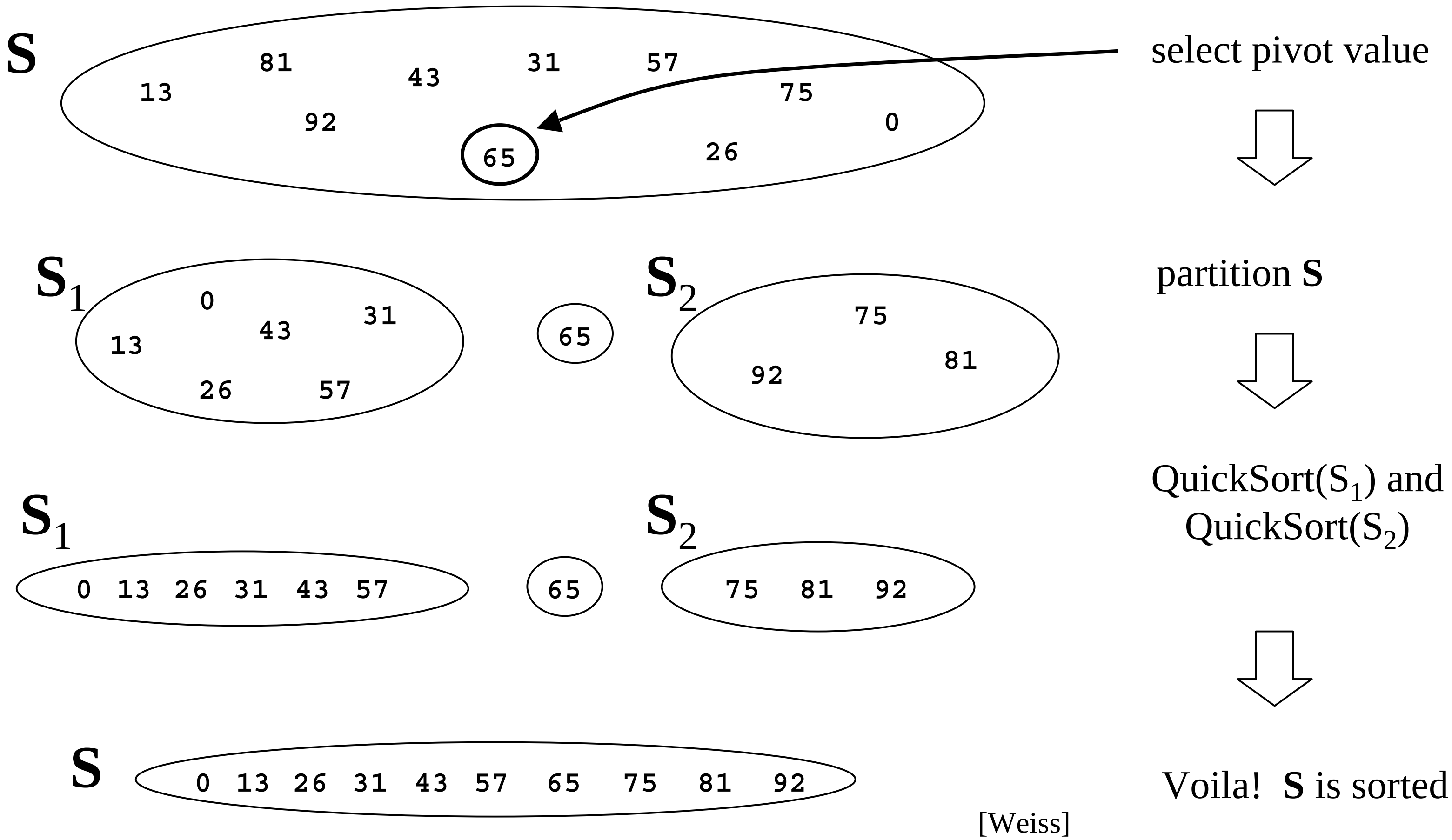
Quicksort

- Quicksort uses a divide and conquer strategy, but does not require the O(N) extra space that MergeSort does
  - › Partition array into left and right sub-arrays
    - Choose an element of the array, called **pivot**
    - the elements in left sub-array are all less than pivot
    - elements in right sub-array are all greater than pivot
  - › Recursively sort left and right sub-arrays
  - › Concatenate left and right sub-arrays in O(1) time

“Four easy steps”

- To sort an array **S**
  1. If the number of elements in **S** is 0 or 1, then return. The array is sorted.
  2. Pick an element *v* in **S**. This is the *pivot* value.
  3. Partition **S**-{*v*} into two disjoint subsets, $\mathbf{S}_1$ = {all values $x \leq v$}, and $\mathbf{S}_2$ = {all values $x \geq v$}.
  4. Return QuickSort($\mathbf{S}_1$), *v*, QuickSort($\mathbf{S}_2$)

The steps of QuickSort



Details, details

- Implementing the actual partitioning
- Picking the pivot
  - › want a value that will cause $|S_1|$ and $|S_2|$ to be non-zero, and close to equal in size if possible
- Dealing with cases where the element equals the pivot

Quicksort Partitioning

- Need to partition the array into left and right sub-arrays
  - the elements in left sub-array are ≤ pivot
  - elements in right sub-array are ≥ pivot
- How do the elements get to the correct partition?
  - Choose an element from the array as the pivot
  - Make one pass through the rest of the array and swap as needed to put elements in partitions

Partitioning:Choosing the pivot

- One implementation (there are others)
  - › median3 finds pivot and sorts left, center, right
    - Median3 takes the median of leftmost, middle, and rightmost elements
    - An alternative is to choose the pivot randomly (need a random number generator; “expensive”)
    - Another alternative is to choose the first element (but can be very bad. Why?)
  - › Swap pivot with next to last element

Partitioning in-place

- Set pointers i and j to start and end of array
- Increment i until you hit element A[i] > pivot
- Decrement j until you hit elmt A[j] < pivot
- Swap A[i] and A[j]
- Repeat until i and j cross
- Swap pivot (at A[N-2]) with A[i]

Example

Choose the pivot as the median of three

0	1	2	3	4	5	6	7	8	9
8	1	4	9	**0**	3	5	2	7	**6**

Median of 0, 6, 8 is 6. Pivot is 6

0	1	4	9	7	3	5	2	6	8
i								j	

Place the largest at the right
and the smallest at the left.
Swap pivot with next to last element.

Example

i								j	
0	1	4	9	7	3	5	2	6	8

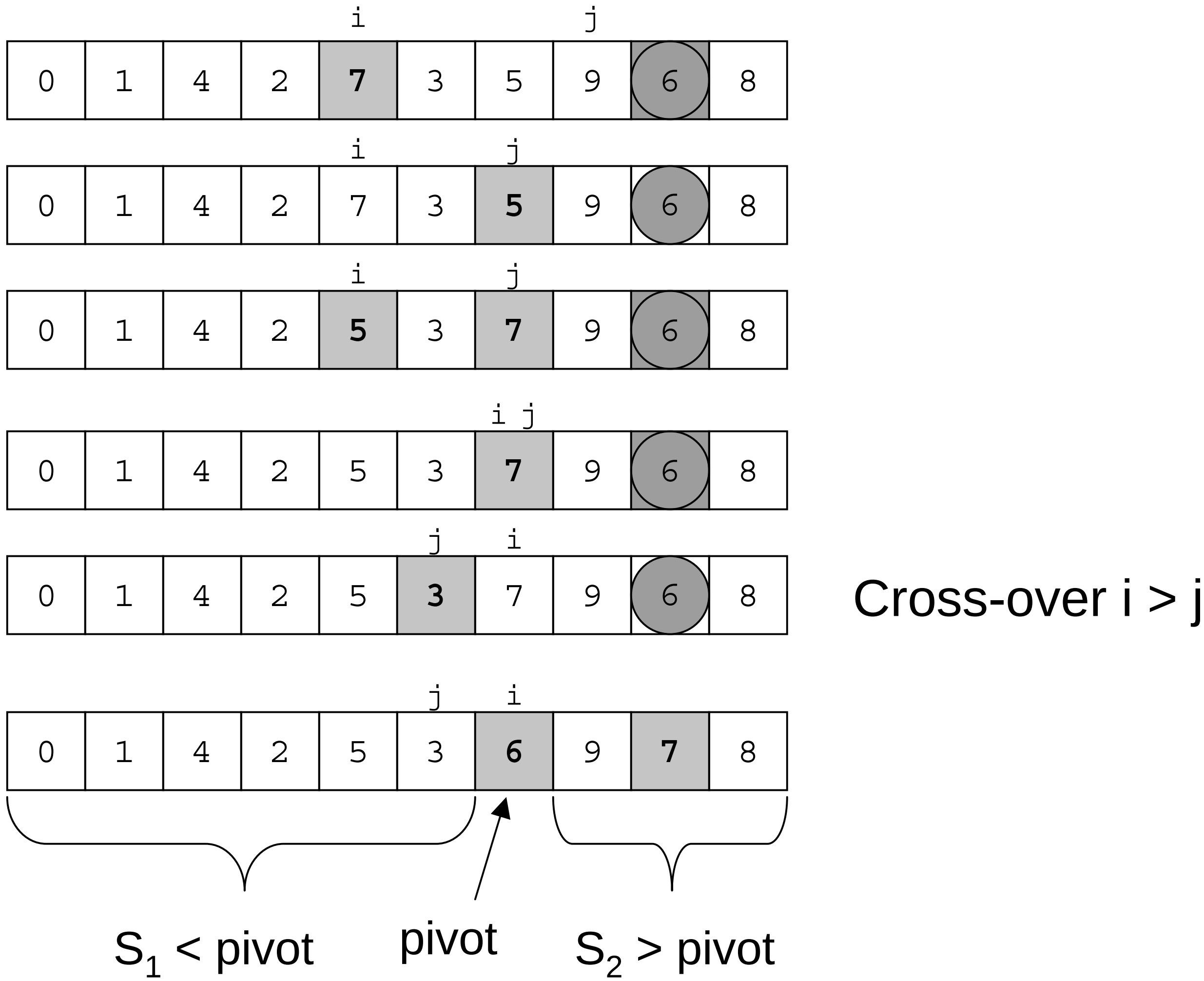
			i					j	
0	1	4	**9**	7	3	5	2	6	8

			i				j		
0	1	4	9	7	3	5	**2**	6	8

			i				j		
0	1	4	**2**	7	3	5	**9**	6	8

Move i to the right up to A[i] larger than pivot.
Move j to the left up to A[j] smaller than pivot.
Swap

Example



i j
0 1 4 2 7 3 5 9 6 8
i j
0 1 4 2 7 3 5 9 6 8
i j
0 1 4 2 5 3 7 9 6 8
i j
0 1 4 2 5 3 7 9 6 8
j i
0 1 4 2 5 3 7 9 6 8
Cross-over i > j
j i
0 1 4 2 5 3 6 9 7 8
S_1 < pivot
pivot
S_2 > pivot

Recursive Quicksort

```
Quicksort(A[]: integer array, left,right : integer): {
pivotindex : integer;
if left + CUTOFF ≤ right then
  pivot := median3(A,left,right);
  pivotindex := Partition(A,left,right-1,pivot);
  Quicksort(A, left, pivotindex – 1);
  Quicksort(A, pivotindex + 1, right);
else
  Insertionsort(A,left,right);
}
```

Don’t use quicksort for small arrays.
CUTOFF = 10 is reasonable.

Quicksort Best Case Performance

- Algorithm always chooses best pivot and splits sub-arrays in half at each recursion
  - › $T(0) = T(1) = O(1)$
    - constant time if 0 or 1 element
  - › For $N > 1$, 2 recursive calls plus linear time for partitioning
  - › $T(N) = 2T(N/2) + O(N)$
    - Same recurrence relation as Mergesort
  - › $T(N) = \underline{O(N \log N)}$

Quicksort Worst Case Performance

- Algorithm always chooses the worst pivot – one sub-array is empty at each recursion
  - $T(N) \leq a$ for $N \leq C$
  - $T(N) \leq T(N-1) + bN$
  - $\leq T(N-2) + b(N-1) + bN$
  - $\leq T(C) + b(C+1) + \ldots + bN$
  - $\leq a + b(C + (C+1) + (C+2) + \ldots + N)$
  - $T(N) = O(N^2)$
- Fortunately, *average case performance* is O(N log N) (see text for proof)

Properties of Quicksort

- Not stable because of long distance swapping.
- No iterative version (without using a stack).
- Pure quicksort not good for small arrays.
- “In-place”, but uses auxiliary storage because of recursive call (O(logn) space).
- O(n log n) average case performance, but O(n^2) worst case performance.

Folklore

- "Quicksort is the best in-memory sorting algorithm."
- Truth
  - › Quicksort uses very few comparisons on average.
  - › Quicksort does have good performance in the memory hierarchy.
    - Small footprint
    - Good locality