# Sorting (Part I)

CSE 373
Data Structures
Unit 16

Reading: Sections 7.1-7.3 and 7.5

# Sorting

- Input
  - › an array A of data records
  - › a key value in each data record
  - › a comparison function which imposes a consistent ordering on the keys (e.g., integers)
- Output
  - › reorganize the elements of A such that
    - For any i and j, if i < j then A[i] ≤ A[j]



# Consistent Ordering

- The comparison function must provided a consistent *ordering* on the set of possible keys
  - › You can compare any two keys and get back an indication of a < b, a > b, or a = b
  - › The comparison functions must be consistent
    - If **`compare(a,b)`** says a<b, then **`compare(b,a)`** must say b>a
    - If **`compare(a,b)`** says a=b, then **`compare(b,a)`** must say b=a
    - If **`compare(a,b)`** says a=b, then **`equals(a,b) and equals(b,a)`** must say a=b



# Why Sort?

- Sorting algorithms are among the most frequently used algorithms in computer science
- Allows binary search of an N-element array in O(log N) time
- Allows O(1) time access to *k*th largest element in the array for any *k*
- Allows easy detection of any duplicates

# Space

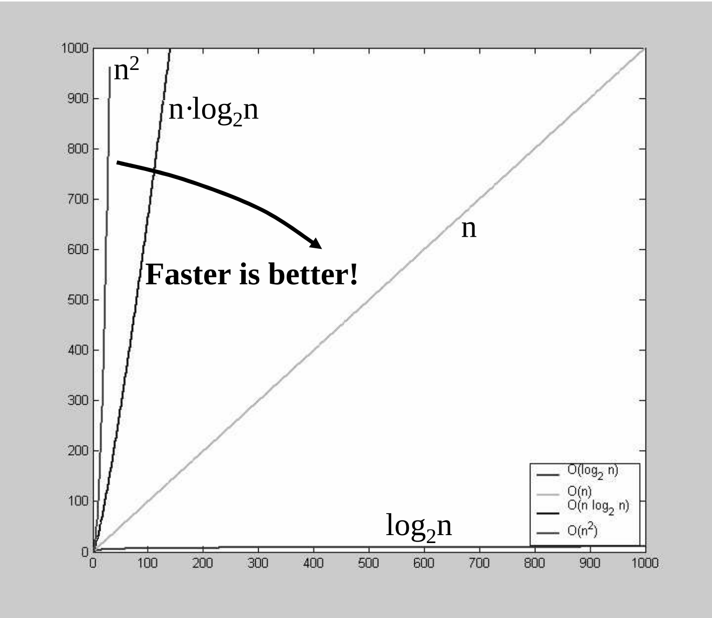
- How much space does the sorting algorithm require in order to sort the collection of items?
  - › Is copying needed? O(n) additional space
  - › In-place sorting – no copying – O(1) additional space
  - › Somewhere in between for "temporary", e.g. O(logn) space
  - › External memory sorting – data so large that does not fit in memory



# Time

- How fast is the algorithm?
  - › The definition of a sorted array A says that for any i<j, A[i] < A[j]
  - › This means that you need to at least check on each element at the very minimum, I.e., at least O(N)
  - › And you could end up checking each element against every other element, which is $O(N^2)$
  - › The big question is: How close to O(N) can you get?

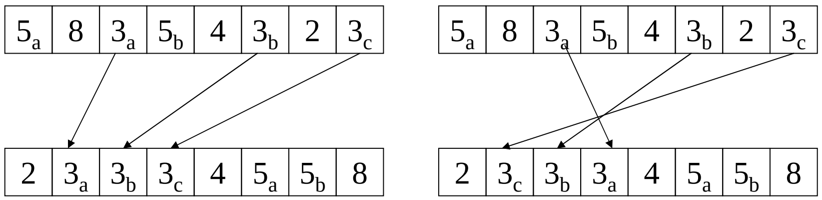





# Stability

- Stability: Does it rearrange the order of input data records which have the same key value (duplicates)?
  - › E.g. Phone book sorted by name. Now sort by county – is the list still sorted by name within each county?
  - › Extremely important property for databases
  - › A stable sorting algorithm is one which does not rearrange the order of duplicate keys

# Example




Stable Sort

Unstable Sort



# Bubble Sort

- “Bubble” elements to to their proper place in the array by comparing elements i and i+1, and swapping if A[i] > A[i+1]
  - › Bubble every element towards its correct position
    - last position has the largest element
    - then bubble every element except the last one towards its correct position
    - then repeat until done or until the end of the quarter, whichever comes first ...



# Bubblesort

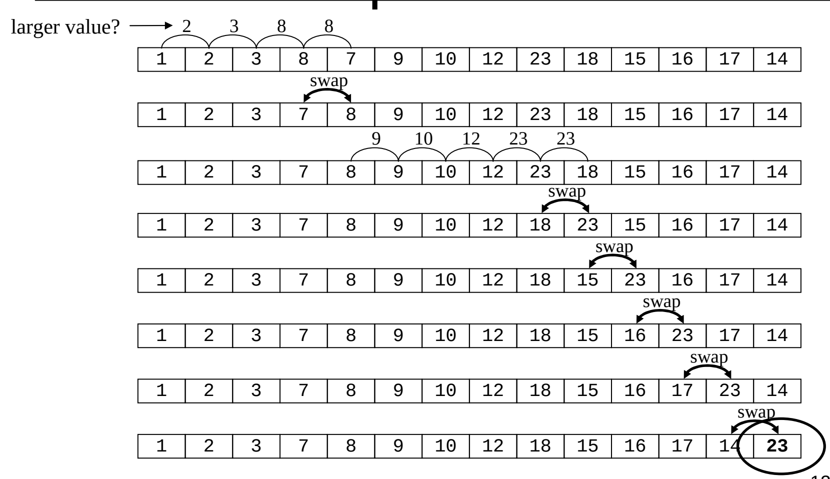
```
bubble(A[1..n]: integer array, n : integer): {
  i, j : integer;
  for i = 1 to n-1 do
    for j = 2 to n–i+1 do
       if A[j-1] > A[j] then SWAP(A[j-1],A[j]);
  }

SWAP(a,b) :  {
  t :integer;
  t:=a; a:=b; b:=t;
}
```

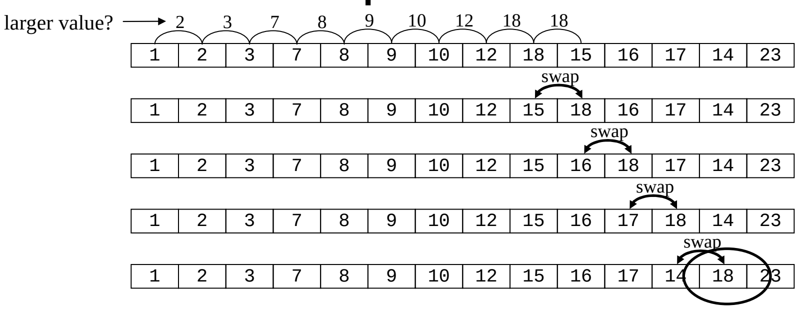
i=1: Largest element is placed at last position
i=k: $k^{th}$ Largest element is placed at $k^{th}$ to last position



# Put the largest element in its place



# Put 2nd largest element in its place




Two elements done, only n-2 more to go ...



# Bubble Sort: **Just Say No**

- "Bubble" elements to to their proper place in the array by comparing elements i and i+1, and swapping if A[i] > A[i+1]
- We bubblize for i=1 to n (i.e, n times)
- Each bubblization is a loop that makes n-i comparisons
- This is $O(n^2)$



# Insertion Sort

- What if first *k* elements of array are already sorted?
  - › <u>4, 7, 12,</u> 5, 19, 16
- We can shift the tail of the sorted elements list down and then *insert* next element into proper position and we get k+1 sorted elements
  - › <u>4, 5, 7, 12,</u> 19, 16



# Insertion Sort

```
InsertionSort(A[1..N]: integer array, N: integer) {
  i, j, temp: integer ;
  for i = 2 to N {
    temp := A[i];
    j := i-1;
    while j > 1 and A[j-1] > temp {
          A[j] := A[j-1]; j := j–1;
          A[j] = temp;
    }
  }
}
```

- Is Insertion sort in place? Stable? Running time = ?
- Have we used this before?

# Example

| | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 8 | 7 | 9 | 10 | 12 | 23 | 18 | 15 | 16 | 17 | 14 |
| 1 | 2 | 3 | 7 | 8 | 9 | 10 | 12 | 23 | 18 | 15 | 16 | 17 | 14 |
| 1 | 2 | 3 | 7 | 8 | 9 | 10 | 12 | 18 | 23 | 15 | 16 | 17 | 14 |
| 1 | 2 | 3 | 7 | 8 | 9 | 10 | 12 | 18 | 15 | 23 | 16 | 17 | 14 |
| 1 | 2 | 3 | 7 | 8 | 9 | 10 | 12 | 15 | 18 | 23 | 16 | 17 | 14 |
| 1 | 2 | 3 | 7 | 8 | 9 | 10 | 12 | 15 | 18 | 16 | 23 | 17 | 14 |
| 1 | 2 | 3 | 7 | 8 | 9 | 10 | 12 | 15 | 16 | 18 | 23 | 17 | 14 |



# Example

| | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 7 | 8 | 9 | 10 | 12 | 15 | 16 | 18 | 17 | 23 | 14 |
| 1 | 2 | 3 | 7 | 8 | 9 | 10 | 12 | 15 | 16 | 17 | 18 | 23 | 14 |
| 1 | 2 | 3 | 7 | 8 | 9 | 10 | 12 | 15 | 16 | 17 | 18 | 14 | 23 |
| 1 | 2 | 3 | 7 | 8 | 9 | 10 | 12 | 15 | 16 | 17 | 14 | 18 | 23 |
| 1 | 2 | 3 | 7 | 8 | 9 | 10 | 12 | 15 | 16 | 14 | 17 | 18 | 23 |
| 1 | 2 | 3 | 7 | 8 | 9 | 10 | 12 | 15 | 14 | 16 | 17 | 18 | 23 |
| 1 | 2 | 3 | 7 | 8 | 9 | 10 | 12 | 14 | 15 | 16 | 17 | 18 | 23 |



# Insertion Sort Characteristics

- In place and Stable
- Running time
  - › Worst case is $O(N^2)$
    - reverse order input
    - must copy every element every time
- Good sorting algorithm for almost sorted data
  - › Each item is close to where it belongs in sorted order.



# Inversions

- An inversion is a pair of elements in wrong order
  - › $i < j$ but $A[i] > A[j]$
- By definition, a sorted array has no inversions
- So you can think of sorting as the process of removing inversions in the order of the elements

# Inversions

- A single value out of place can cause several inversions

| | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| value | 1 | 2 | 3 | 8 | 7 | 9 | 10 | 12 | 23 | 14 | 15 | 16 | 17 | 18 |
| index | **0** | **1** | **2** | **3** | **4** | **5** | **6** | **7** | **8** | **9** | **10** | **11** | **12** | **13** |



# Reverse order

- All values out of place (reverse order) causes numerous inversions

| | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| value | 1 | 2 | 3 | 8 | 7 | 9 | 10 | 12 | 23 | 18 | 17 | 16 | 15 | 14 |
| index | **0** | **1** | **2** | **3** | **4** | **5** | **6** | **7** | **8** | **9** | **10** | **11** | **12** | **13** |



# Inversions

- Our simple sorting algorithms so far swap adjacent elements (explicitly or implicitly) and remove just 1 inversion at a time
  - › Their running time is proportional to number of inversions in array
- Given N distinct keys, the maximum possible number of inversions is

$$(n-1)+(n-2)+...+1=\sum_{i=1}^{n-1} i = \frac{(n-1)n}{2}$$



# Inversions and Adjacent Swap Sorts

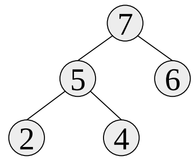
- "Average" list will contain half the max number of inversions = $\frac{(n-1)n}{4}$
  - › So the average running time of Insertion sort is $\Theta(N^2)$ (i.e, $O(N^2)$ is a tight bound)
- Any sorting algorithm that only swaps adjacent elements requires $\Omega(N^2)$ time because each swap removes only one inversion (lower bound)

# Heap Sort

- We use a Max-Heap
- Root node = A[1]
- Children of A[i] = A[2i], A[2i+1]
- Keep track of current size N (number of nodes)

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| value | 7 | 5 | 6 | 2 | 4 | | | |
| index | **1** | **2** | **3** | **4** | **5** | **6** | **7** | **8** |

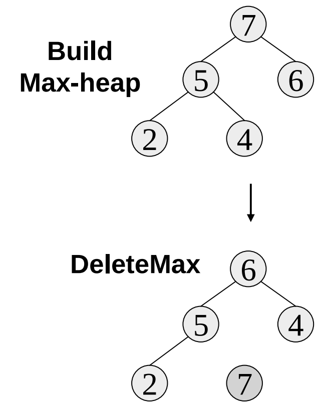
N = 5





# Using Binary Heaps for Sorting

- Build a <u>max-heap</u>
- Do N <u>DeleteMax</u> operations and store each Max element as it comes out of the heap
- Data comes out in largest to smallest order
- Where can we put the elements as they are removed from the heap?



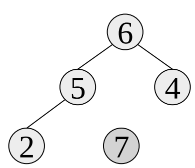



# 1 Removal = 1 Addition

- Every time we do a DeleteMax, the heap gets smaller by one node, and we have one more node to store
  - › Store the data at the end of the heap array
  - › Not "in the heap" but it is in the heap array

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| value | 6 | 5 | 4 | 2 | 7 | | | |
| index | **1** | **2** | **3** | **4** | **5** | **6** | **7** | **8** |

N = 4

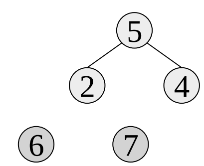




# Repeated DeleteMax

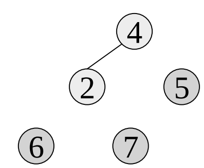
| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 5 | 2 | 4 | 6 | 7 | | | |
| **1** | **2** | **3** | **4** | **5** | **6** | **7** | **8** |

N = 3



| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 4 | 2 | 5 | 6 | 7 | | | |
| **1** | **2** | **3** | **4** | **5** | **6** | **7** | **8** |

N = 2

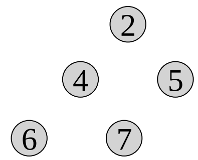


# Heap Sort is In-place

- After all the DeleteMaxs, the heap is gone but the array is full and is in sorted order

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| value | 2 | 4 | 5 | 6 | 7 | | | |
| index | **1** | **2** | **3** | **4** | **5** | **6** | **7** | **8** |

N = 0






# Heapsort: Analysis

- Running time
  - › time to build max-heap is O(N)
  - › time for N DeleteMax operations is N O(log N)
  - › total time is **O(N log N)**
- Can also show that running time is Ω(N log N) for some inputs,
  - › so *worst case* is Θ**(N log N)**
  - › *Average case* running time is also O(N log N)
- Heapsort is in-place but not stable (why?)