Graph Paths

CSE 373 - Data Structures
May 24, 2002

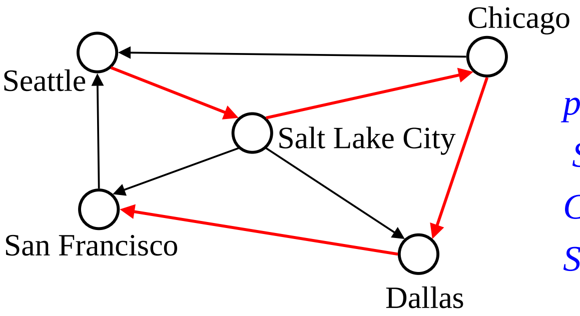
Readings and References

- Reading
  - › Section 9.3, *Data Structures and Algorithm Analysis in C*, Weiss
- Other References

Some slides based on: CSE 326 by S. Wolfman, 2000

Path

- A *path* is a list of vertices $\{v_1, v_2, \dots, v_n\}$ such that (v_i, v_{i+1}) is in E for all $0 \le i < n$.



p = {Seattle, Salt Lake City, Chicago, Dallas, San Francisco}

Simple Paths and Cycles

- A *simple path* repeats no vertices
  - › eg: {Seattle, Salt Lake City, San Francisco}
- A *cycle* is a path that starts and ends at the same vertex:
  - › {Seattle, Salt Lake City, San Francisco, Seattle}
- A *simple cycle* is a cycle that repeats no vertices and the first vertex is also the last
- A *directed acyclic graph* (DAG) is a directed graph with no cycles

Connected

- G is *connected* if there is a path between every pair of distinct vertices in the graph
- A graph which is not connected is the union of two or more connected subgraphs
    - › the subgraphs partition the graph G
    - › the subgraphs are the *connected components* of G
    - › note that the connected components are *not* connected to each other, but are themselves connected graphs

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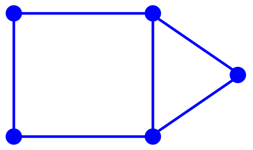
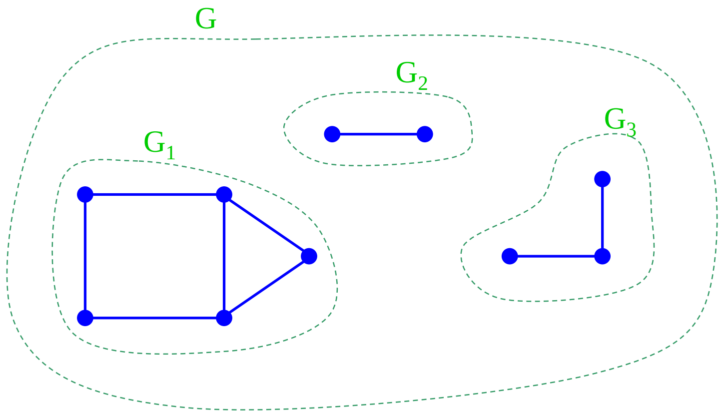
Undirected Connected Graph

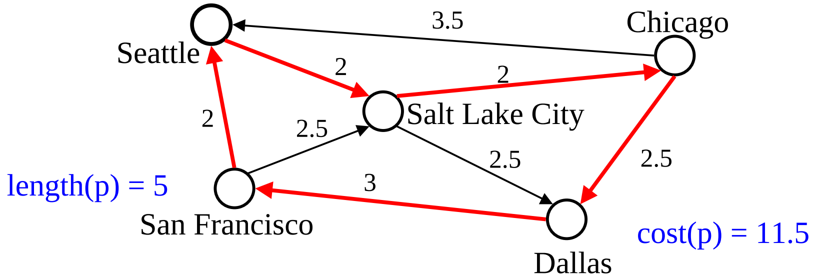


Connected Components of G



Path cost and Path length

- *Path cost*: the sum of the costs of each edge
- *Path length*: the number of edges in the path
    - › Path length is the unweighted path cost (each edge = 1)



Shortest Path Problems

- Given a graph G = (*V*, *E*) and a “source” vertex *s* in *V*, find the minimum cost paths from *s* to every vertex in *V*
- Many variations:
  - › unweighted vs. weighted
  - › cyclic vs. acyclic
  - › pos. weights only vs. pos. and neg. weights
  - › etc

Why study shortest path problems?

- Traveling on a budget: What is the cheapest airline schedule from Seattle to city X?
- Optimizing routing of packets on the internet:
  - › Vertices are routers and edges are network links with different delays. What is the routing path with smallest total delay?
- Shipping: Find which highways and roads to take to minimize total delay due to traffic

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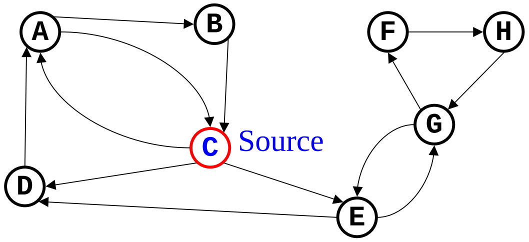
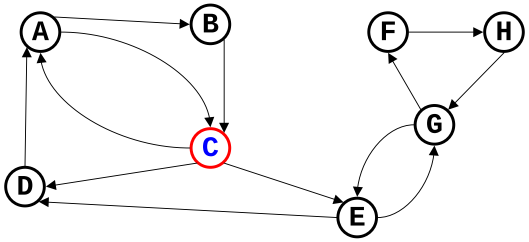
Unweighted Shortest Path Problem

Problem: Given a “source” vertex *s* in an unweighted graph G = (*V*,*E*), find the shortest path from *s* to all vertices in G



Breadth-First Search Solution

- Basic Idea: Starting at node s, find vertices that can be reached using 0, 1, 2, 3, …, N-1 edges (works even for cyclic graphs!)



Breadth-First Search Algorithm

- Uses a queue to track vertices that are “nearby”
- source vertex is **s**

```
Distance[s] = 0
Enqueue(s)
While queue is not empty
    X = dequeue a vertex
    For each vertex Y that is (adjacent to X and not
       previously visited)
        Distance[Y] = Distance[X] + 1
        Previous[Y] = X
        Enqueue Y
```

For each vertex

For each edge incident with that vertex

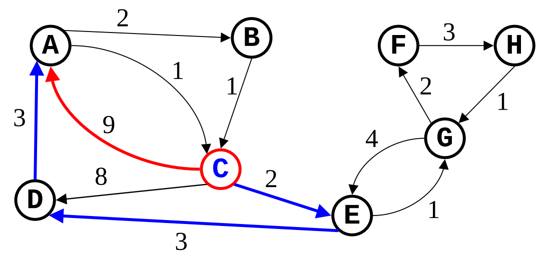
- Running time (same as topological sort) = **O(|*V*| + |*E*|)**

What if edges have weights?

- Breadth First Search does not work anymore
  - › minimum *cost* path may have more edges than minimum *length* path

Shortest path from C to A:
C→A (cost = 9)

Minimum Cost Path = C→E→D→A
(cost = 8)



Dijkstra’s Algorithm for Weighted Shortest Path

- Classic algorithm for solving shortest path in weighted graphs (without negative weights)
- A *greedy* algorithm (irrevocably makes decisions without considering future consequences)
- Each vertex has a cost for path from initial vertex
- Greedy choice – always expand to the least cost vertex
  - › a vertex already visited may be updated if a better path to it is found before it is added to the distinguished set

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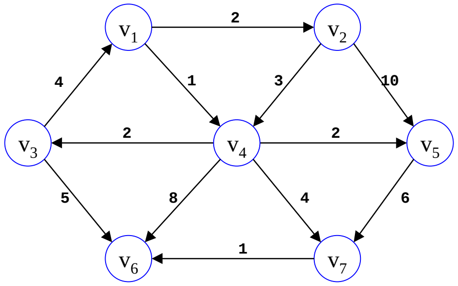
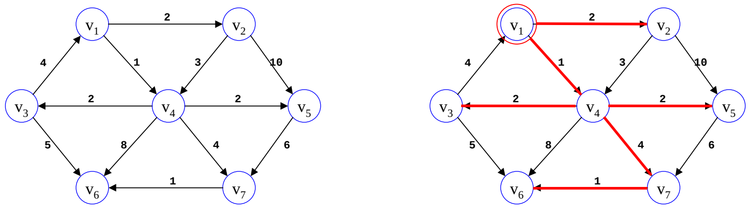
Dijkstra’s Shortest Path Algorithm

- Initialize the cost of initial node to 0, and all the rest of the nodes to ∞
- Initialize set S to be ∅
- While there are nodes left in the graph but not in S
  - › Select the node K with the lowest cost that is not in S and identify the node as now being in S
  - › for each node A adjacent to K
    - if (cost(K)+cost(K,A) < A’s currently known cost
      - – set cost(A) = cost(K)+cost(K,A)
      - – set previous(A) = K so that we can remember the path

A weighted directed graph



Dijkstra example



	S ?	d_v	P	S ?	d_v	P	S ?	d_v	P	S ?	d_v	P	S ?	d_v	P	S ?	d_v	P	S ?	d_v	P
v_1	*	0	–	*	0	–	*	0	–	*	0	–	*	0	–	*	0	–	*	0	–
v_2		2	v_1		2	v_1	*	2	v_1	*	2	v_1	*	2	v_1	*	2	v_1	*	2	v_1
v_3		∞			3	v_4		3	v_4		3	v_4	*	3	v_4	*	3	v_4	*	3	v_4
v_4		1	v_1	*	1	v_1	*	1	v_1	*	1	v_1	*	1	v_1	*	1	v_1	*	1	v_1
v_5		∞			3	v_4		3	v_4	*	3	v_4	*	3	v_4	*	3	v_4	*	3	v_4
v_6		∞			9	v_4		9	v_4		9	v_4		8	v_3		6	v_7	*	6	v_7
v_7		∞			5	v_4		5	v_4		5	v_4		5	v_4	*	5	v_4	*	5	v_4

Analysis of Dijkstra's Algorithm

```
While there are nodes left in the graph but not in S          ← |V| times
    Select the node K with the lowest cost that is not in S and   ← O(|V|)
    identify the node as now being in S
    for each node A adjacent to K                             ← O(|E|) total
        if (cost(K)+cost(K,A) < A's currently known cost
            set cost(A) = cost(K)+cost(K,A)
            set  previous(A) = K so that we can remember the
            path
```

Total time = $|V|\ (O(|V|)) + O(|E|) = O(|V|^2 + |E|)$

Dense graph: $|E| = \Theta(|V|^2) \rightarrow$ Total time = $O(|V|^2) = O(|E|)$

Sparse graph: $|E| = \Theta(|V|) \rightarrow$ Total time = $O(|V|^2) = O(|E|^2)$

Quadratic! Can we do better?

Analysis of Dijkstra's Algorithm

Yes! Use a priority queue to store vertices with key = cost

|*V*| times:

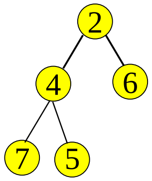
Select the unknown node *N* with the *lowest cost*

deleteMin

|*E*| times:

A's cost = *N*'s cost + cost of (*N*, *A*)

decreaseKey

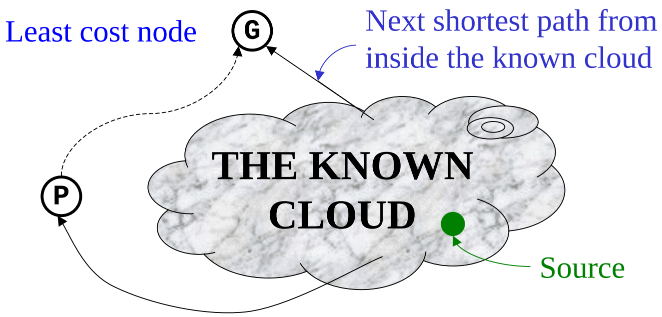


Total run time = $O(|V| \log |V| + |E| \log |V|)$

Does It Always Work?

- Dijkstra’s algorithm is an example of a greedy algorithm
- Greedy algorithms always make choices that currently seem the best
  - › Short-sighted – no consideration of long-term or global issues
  - › Locally optimal does not always mean globally optimal
- In Dijkstra’s case – choose the least cost node, but what if there is another path through other vertices that is cheaper?

“Cloudy” Proof



If the path to **G** is the next shortest path, the path to **P** must be at least as long. Note - no negative path weights!

Therefore, any path through **P** to **G** cannot be shorter!

Inside the Cloud (Proof)

Everything inside the cloud has the correct shortest path

Proof is by induction on the # of nodes in the cloud:

- › Base case: Initial cloud is just the source with shortest path 0
- › Inductive hypothesis: cloud of k-1 nodes all have shortest paths
- › Inductive step: choose the least cost node G → has to be the shortest path to G (previous slide). Add k^{th} node G to the cloud