Topological Sort of a Graph

CSE 373 - Data Structures

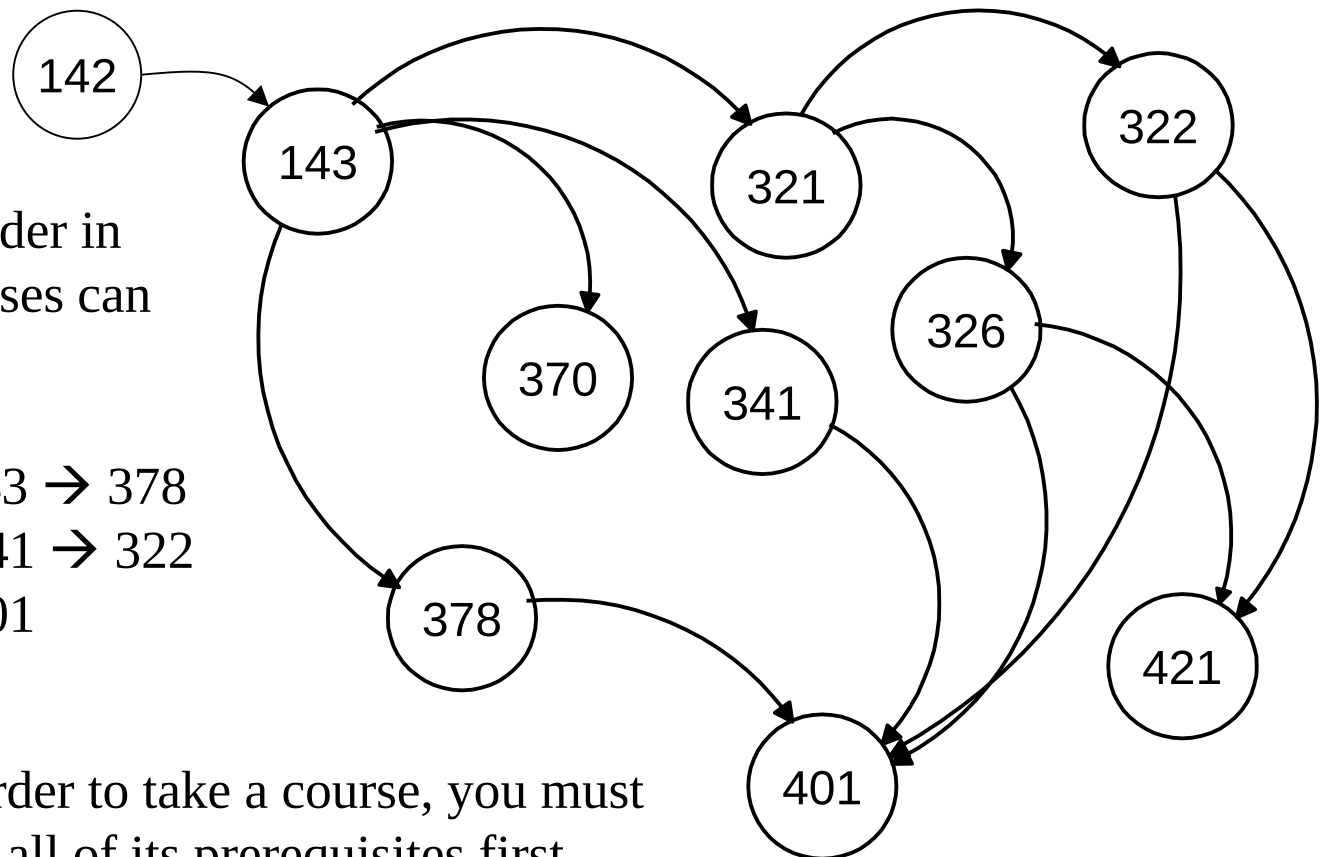
May 24, 2002

Readings and References

- Reading
  - › Section 9.2, *Data Structures and Algorithm Analysis in C*, Weiss
- Other References

Some slides based on: CSE 326 by S. Wolfman, 2000

Topological Sort



Problem: Find an order in which all these courses can be taken.

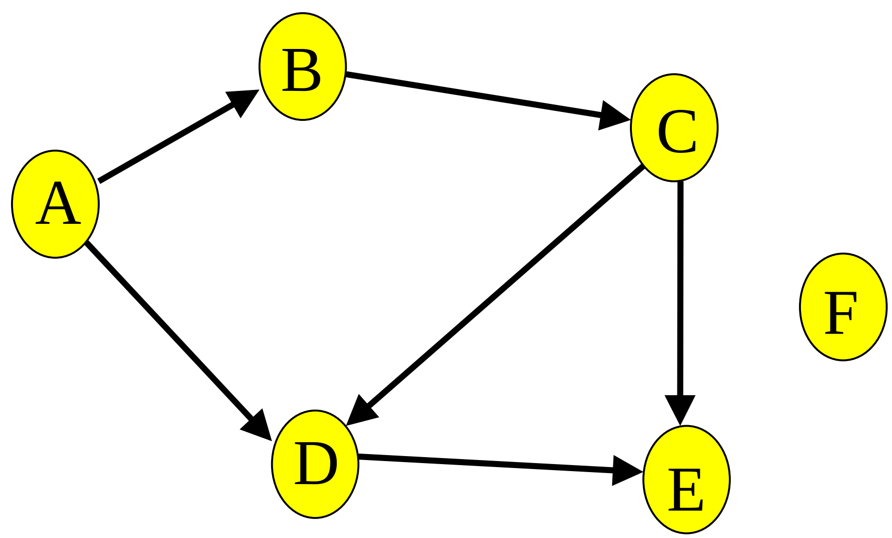
Example: 142 → 143 → 378 → 370 → 321 → 341 → 322 → 326 → 421 → 401

In order to take a course, you must take all of its prerequisites first

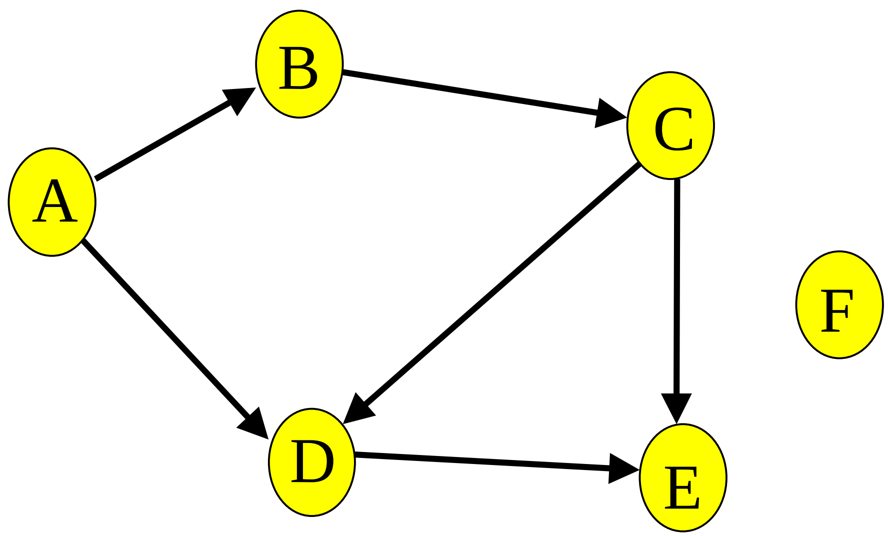
Topological Sort

Given a digraph $G = (V, E)$, find a linear ordering of its vertices such that:

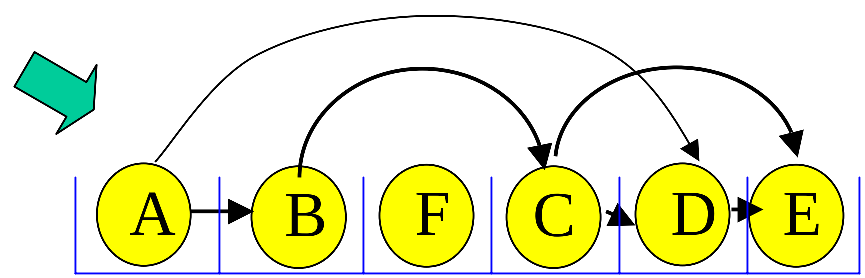
for any edge (v, w) in E, v precedes w in the ordering



Topo sort - good example

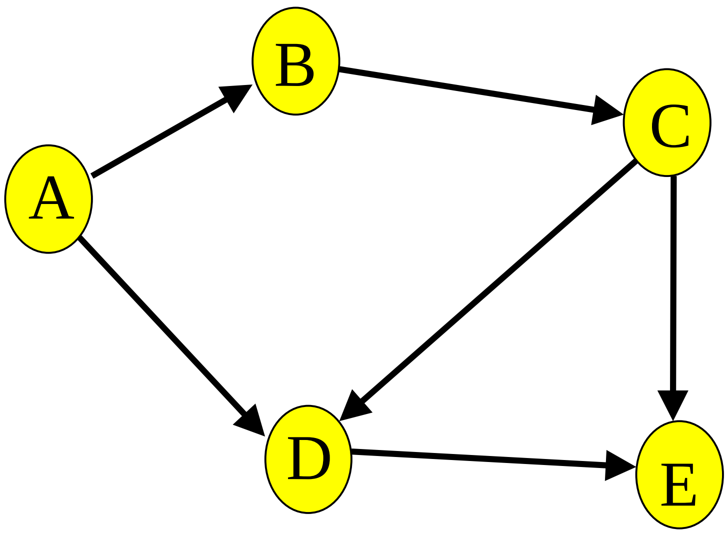


Any linear ordering in which all the arrows go to the right is a valid solution

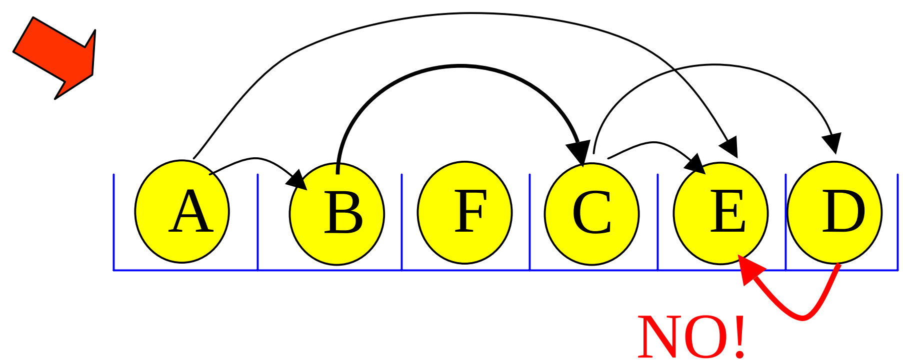


Note that F can go anywhere in this list because it is not connected.

Topo sort - bad example



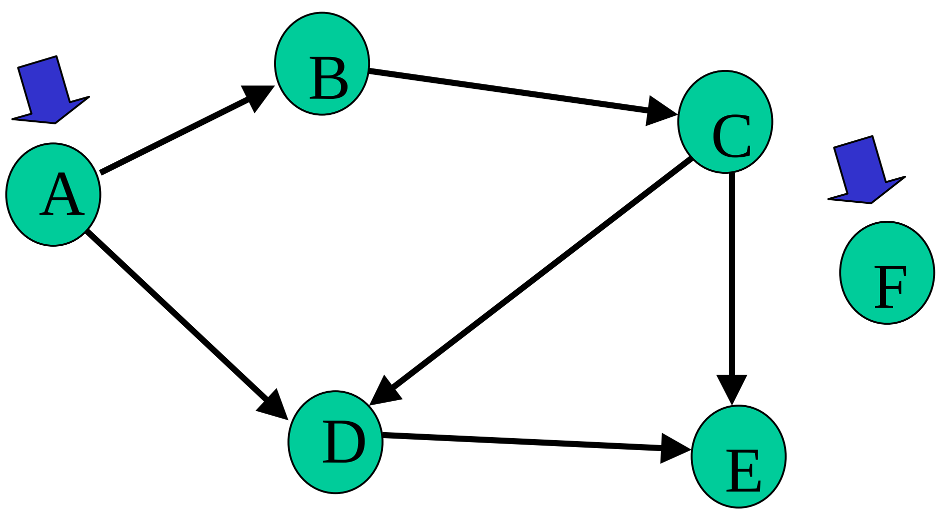
Any linear ordering in which an arrow goes to the left is not a valid solution



Topo sort algorithm - 1

Step 1: Identify vertices that have no incoming edges

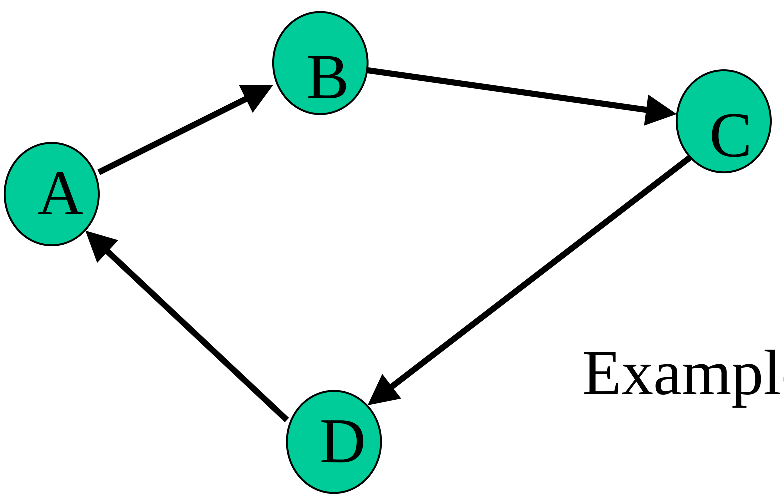
- The "in-degree" of these vertices is zero



Topo sort algorithm - 1a

Step 1: Identify vertices that have no incoming edges

- If *no such vertices*, graph has cycle(s) (cyclic graph)
- Topological sort not possible – Halt.

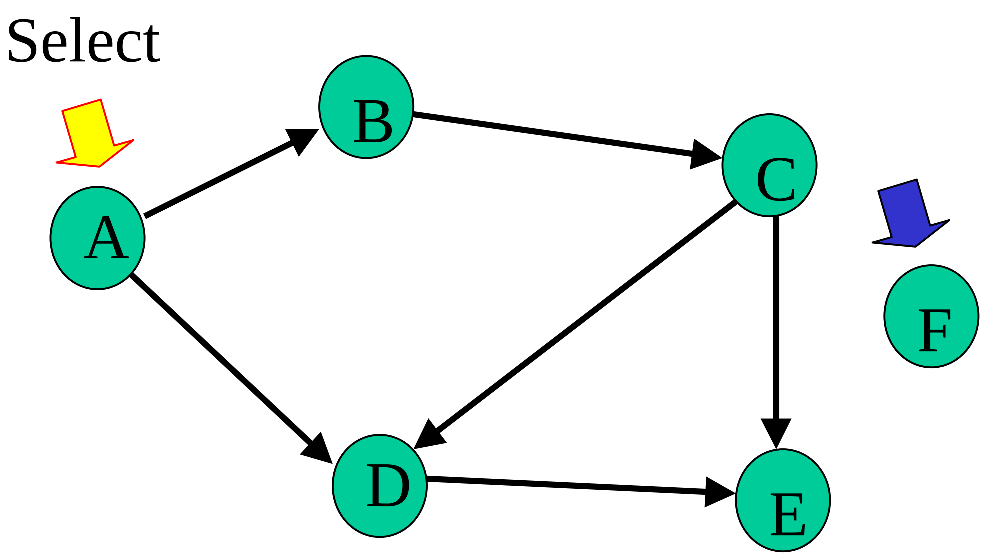


Example of a cyclic graph

Topo sort algorithm - 1b

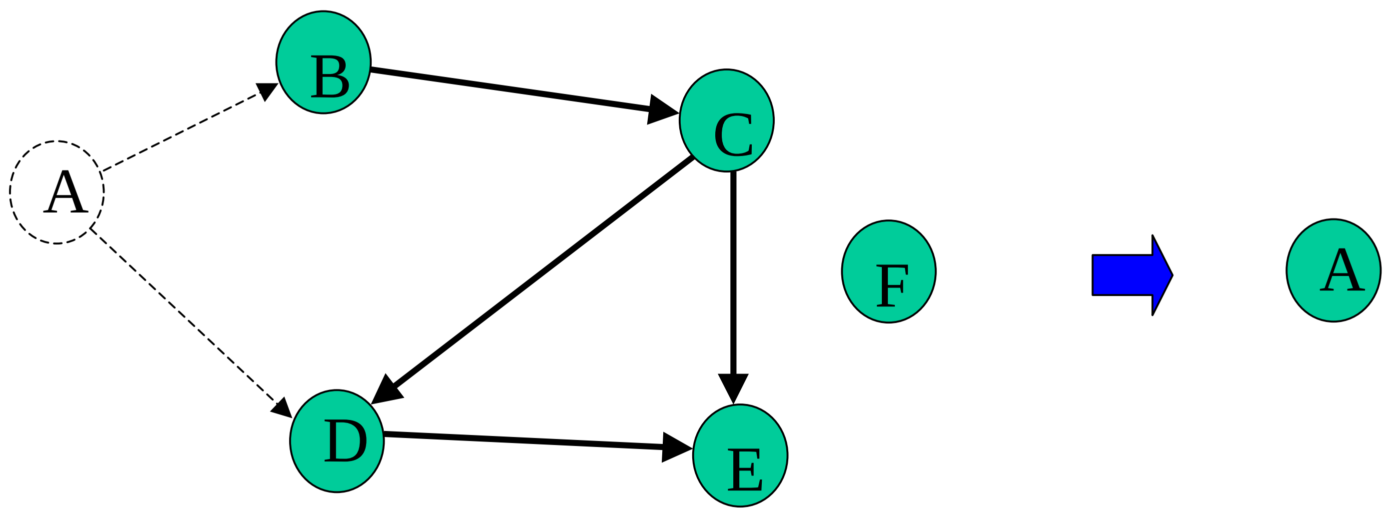
Step 1: Identify vertices that have no incoming edges

- Select one such vertex



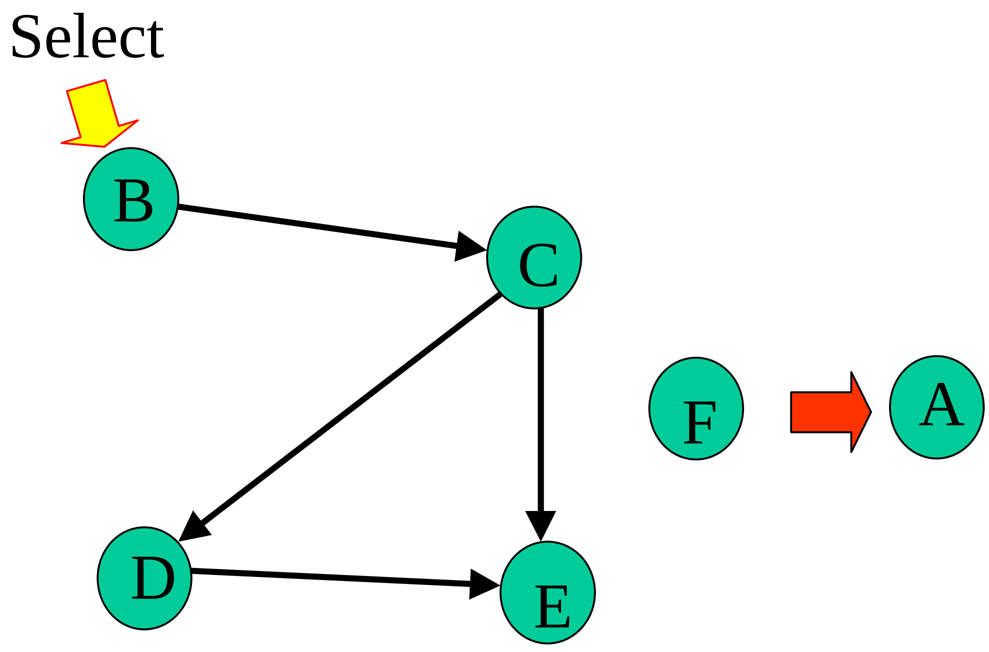
Topo sort algorithm - 2

Step 2: Delete this vertex of in-degree 0 and all its outgoing edges from the graph. Place it in the output.



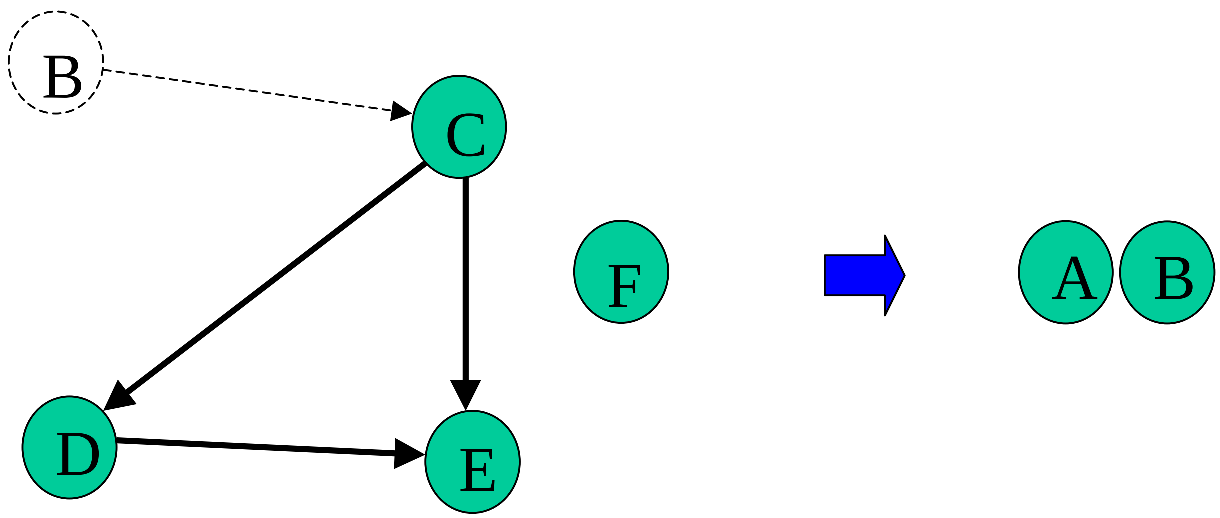
Cook until done

Repeat Step 1 and Step 2 until graph is empty



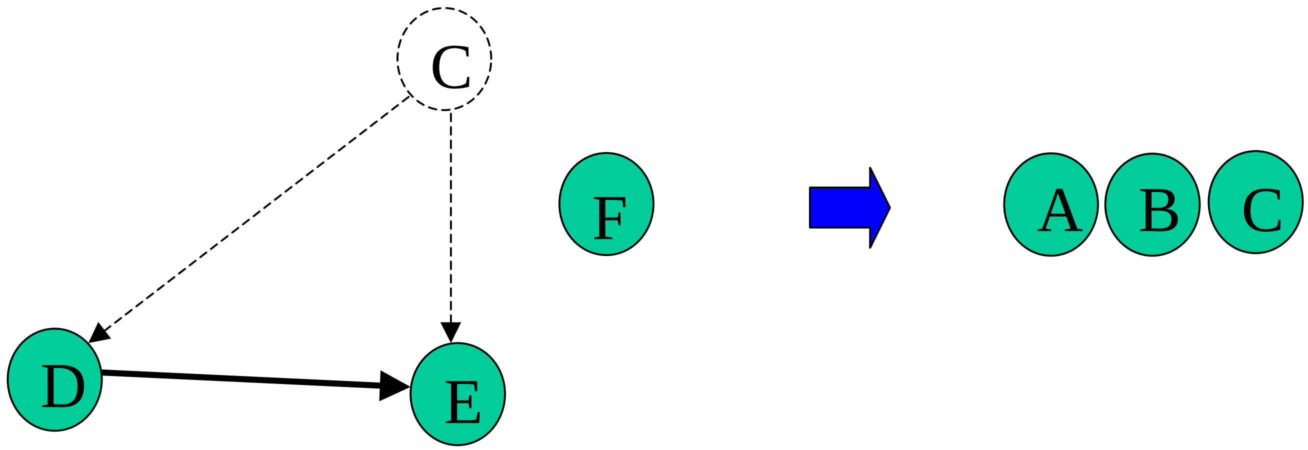
B

Select B. Copy to sorted list. Delete B and its edges.



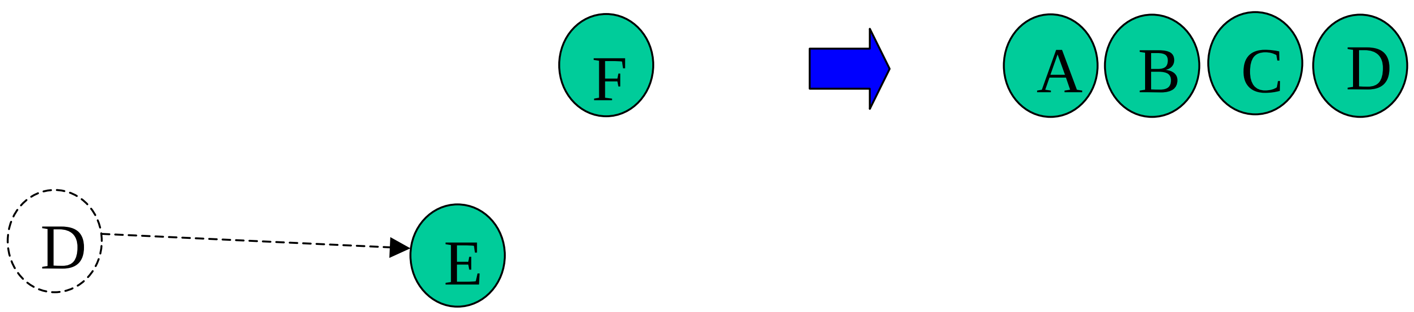
C

Select C. Copy to sorted list. Delete C and its edges.



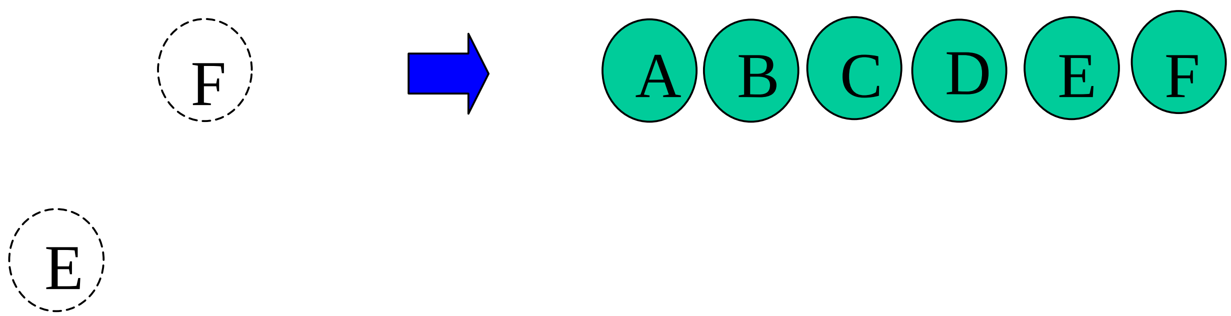
D

Select D. Copy to sorted list. Delete D and its edges.

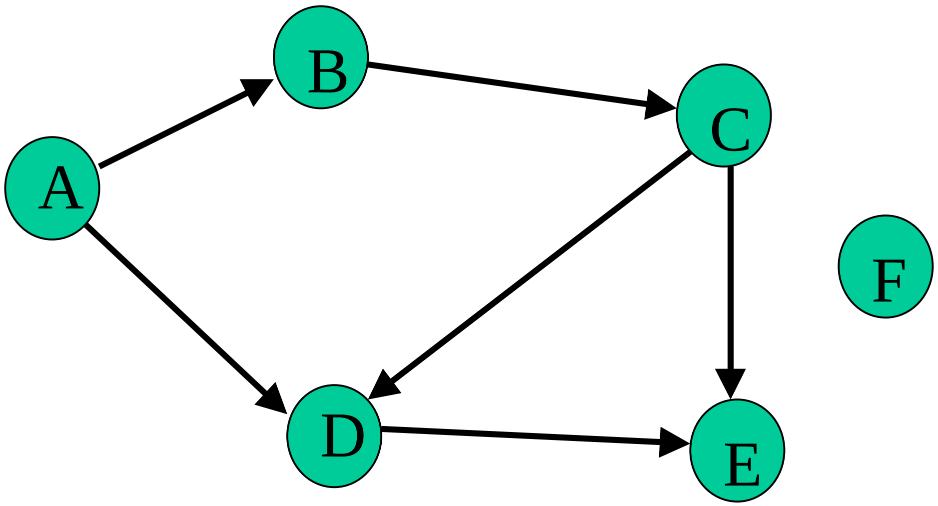


E, F

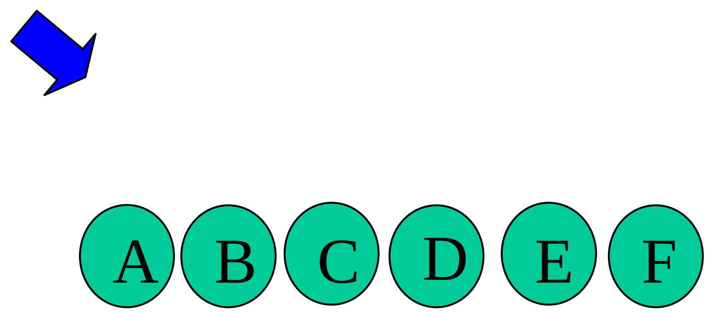
Select E. Copy to sorted list. Delete E and its edges.
Select F. Copy to sorted list. Delete F and its edges.



Done



Remove from algorithm and serve.



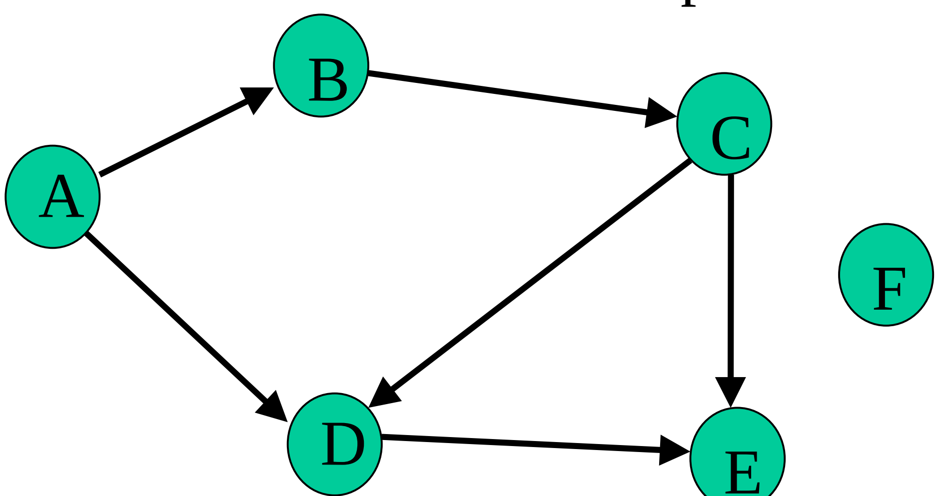
Topo sort run time analysis

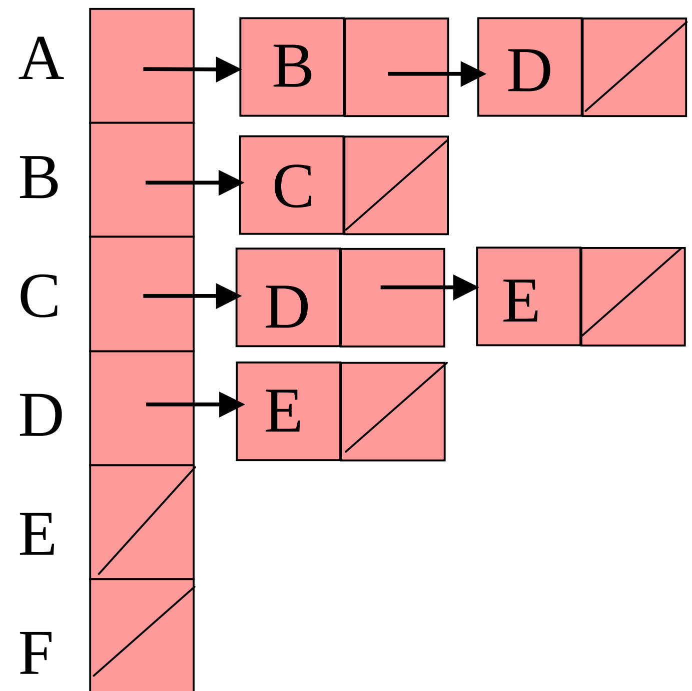
For input graph G = (*V*,*E*), Run Time = O(?)

Break down into total time to:

- → Find a vertex with in-degree 0
- → Remove its edges
- → Place vertex in output

Assume adjacency list representation





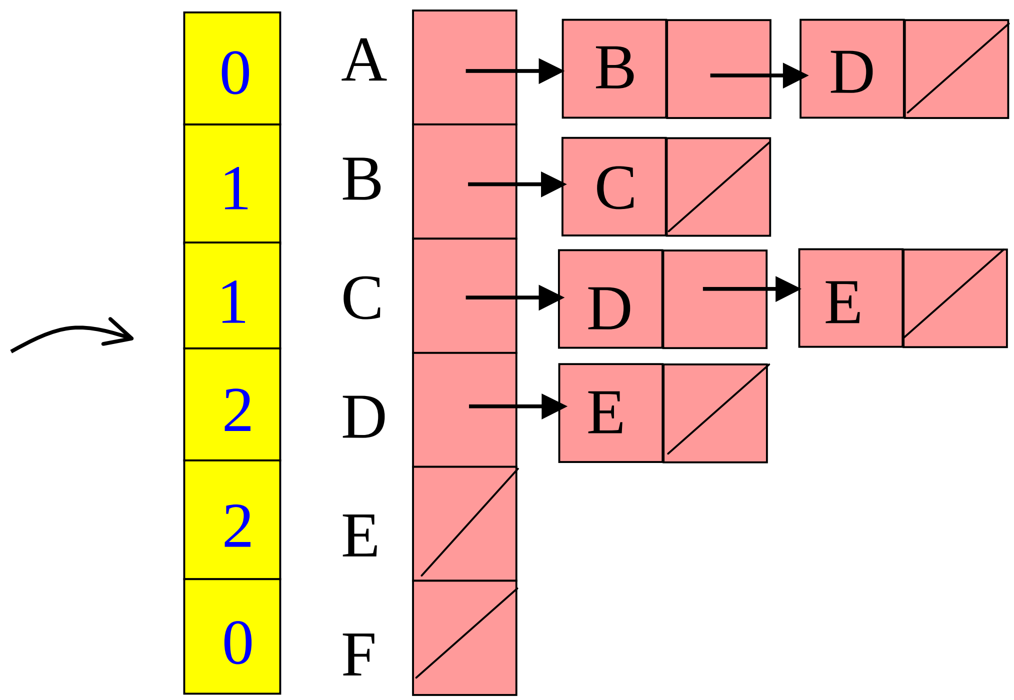
Tracking "in-degree"

Calculate and store In-Degree of all vertices in an array

→ Find vertex with in-degree 0: Search the array

→ Remove its edges: Update the array

In-Degree array



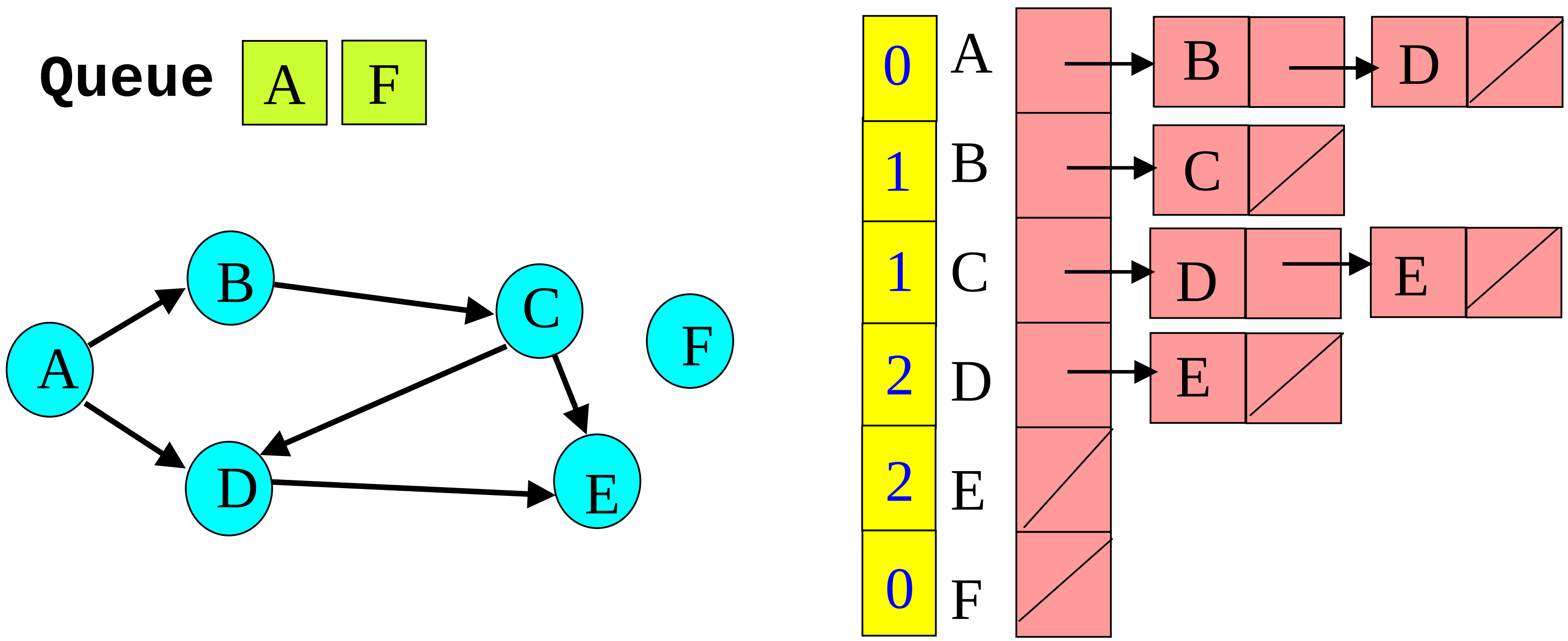
Topo Sort_1 run time

- Find vertices with in-degree 0:
  - › |V| vertices, and for each vertex it takes O(|V|) to search the In-Degree array = $O(|V|^2)$
- Remove edges:
  - › |E| edges
- Place vertices in output:
  - › |V| vertices
- For input graph G = (V,E)
  - › Run Time = $O(|V|^2 + |E|)$
  - › Quadratic in |V|

We need a better way to find the next vertex with degree(v)=0 ...

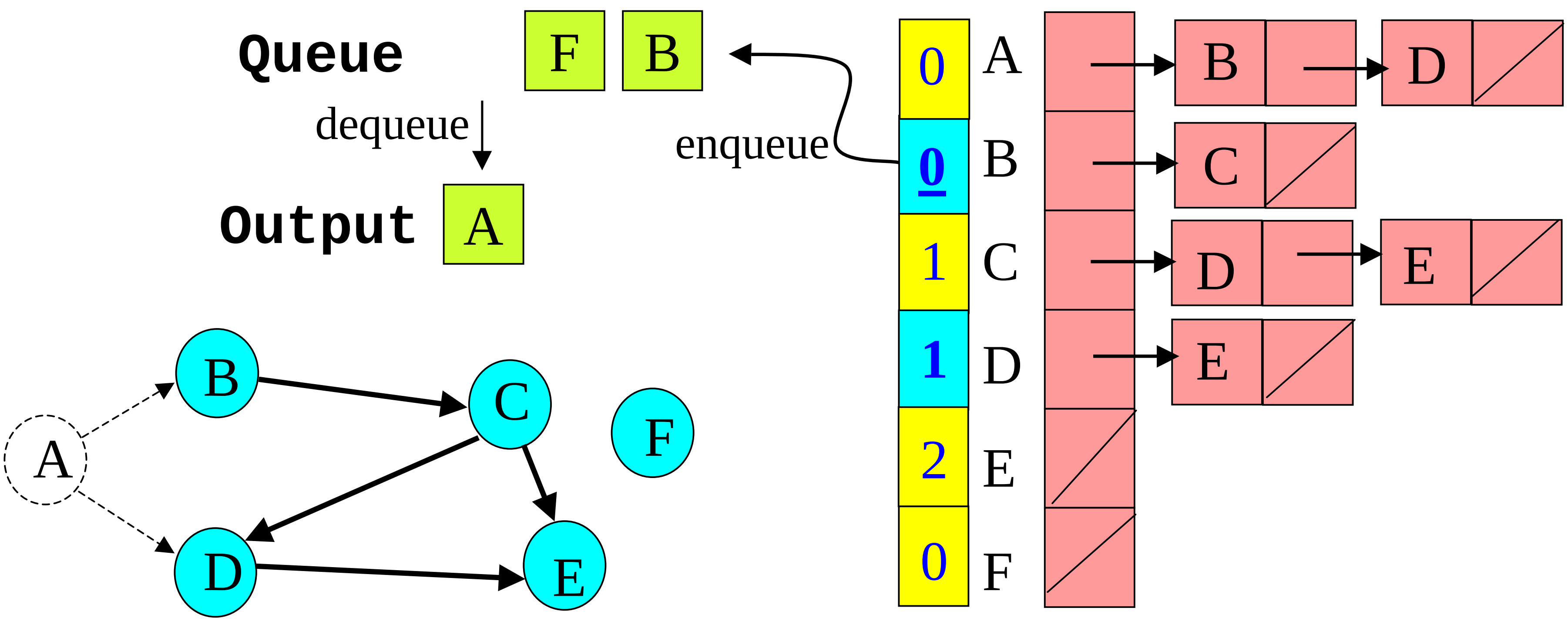
Topo Sort with queue

Key idea: Initialize and maintain a *queue (or stack)* of vertices with In-Degree 0



Topo Sort with queue

After each vertex is output, when updating In-Degree array, *enqueue any vertex whose In-Degree has become zero*



Topological Sort Algorithm #2

- Store each vertex's In-Degree in an array
- Initialize queue with all "in-degree=0" vertices
- While there are vertices remaining in the queue:
  - › Dequeue and output a vertex
  - › Reduce In-Degree of all vertices adjacent to it by 1
  - › Enqueue any of these vertices whose In-Degree became zero

Topo Sort$_2$ run time

- Initialize In-Degree array: O(|E|)
- Initialize Queue with In-Degree 0 vertices: O(|V|)
- Dequeue and output vertex:
  - › |V| vertices, each takes only O(1) to dequeue and output: O(|V|)
- Reduce In-Degree of all vertices adjacent to a vertex and Enqueue any In-Degree 0 vertices:
  - › O(|E|)
- For input graph G=(V,E) run time = O(|V| + |E|)
  - › Linear in |V|