Graph Intro

CSE 373 - Data Structures

May 22, 2002

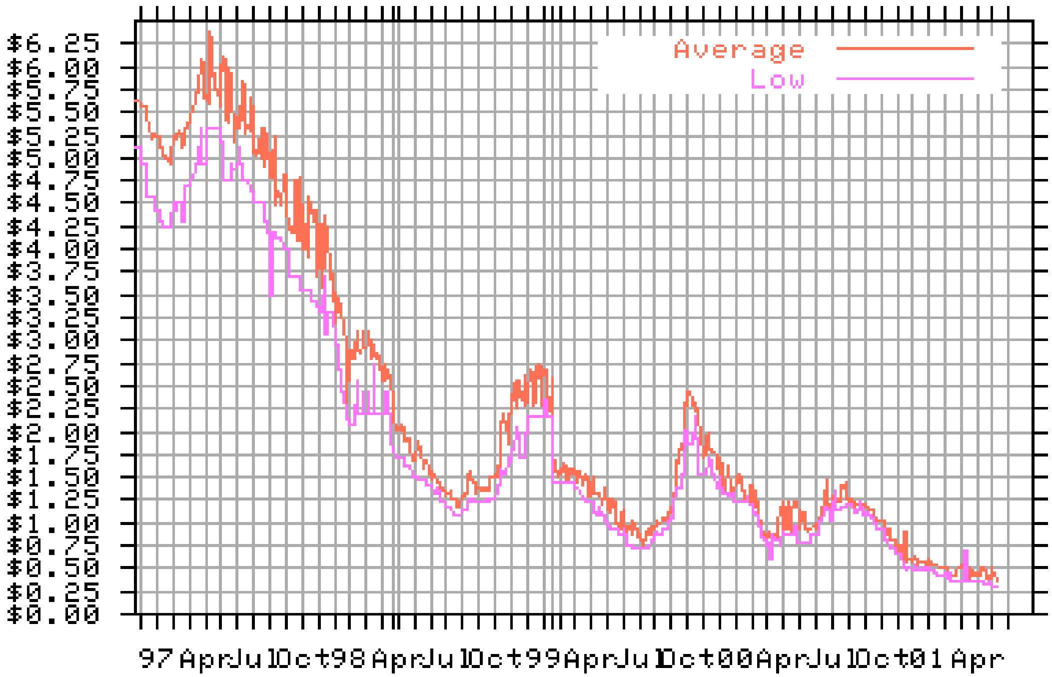
Readings and References

- Reading
  - › Section 9.1, *Data Structures and Algorithm Analysis in C*, Weiss
- Other References
  - › Section 23.1, Representation of Graphs, *Intro to Algorithms*, Cormen, Leiserson, Rivest

Some slides based on: CSE 326 by S. Wolfman, 2000

What are graphs?

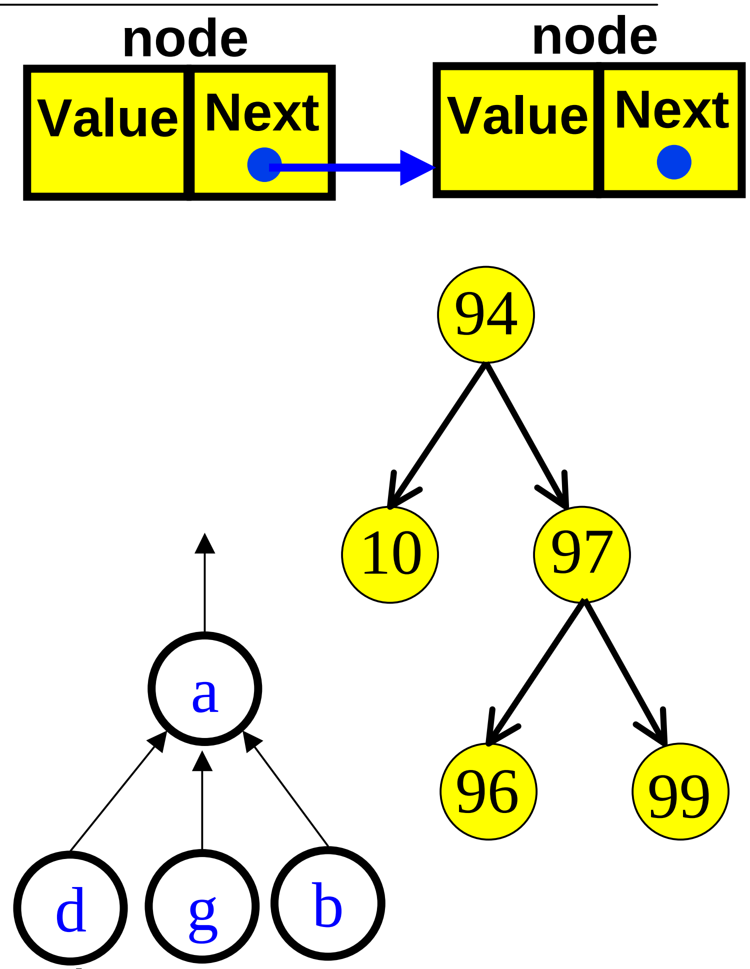
- Yes, this is a graph….



- But we are interested in a different kind of “graph”

Motivation for Graphs

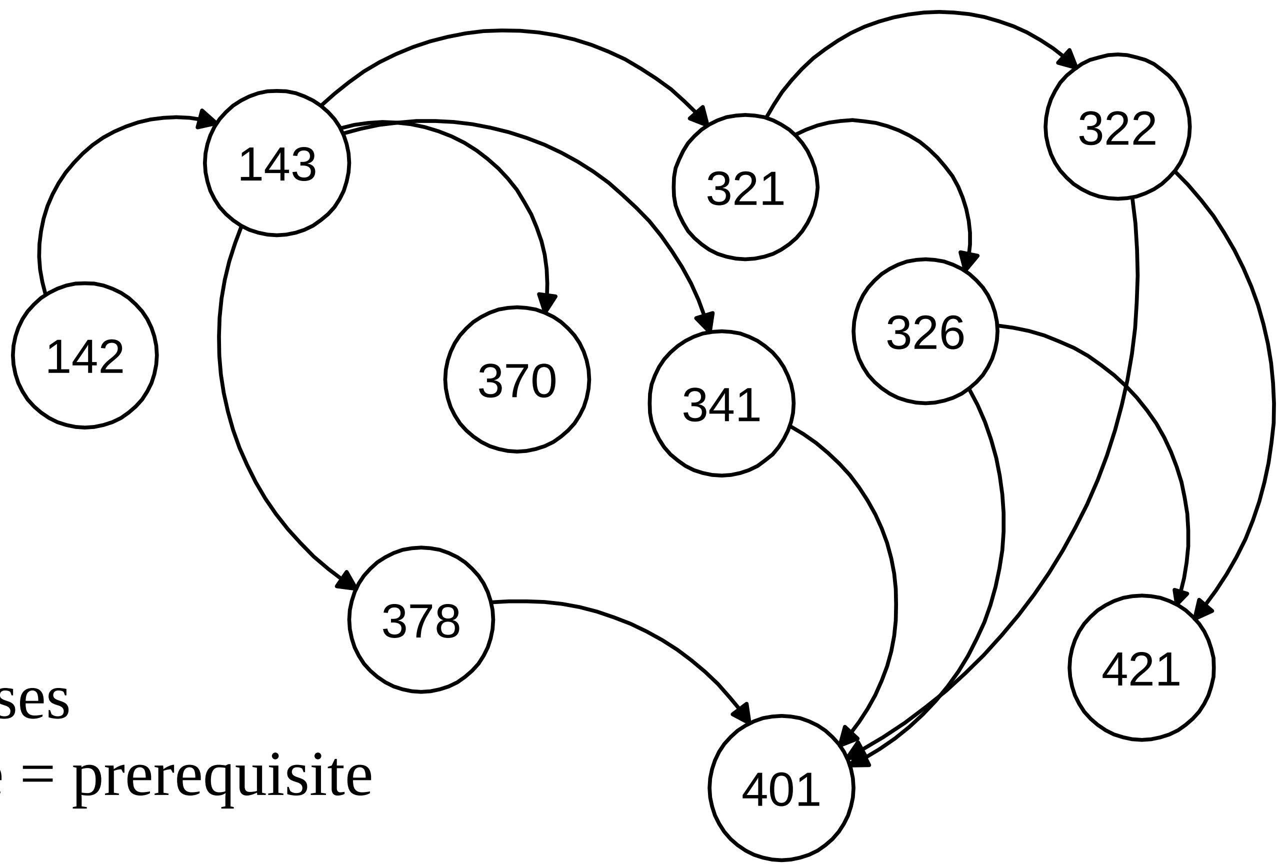
- Consider the data structures we have looked at so far…
- Linked list: nodes with 1 incoming edge + 1 outgoing edge
- Binary trees/heaps: nodes with 1 incoming edge + 2 outgoing edges
- Binomial trees/B-trees: nodes with 1 incoming edge + multiple outgoing edges
- Up-trees: nodes with multiple incoming edges + 1 outgoing edge



Motivation for Graphs

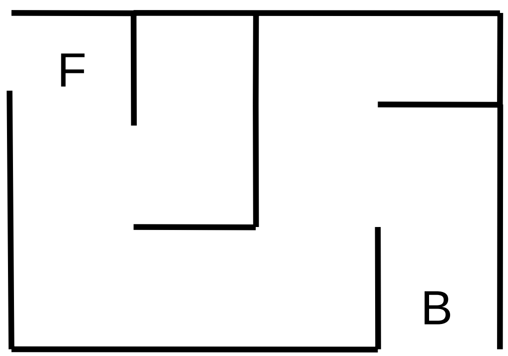
- What is common among these data structures?
- How can you generalize them?
- Consider data structures for representing the following problems…

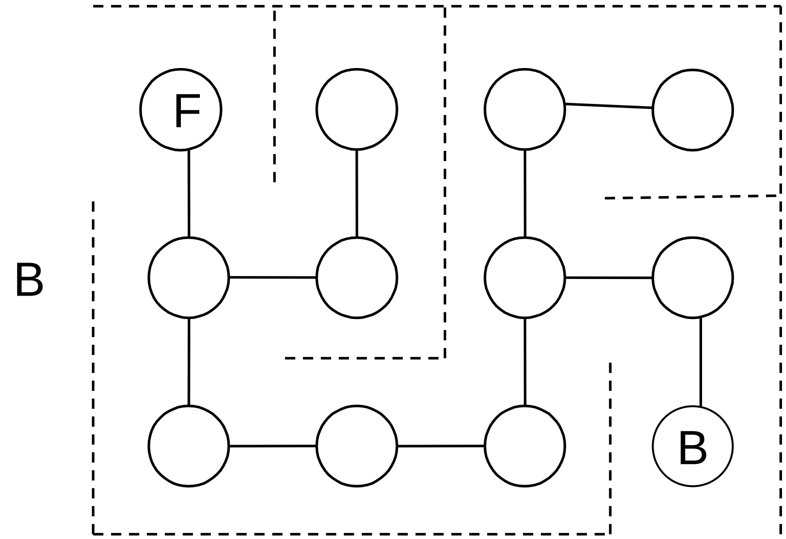
CSE Course Prerequisites at UW



Nodes = courses
Directed edge = prerequisite

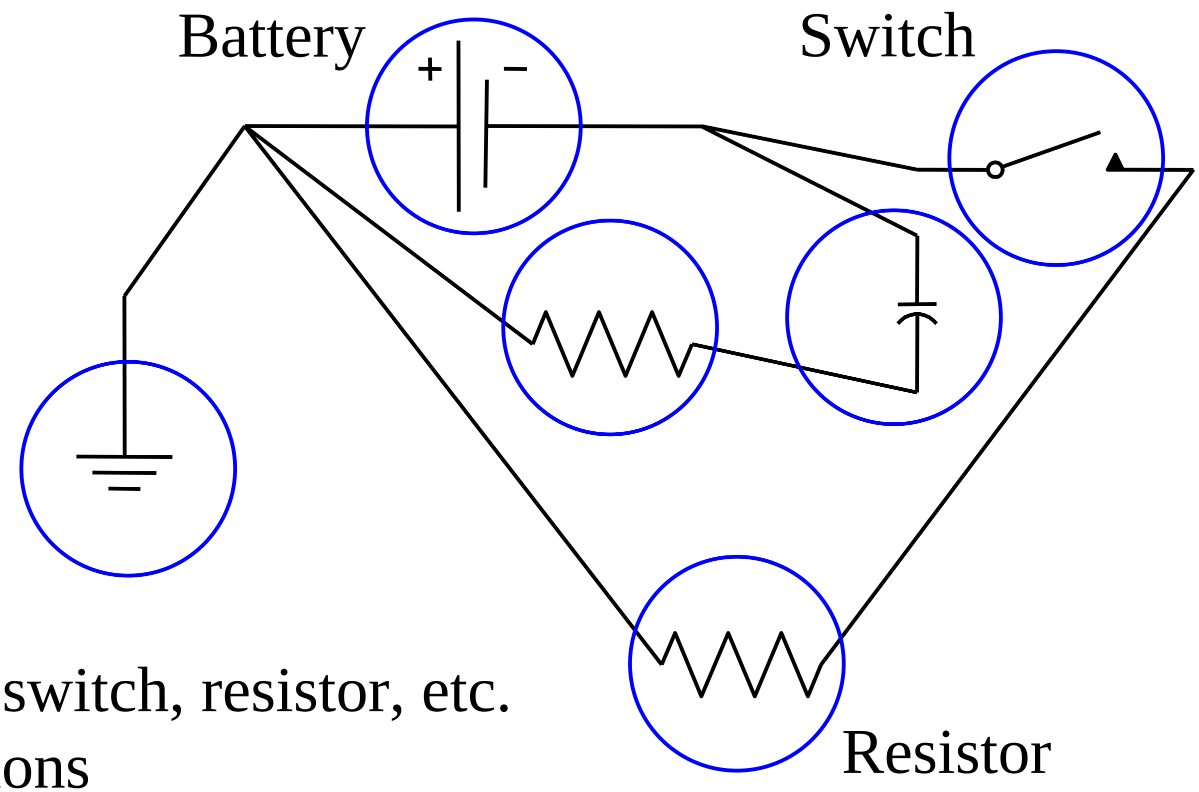
Representing a Maze





Nodes = rooms
Edge = door or passage

Representing Electrical Circuits

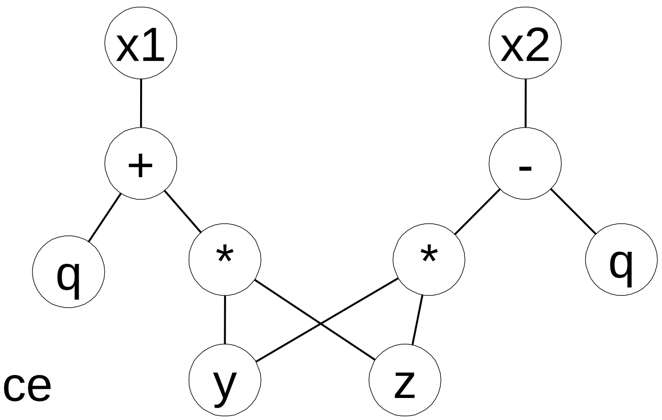


Nodes = battery, switch, resistor, etc.
Edges = connections

Program statements

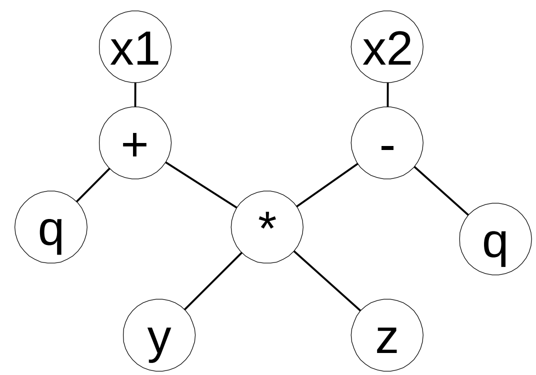
```
x1=q+y*z
x2=y*z-q
```

Naive:



y*z calculated twice

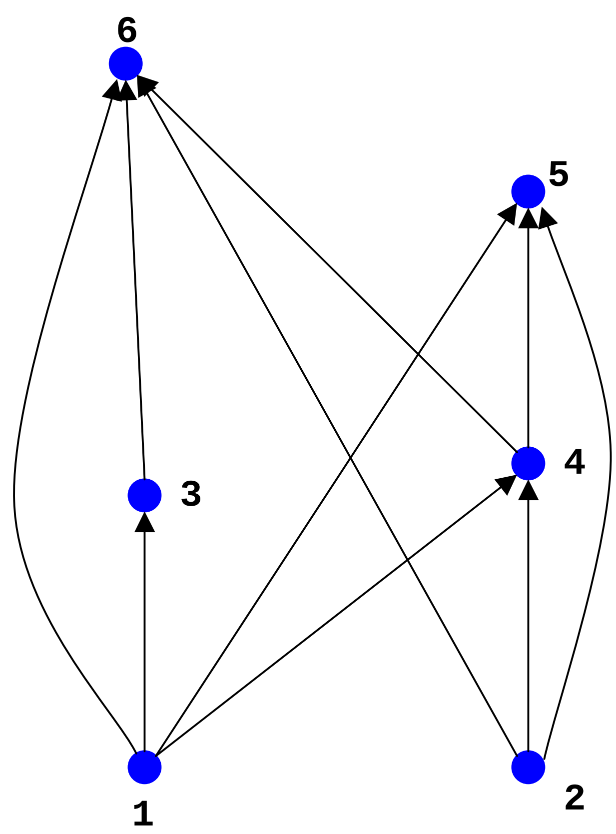
common
subexpression
eliminated:



Nodes = symbols/operators
Edges = relationships

Precedence

S_1	`a=0;`
S_2	`b=1;`
S_3	`c=a+1`
S_4	`d=b+a;`
S_5	`e=d+1;`
S_6	`e=c+d;`



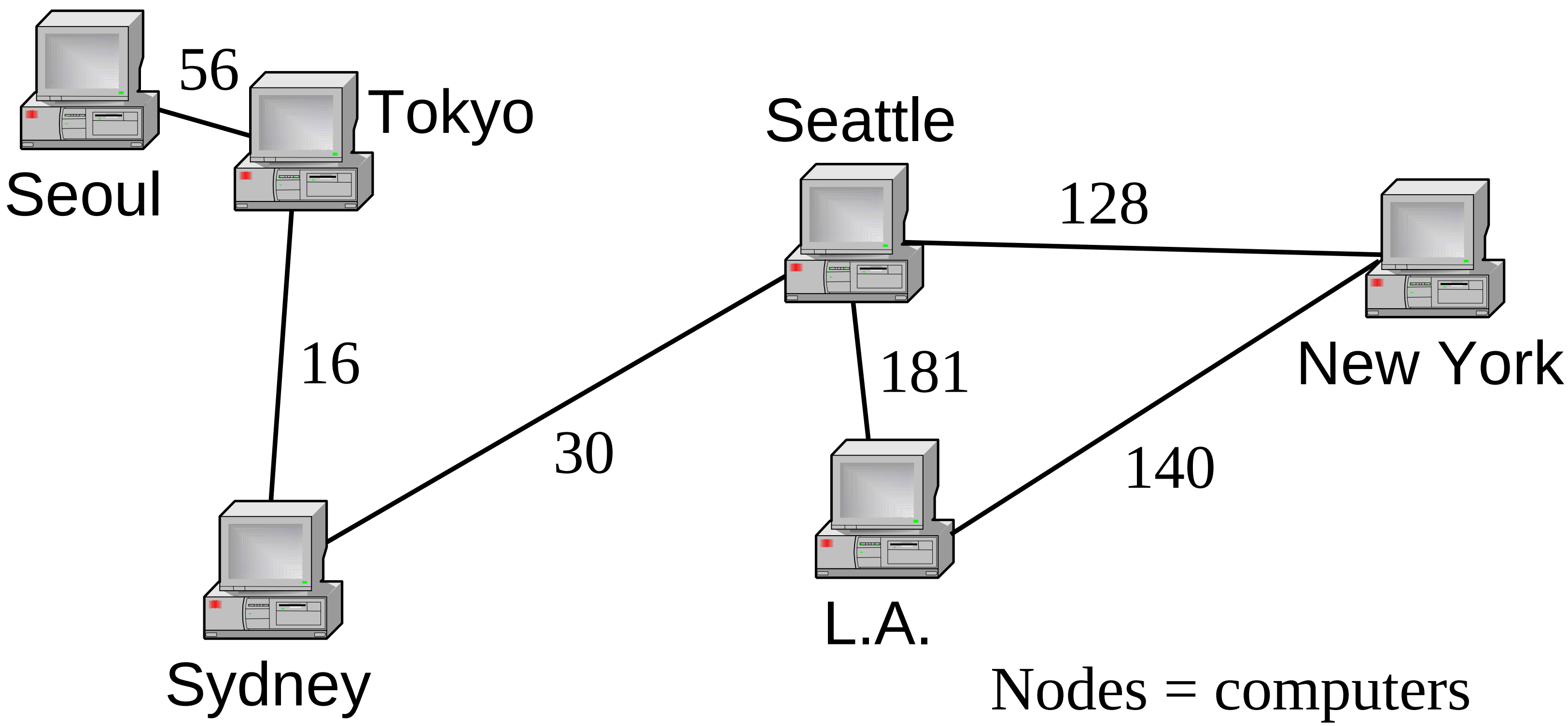
Which statements must execute before S_6?

S_1, S_2, S_3, S_4

Nodes = statements

Edges = precedence requirements

Information Transmission in a Computer Network

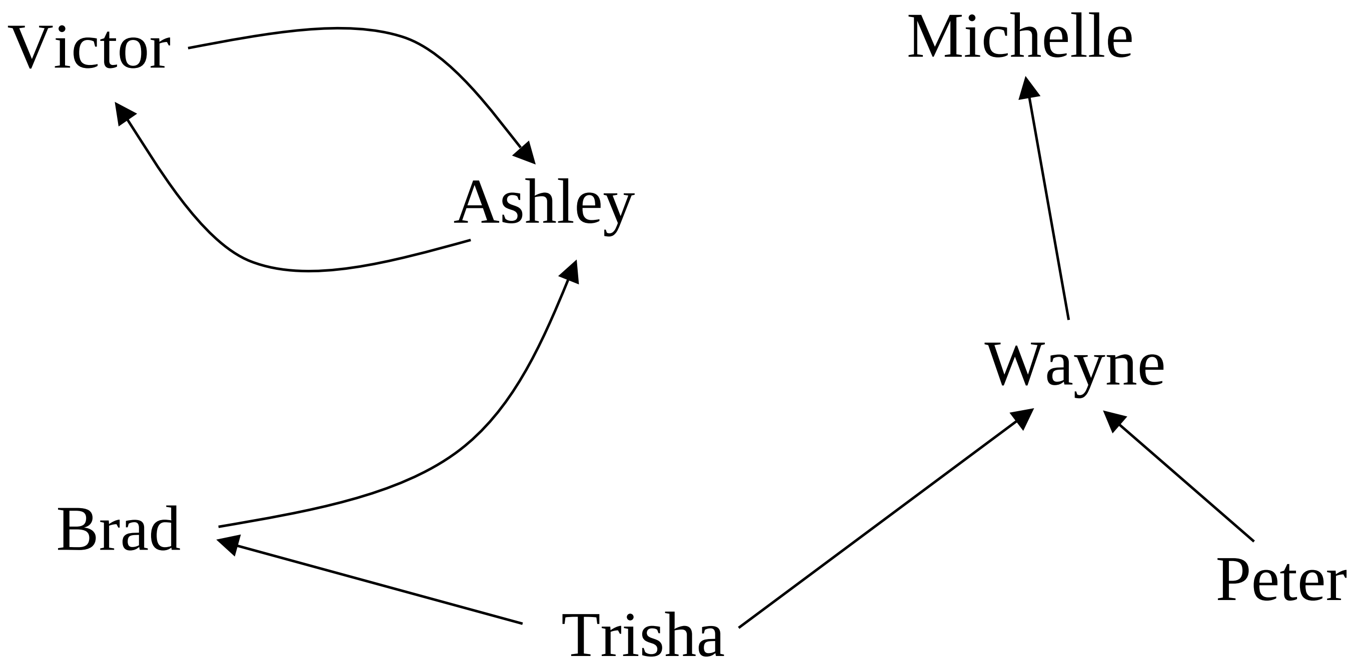


Nodes = computers
Edges = transmission rates

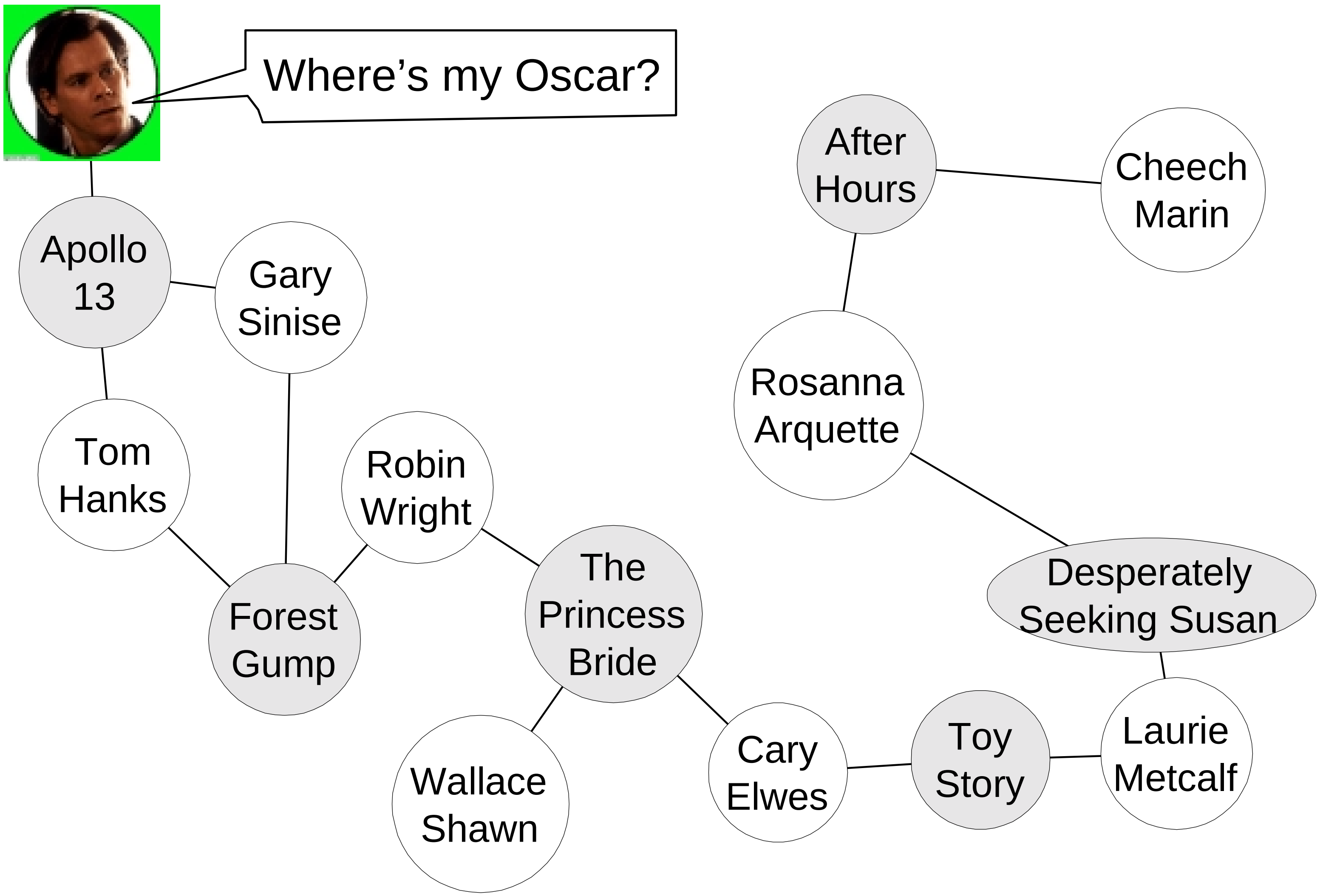
Traffic Flow on Highways

Nodes = cities
Edges = # vehicles on connecting highway

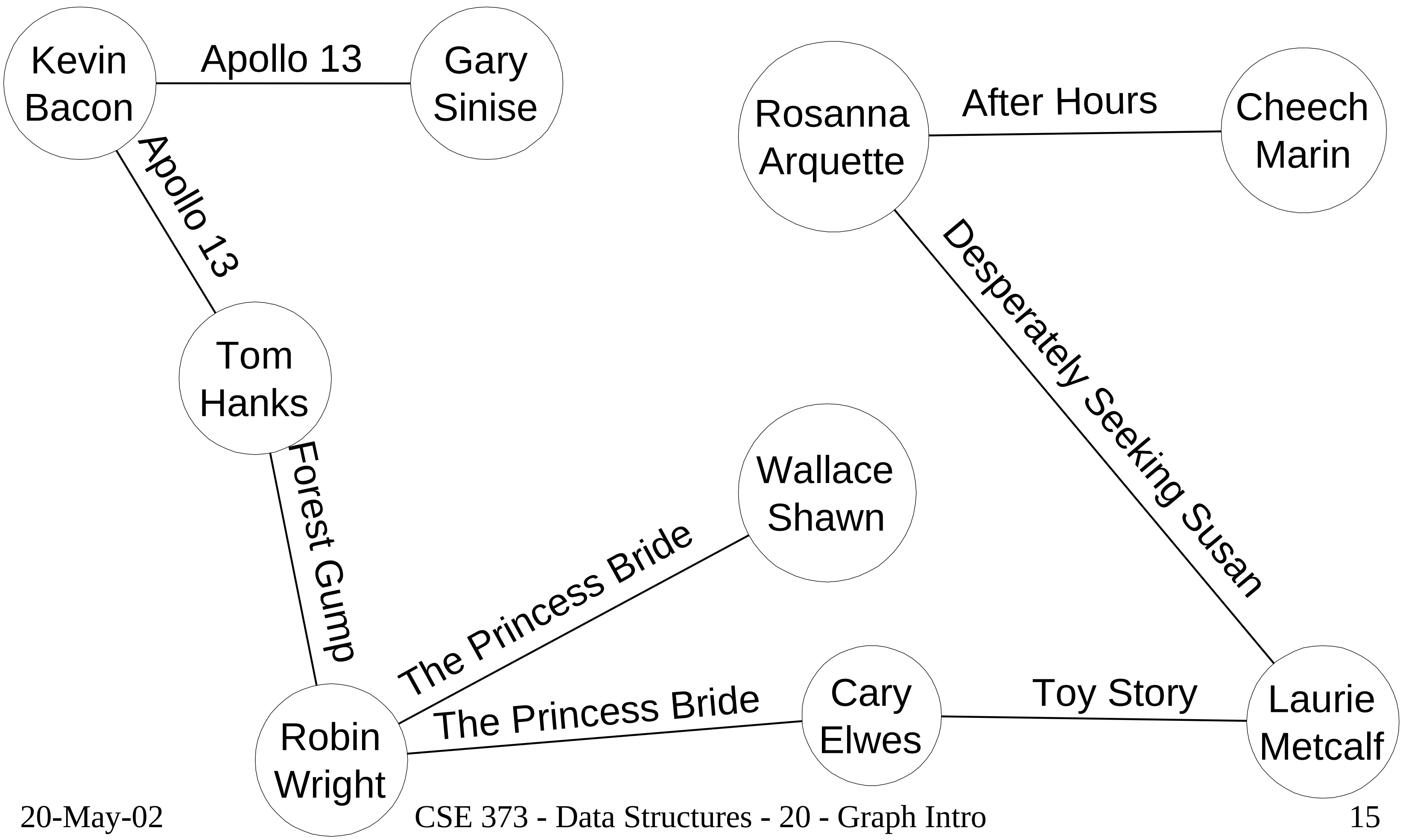
Soap Opera Relationships



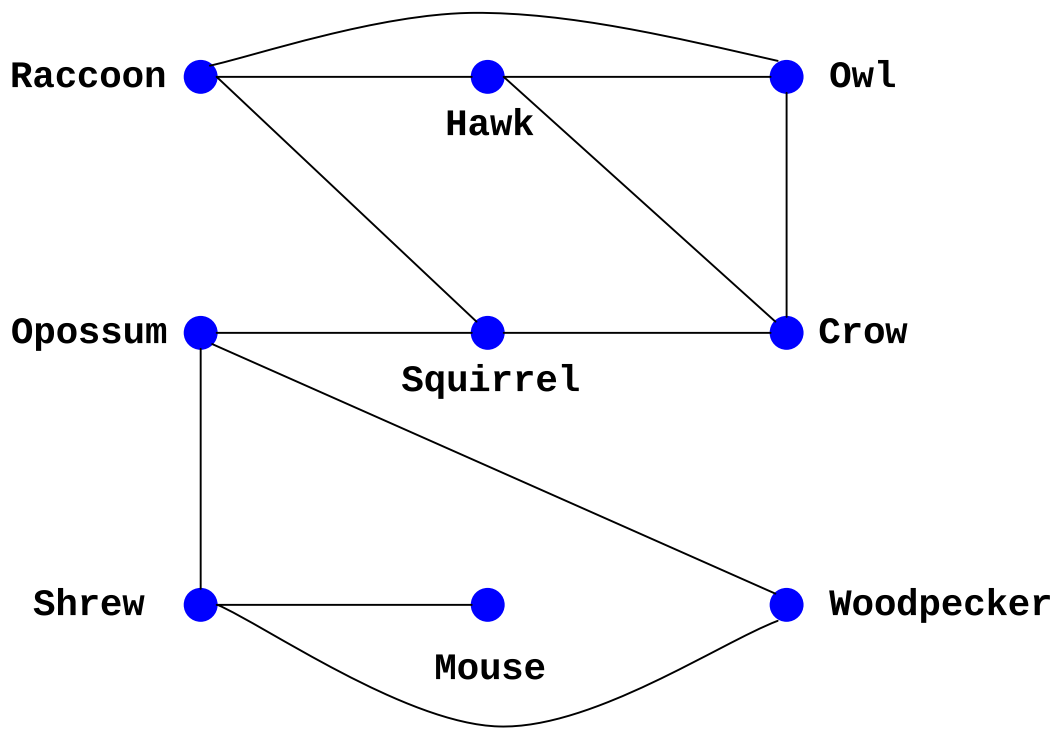
Six Degrees of Separation from Kevin Bacon



Six Degrees of Separation from Kevin Bacon



Niche overlaps



Graph Definition

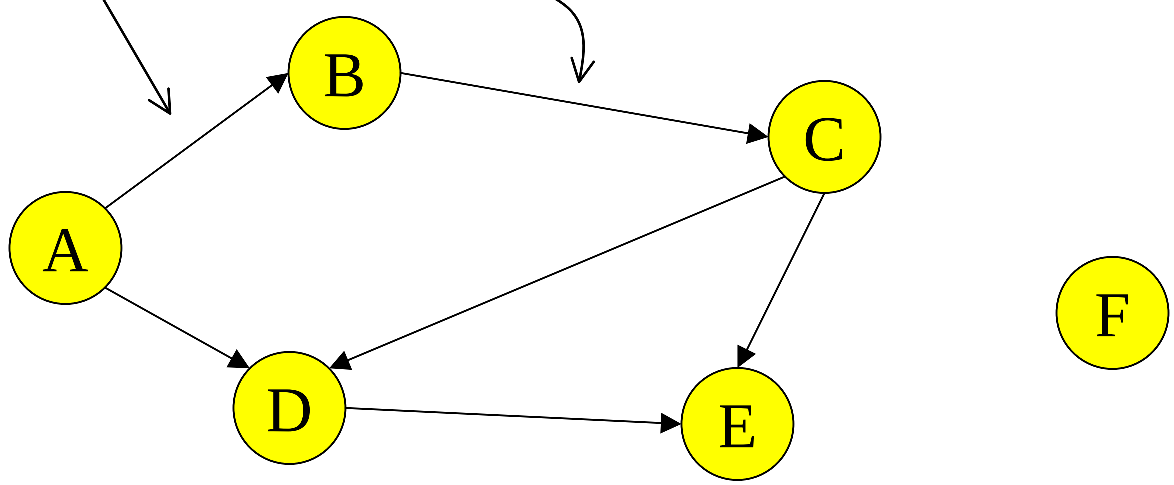
- A graph is simply a collection of nodes plus edges
  - Linked lists, trees, and heaps are all special cases of graphs
- The nodes are known as vertices (node = “vertex”)
- Formal Definition: A graph G is a pair (V, E) where
  - V is a set of vertices or nodes
  - E is a set of edges that connect vertices

Graph Example

- Here is a graph $G = (V, E)$
  - Each edge is a pair (v_1, v_2), where v_1, v_2 are vertices in V

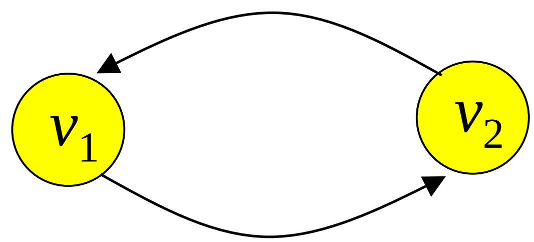
$V = \{A, B, C, D, E, F\}$

$E = \{(A,B), (A,D), (B,C), (C,D), (C,E), (D,E)\}$

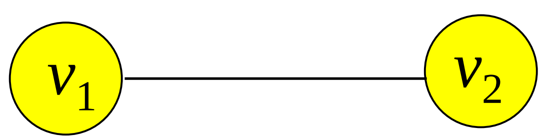


Directed vs Undirected Graphs

- If the order of edge pairs (v_1, v_2) matters, the graph is directed (also called a digraph): $(v_1, v_2) \neq (v_2, v_1)$



- If the order of edge pairs (v_1, v_2) does not matter, the graph is called an undirected graph: in this case, $(v_1, v_2) = (v_2, v_1)$



Undirected Terminology

- Two vertices *u* and *v* are *adjacent* in an undirected graph G if {*u*,*v*} is an edge in G
  - › edge e = {*u*,*v*} is *incident with* vertex *u* and vertex *v*
- The degree of a vertex in an undirected graph is the number of edges incident with it
  - › a loop counts twice (both ends count)
  - › denoted with deg(*v*)

Directed Terminology

- Vertex u is *adjacent to* vertex v in a directed graph G if (u,v) is an edge in G
  - › vertex u is the initial vertex of (u,v)
- Vertex v is *adjacent from* vertex u
  - › vertex v is the terminal (or end) vertex of (u,v)
- Degree
  - › *in-degree* is the number of edges with the vertex as the terminal vertex
  - › *out-degree* is the number of edges with the vertex as the initial vertex
  - › a loop adds 1 to in-degree and 1 to out-degree

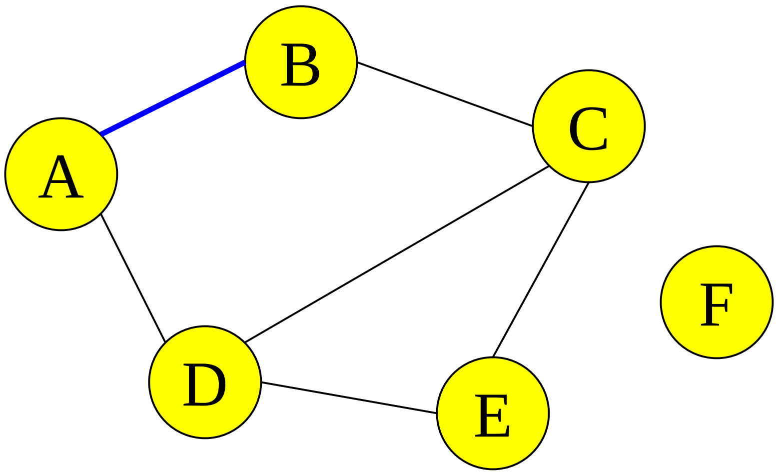
Handshaking Theorem

- Let G=(V,E) be an undirected graph with |E|=e edges
- Then $2e = \sum_{v \in V} \deg(v)$
- Every edge contributes +1 to the degree of each of the two vertices it is incident with
  - › number of edges is exactly half the sum of deg(*v*)
  - › the sum of the deg(*v*) values must be even

Graph Representations

- Space and time are analyzed in terms of:
  - Number of vertices = $|V|$ and
  - Number of edges = $|E|$
- There are two ways of representing graphs:
  - The *adjacency matrix* representation
  - The *adjacency list* representation

Adjacency Matrix

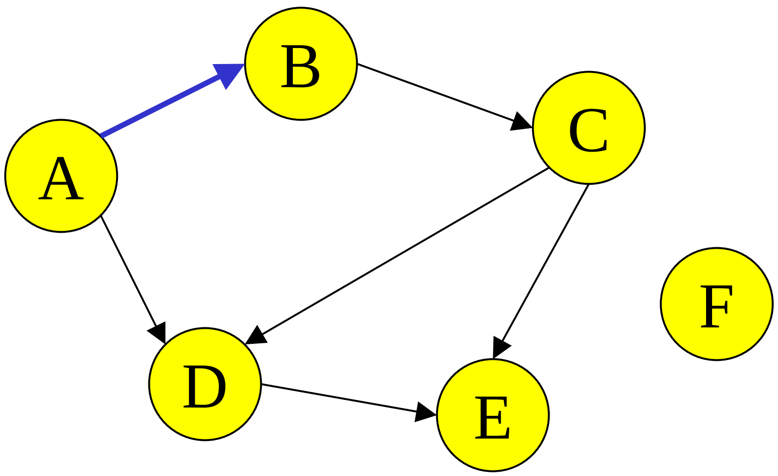


$$
\begin{array}{c|c}
 & \begin{array}{cccccc} A & B & C & D & E & F \end{array} \\
\begin{array}{c} A \\ B \\ C \\ D \\ E \\ F \end{array} &
\begin{pmatrix}
0 & 1 & 0 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 1 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\end{array}
$$

$$
M(v, w) = \begin{cases} 1 \text{ if } (v, w) \text{ is in E} \\ 0 \text{ otherwise} \end{cases}
$$

Space = $|V|^2$

Adjacency Matrix for a Digraph



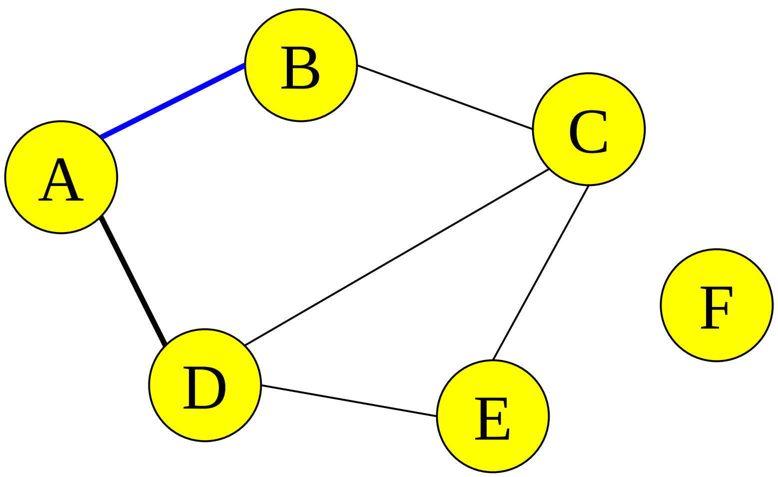
$$\begin{array}{c} \\ A \\ B \\ C \\ D \\ E \\ F \end{array} \begin{array}{c} \begin{array}{cccccc} A & B & C & D & E & F \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

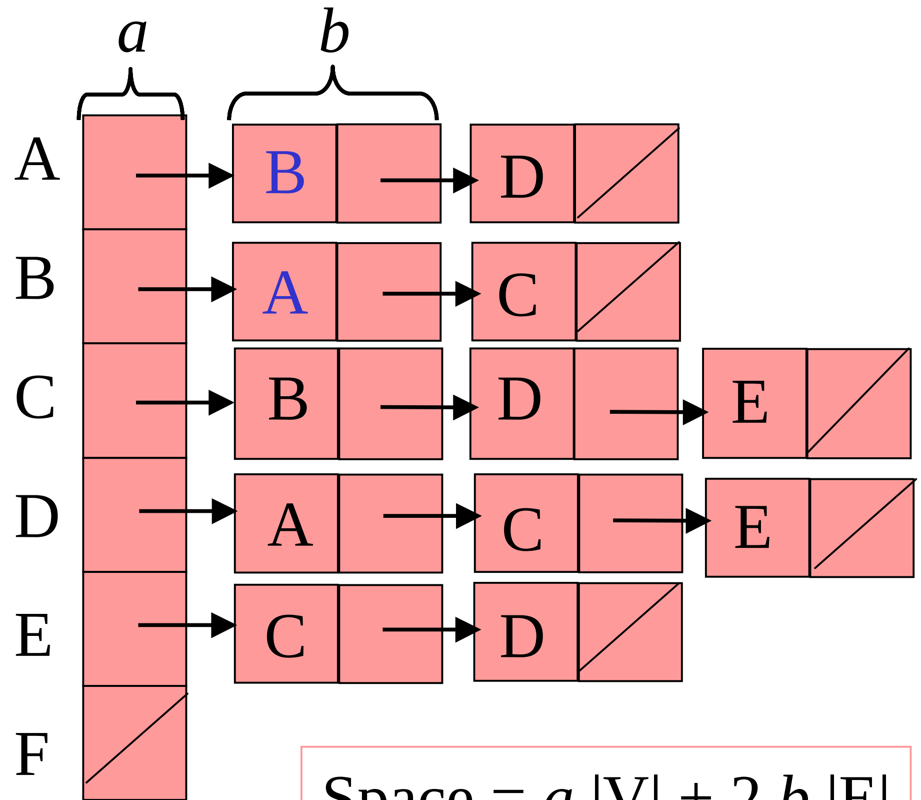
$$M(v, w) = \begin{cases} 1 \text{ if } (v, w) \text{ is in E} \\ 0 \text{ otherwise} \end{cases}$$

Space = $|V|^2$

Adjacency List

For each v in V, $L(v)$ = list of w such that (v, w) is in E

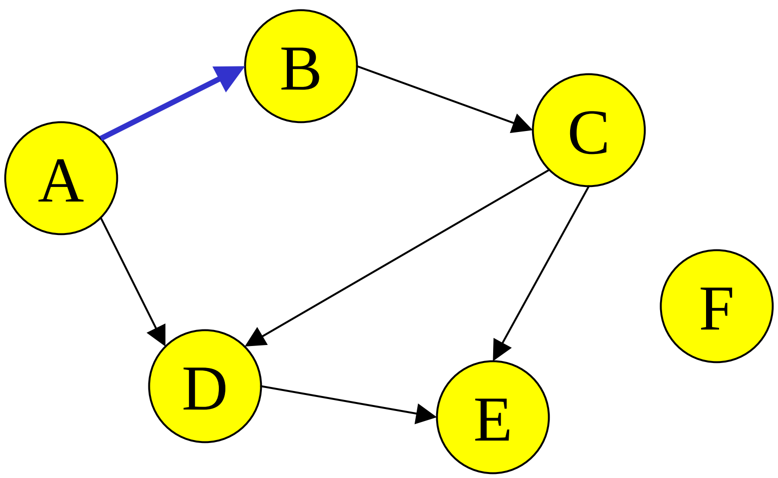


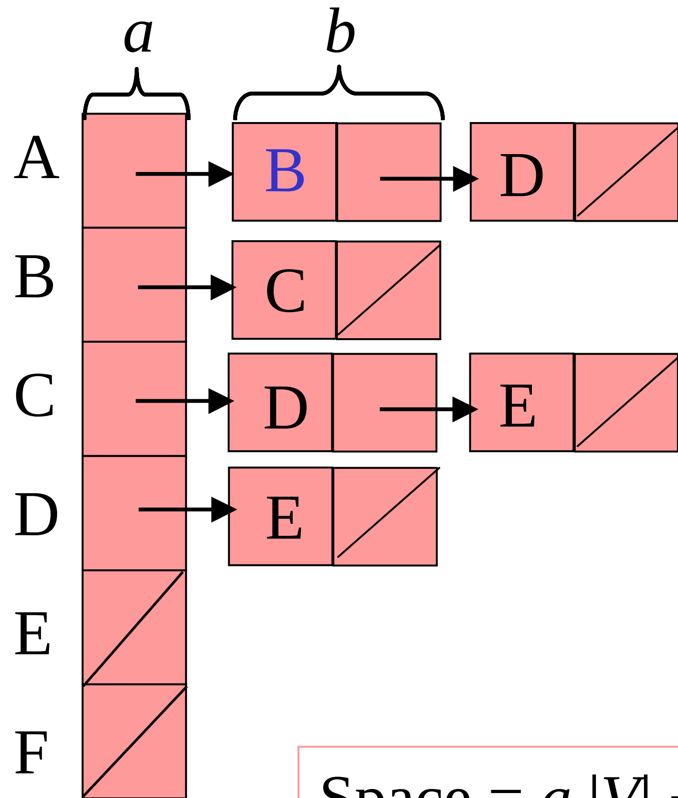


Space = $a\,|V| + 2\,b\,|E|$

Adjacency List for a Digraph

For each *v* in *V*, *L*(*v*) = list of *w* such that (*v*, *w*) is in *E*



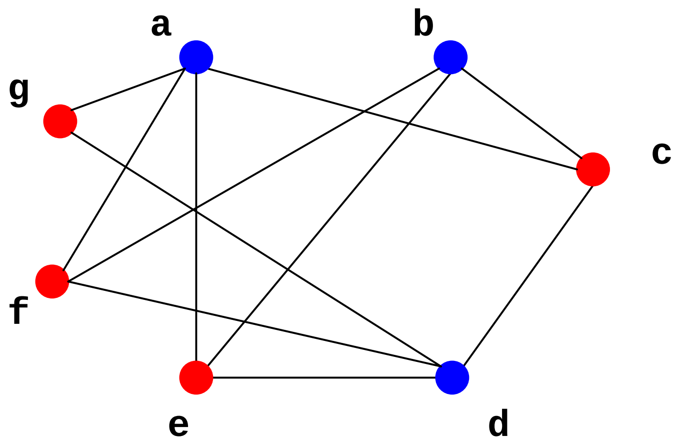


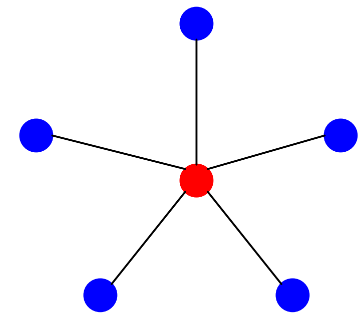
Space = $a\ |V| + b\ |E|$

Bipartite

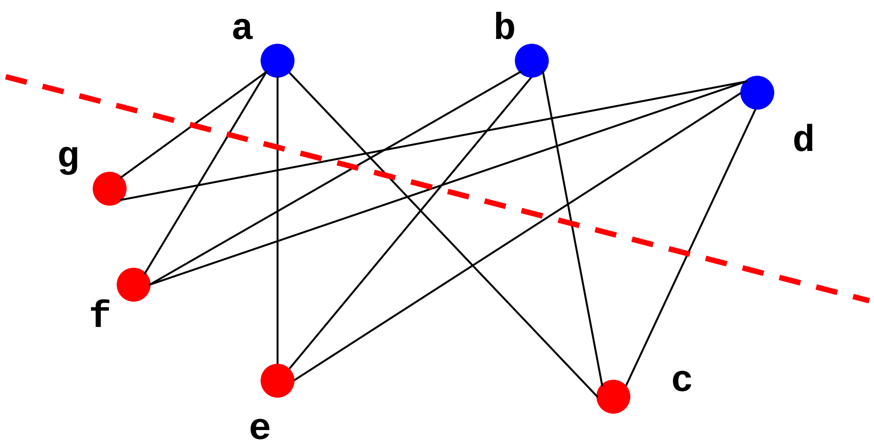
- A simple graph is bipartite if:
  - › its vertex set V can be partitioned into two disjoint non-empty sets such that
    - every edge in the graph connects a vertex in one set to a vertex in the other set
    - which also means that no edge connects a vertex in one set to another vertex in the same set
  - › no triangular connections

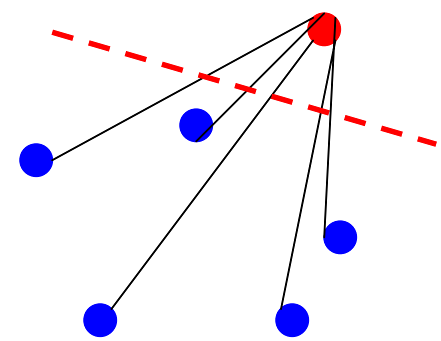
Bipartite examples



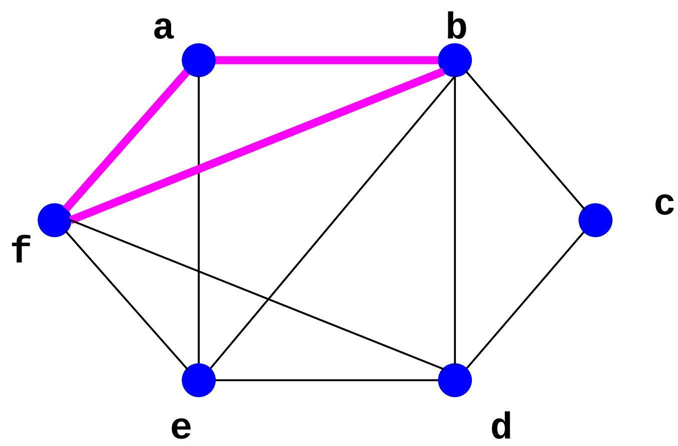


{a b d}
{c e f g}





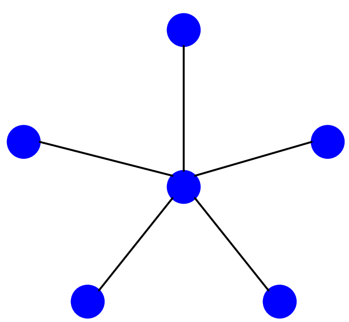
Bipartite example - not



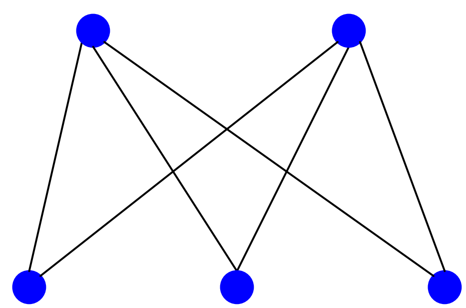
a says that *b* and *f* should be in S_2,
but *b* says *a* and *f* should be in S_1.
TILT!

Complete bipartite graph $K_{m,n}$

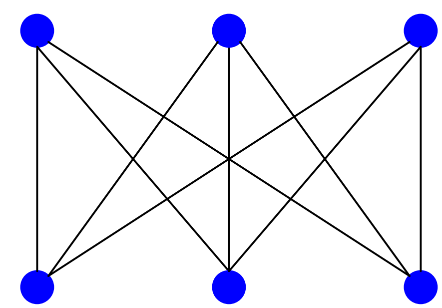
- vertex set partitioned into two subsets of sizes *m* and *n*
- all vertices in one subset are connected to all vertices in the other subset



$K_{1,5}$



$K_{2,3}$



$K_{3,3}$