Disjoint Sets

CSE 373 - Data Structures

May 20, 2002

Readings and References

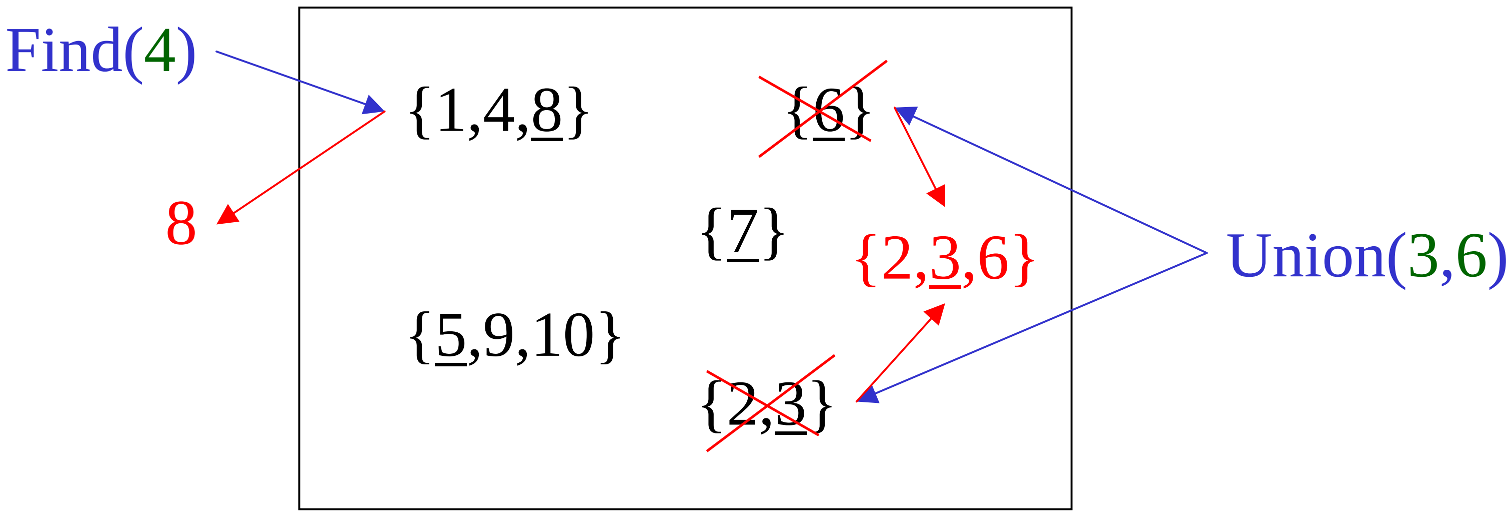
- Reading
  - › Chapter 8, *Data Structures and Algorithm Analysis in C*, Weiss
- Other References

Disjoint Set ADT

- Find: Given an element, return the “name” of its equivalence class
    - note that we are finding the equivalence class, not the element
- Union: Given the “names” of two equivalence classes, merge them into one class
    - › may have a new name or one of the two old names

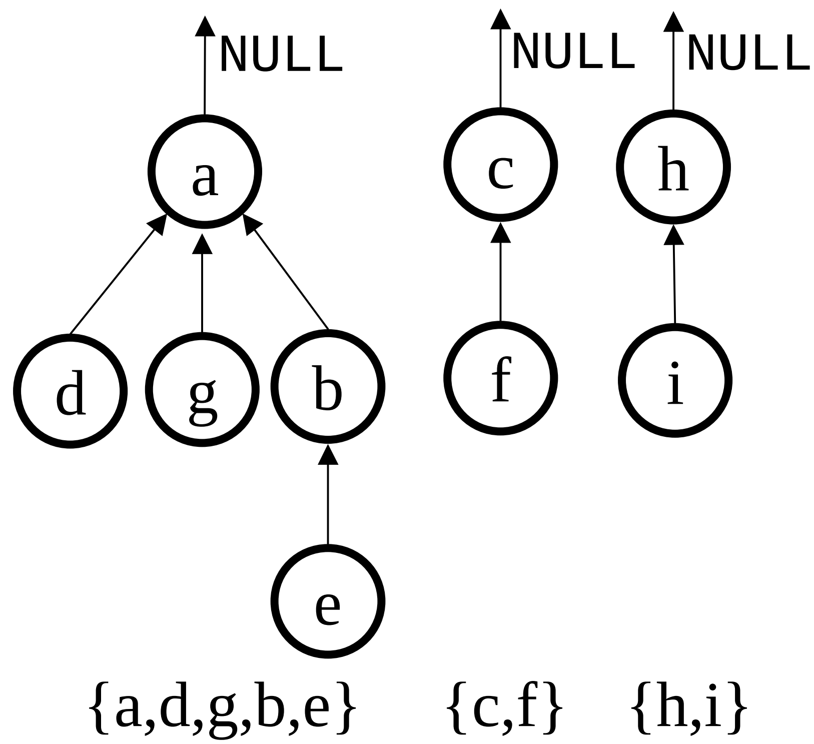
Disjoint Set Example

Equivalence Classes = {1,4,8}, {2,3}, {6}, {7}, {5,9,10}
Name of equivalence class underlined



Up-Tree Virtual Data Structure

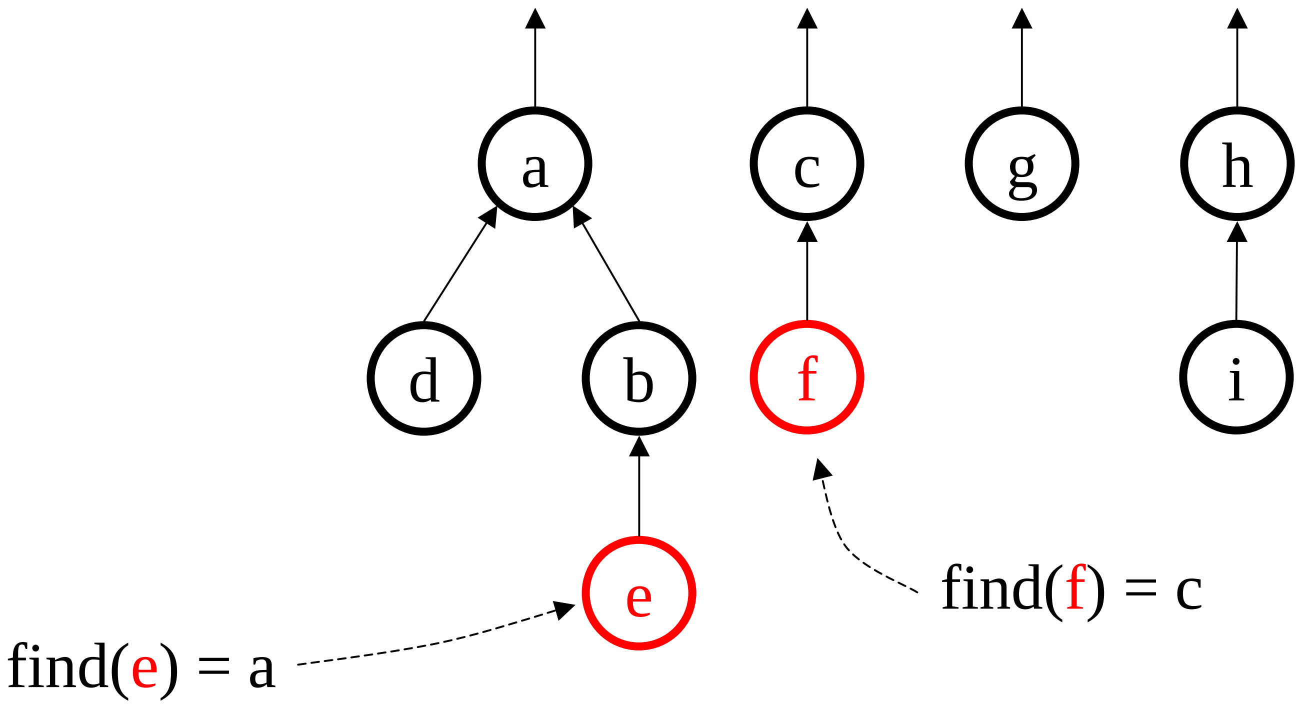
- Each equivalence class (or discrete set) is an up-tree with its root as its representative member
- All members of a given set are nodes in that set's up-tree
- Hash table maps input data to the node associated with that data
  - input string → integer



Up-trees are usually **not** binary!

Example of Find

Find: Just traverse from the node to the root.

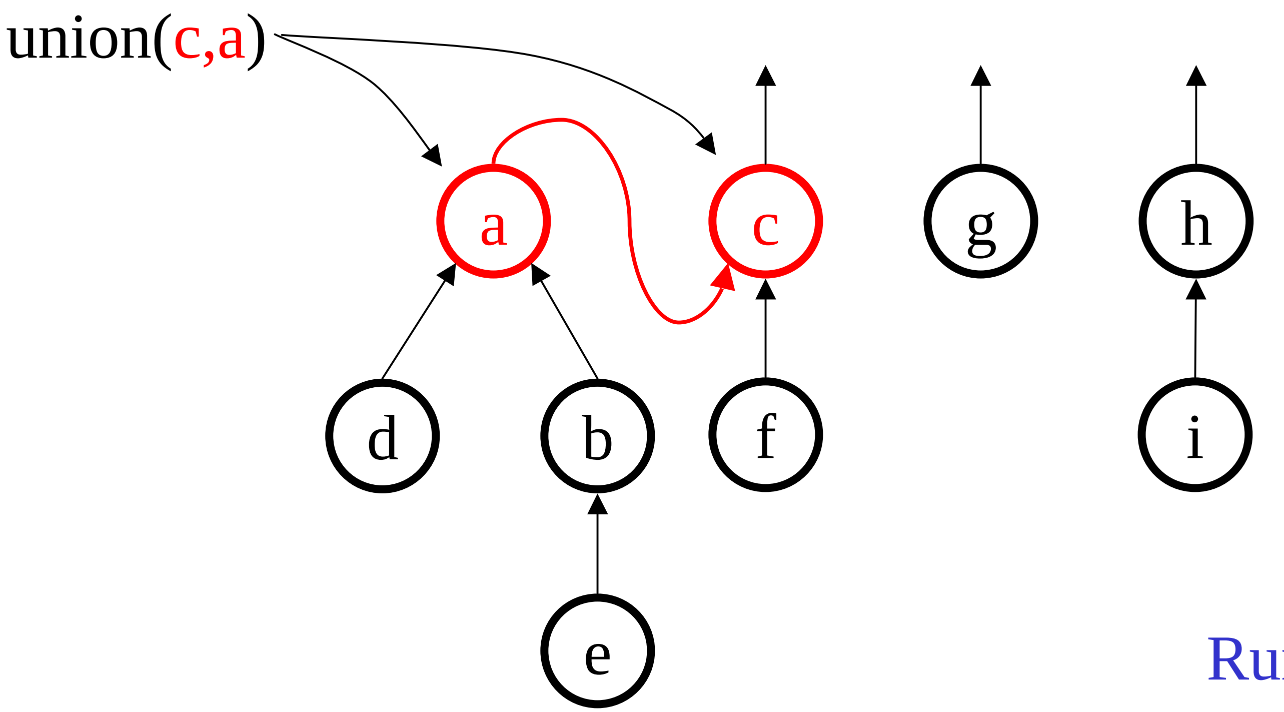


Runtime = ?

Example of Union

Union: Just hang one root from the other.

Now find(f) = c
and find(e) = c

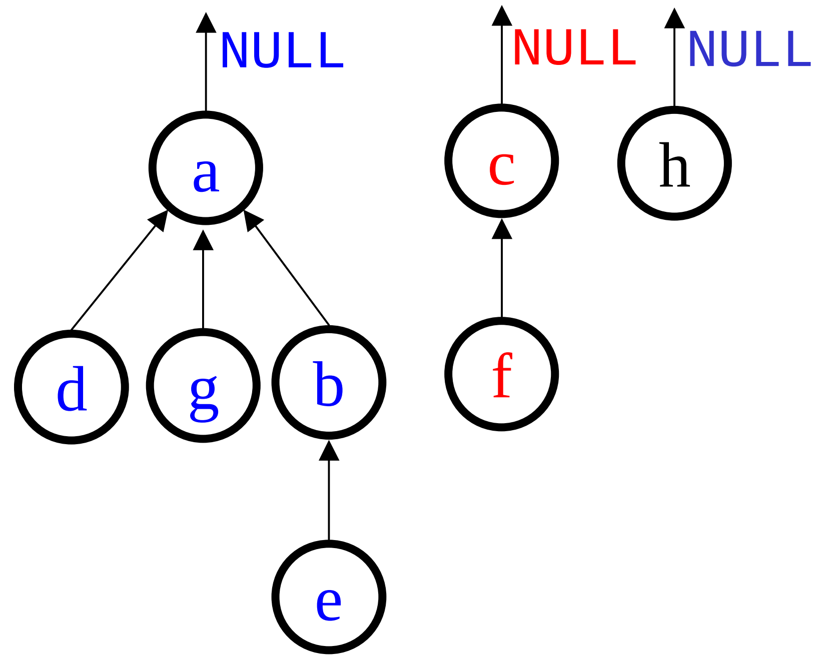


Runtime = ?

An Up-Tree Implementation

- Forest of up-trees can easily be stored in an array “up”
- If node names are pos integers or characters, can use a very simple, perfect hash function: Hash(X) = X
- up[X] = parent of X;

  = 0 if X is a root



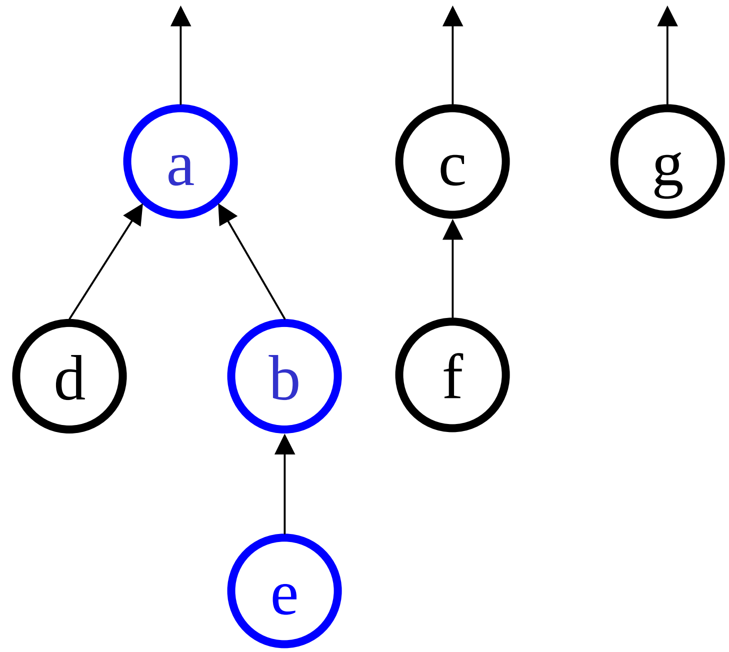
	0	1/a	2/b	3/c	4/d	5/e	6/f	7/g	8/h
Array up:	-	**0**	1	**0**	1	2	3	1	**0**

Example of Find

Traverse to the root

Find(e) = a

Runtime = ?



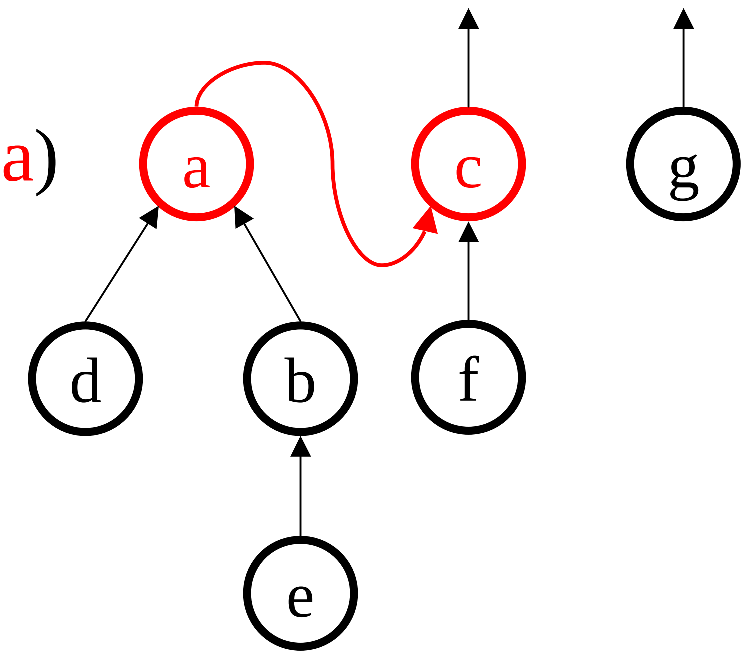
Array up:

0	1/a	2/b	3/c	4/d	5/e	6/f	7/g	8/h
-	0	1	0	1	2	3	1	0

Example of Union

Hang one root from another

Union(c,a)



Runtime = ?

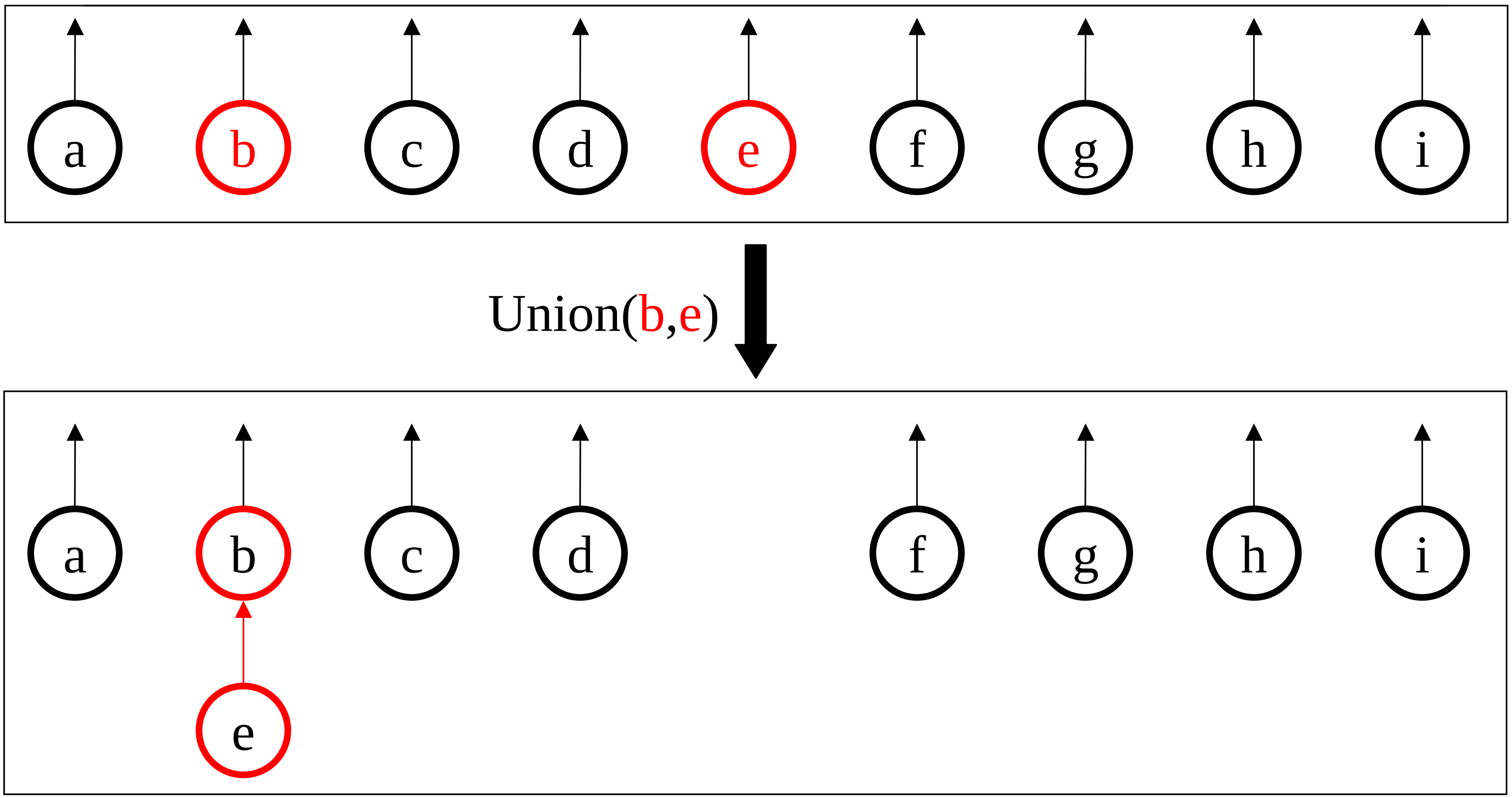
Now:
Find(f) = c
Find(e) = c

Array up:

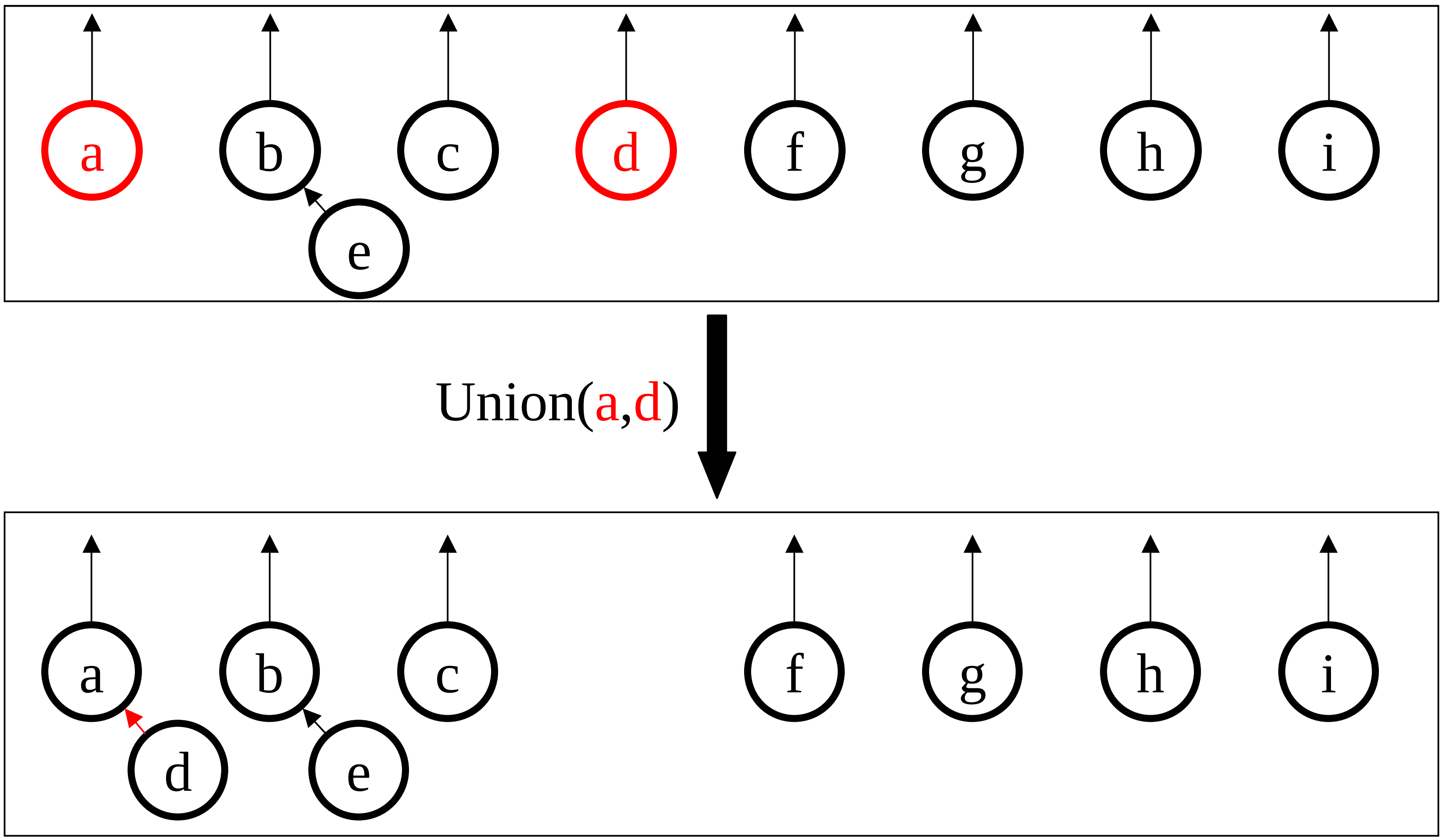
	0	1/a	2/b	3/c	4/d	5/e	6/f	7/g	8/h
Array up:	-	3	1	0	1	2	3	1	0

Change a (from 0) to point to c (= 3)

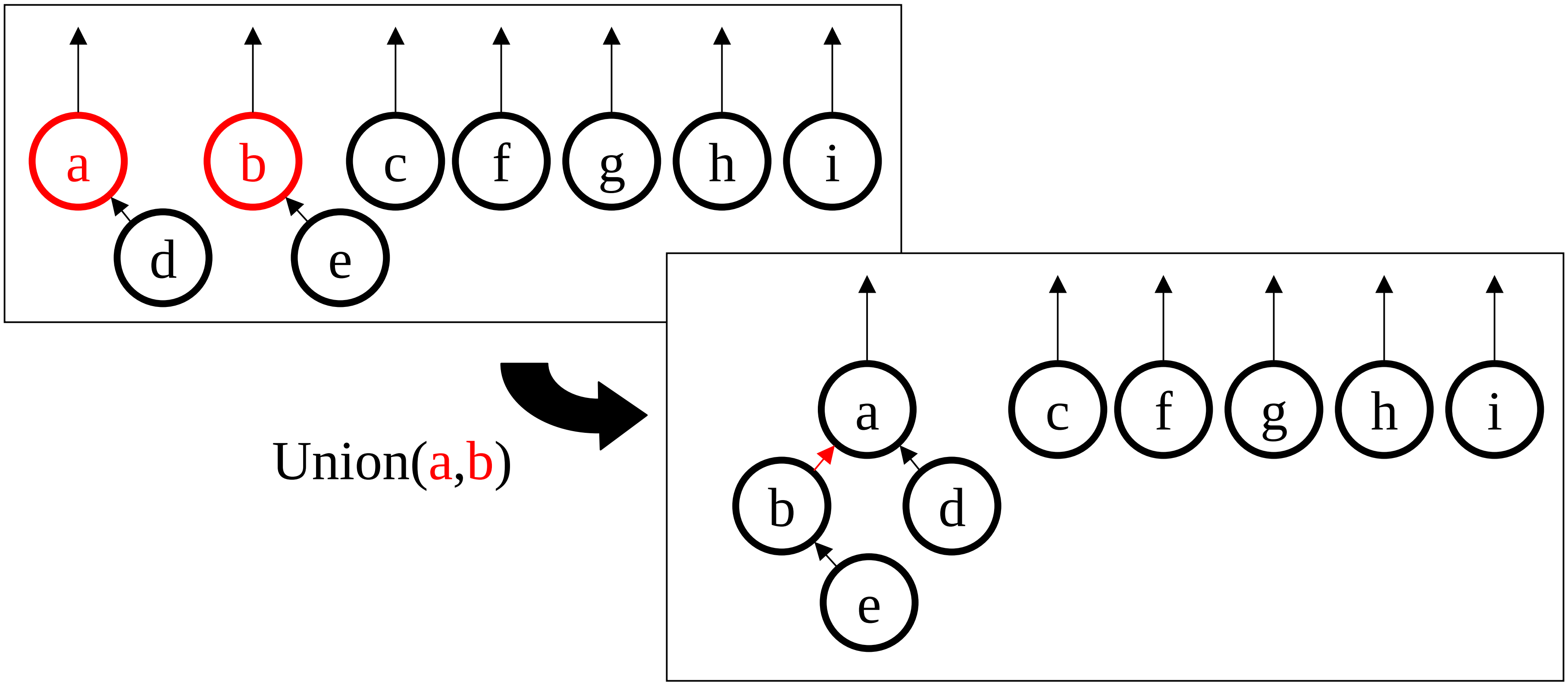
Example: Union(b,e)



Example: Union(a,d)



Example: Union(a,b)

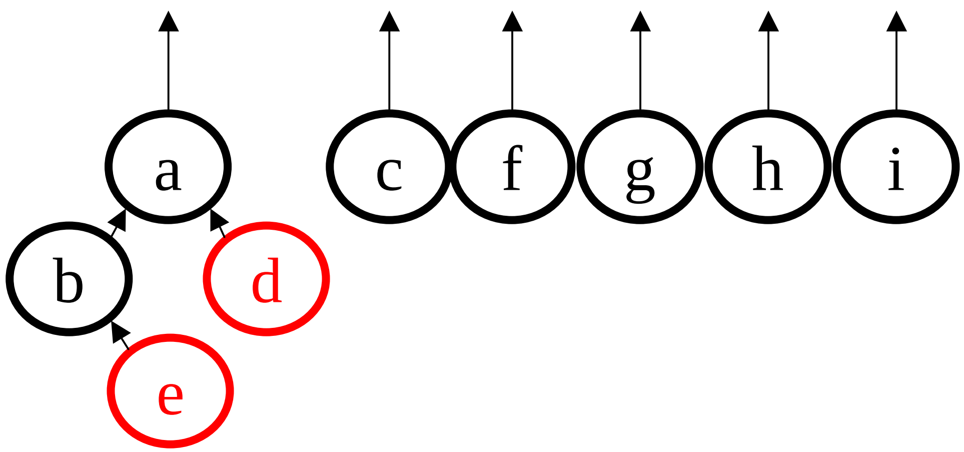


Example: Union(d,e)

Union(d,e) – But (you say) d and e are not roots!

May be allowed in some implementations – do Find first to get roots

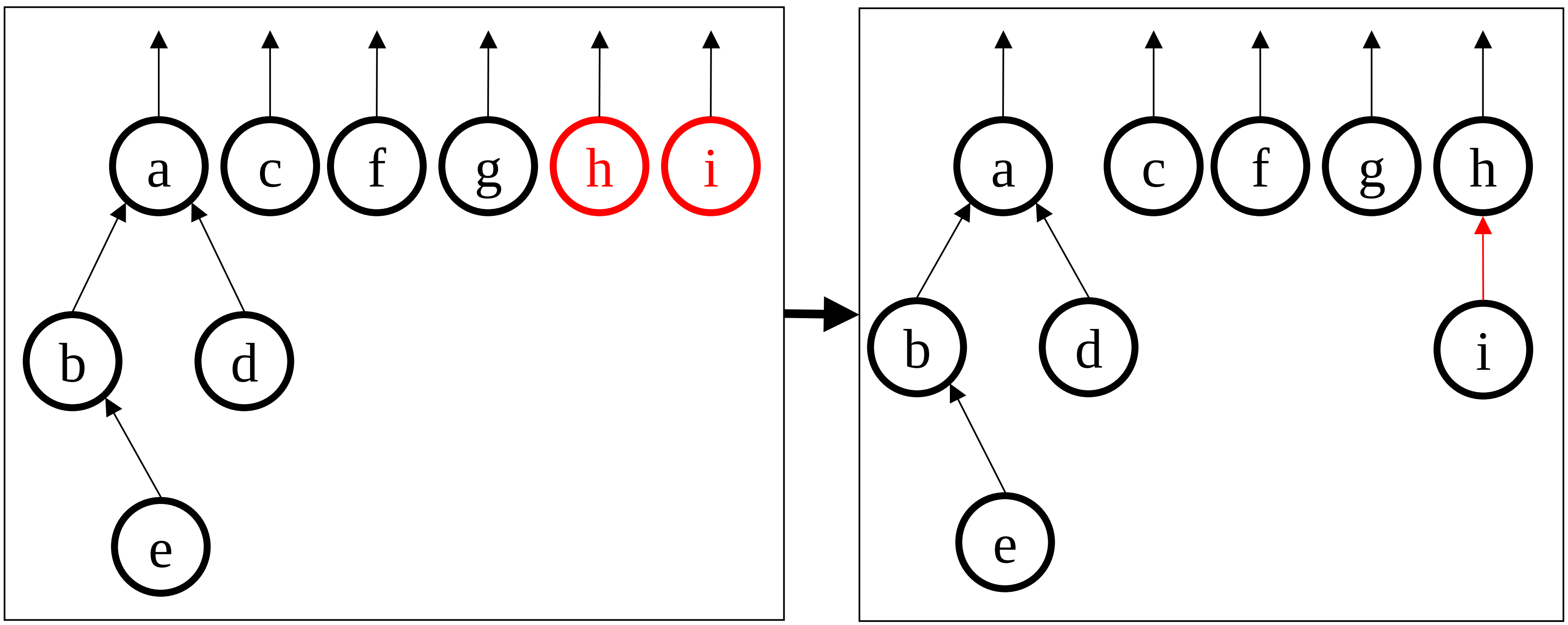
Since Find(d) = Find(e), union already done!



But: while we're finding **e**, could we do something to speed up Find(e) next time? (hold that thought!)

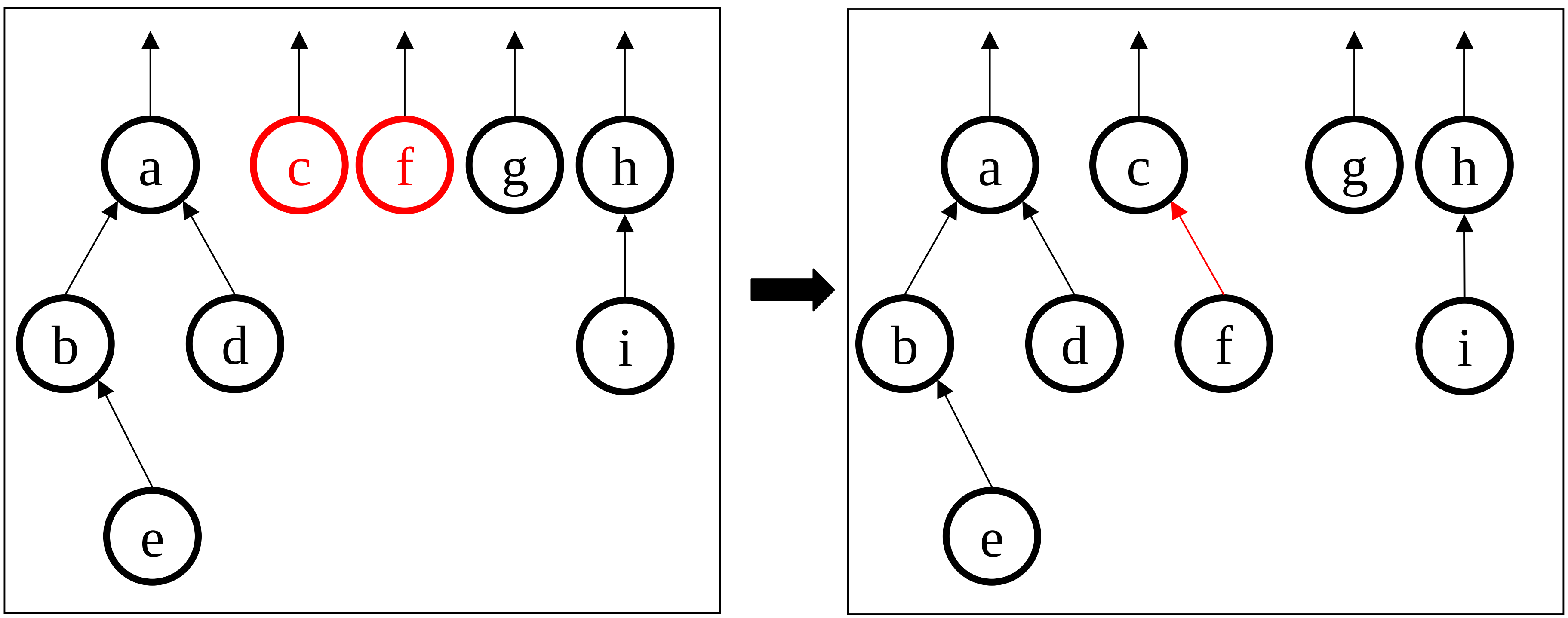
Example: Union(h,i)

Union(h,i)



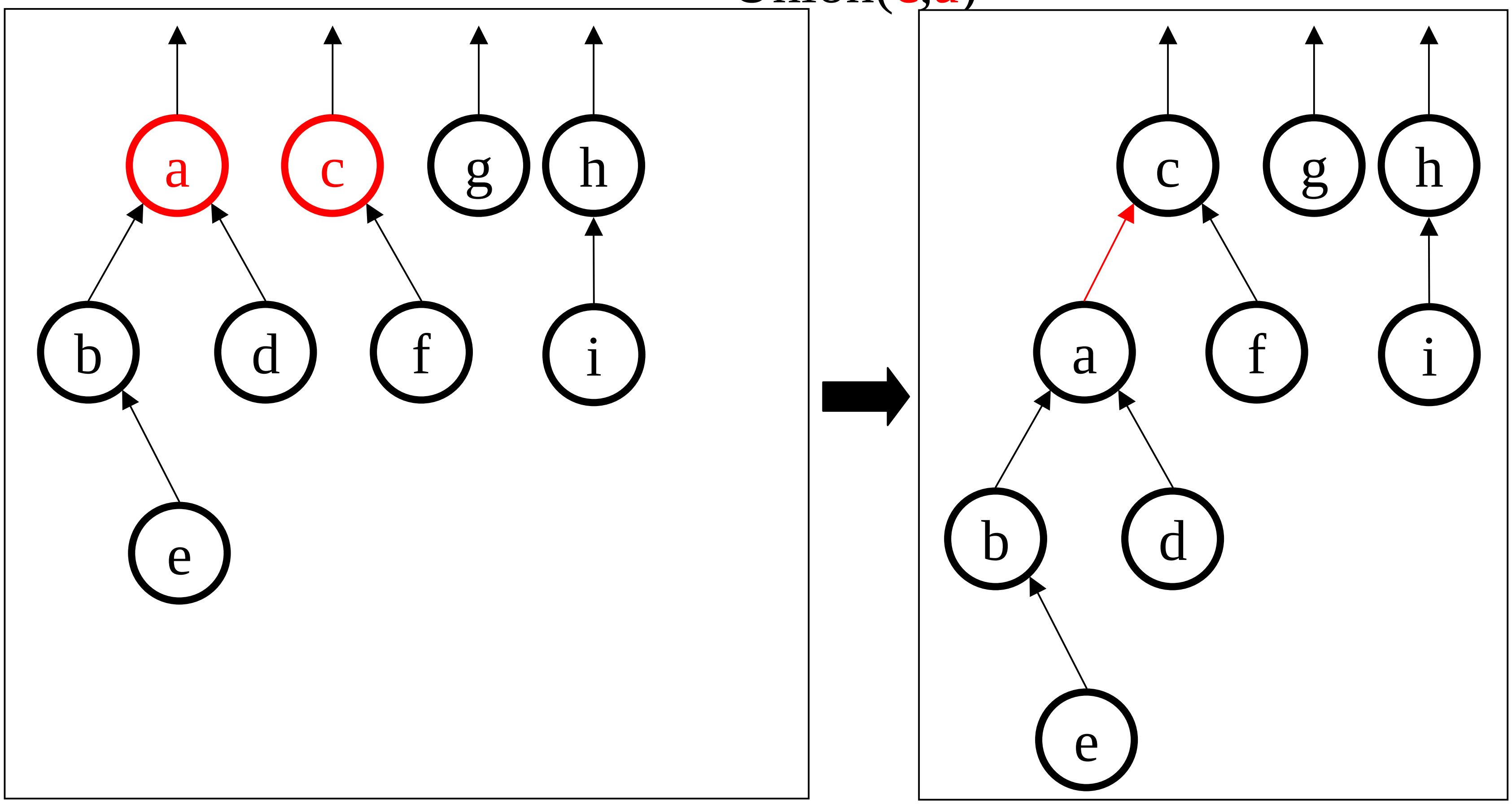
Example: Union(c,f)

Union(c,f)



Example: Union(c,a)

Union(c,a)



An Implementation of Find

```
int Find(int X, DisjSet up) {
// Assumes X = Hash(X_Element)
// X_Element could be str/char etc.
  if (up[X] <= 0)    // Parent is flag value
       return X;     // so X is a root
  else               // else find root recursively
       return Find(up[X],up);
}
```

Runtime of Find: O(max height)

Height of tree depends on the previous Unions that built the particular tree

→Best case: U(1,2), U(1,3), U(1,4) ,…   O(1)

→Worst case: U(2,1), U(3,2), U(4,3),… O(N)

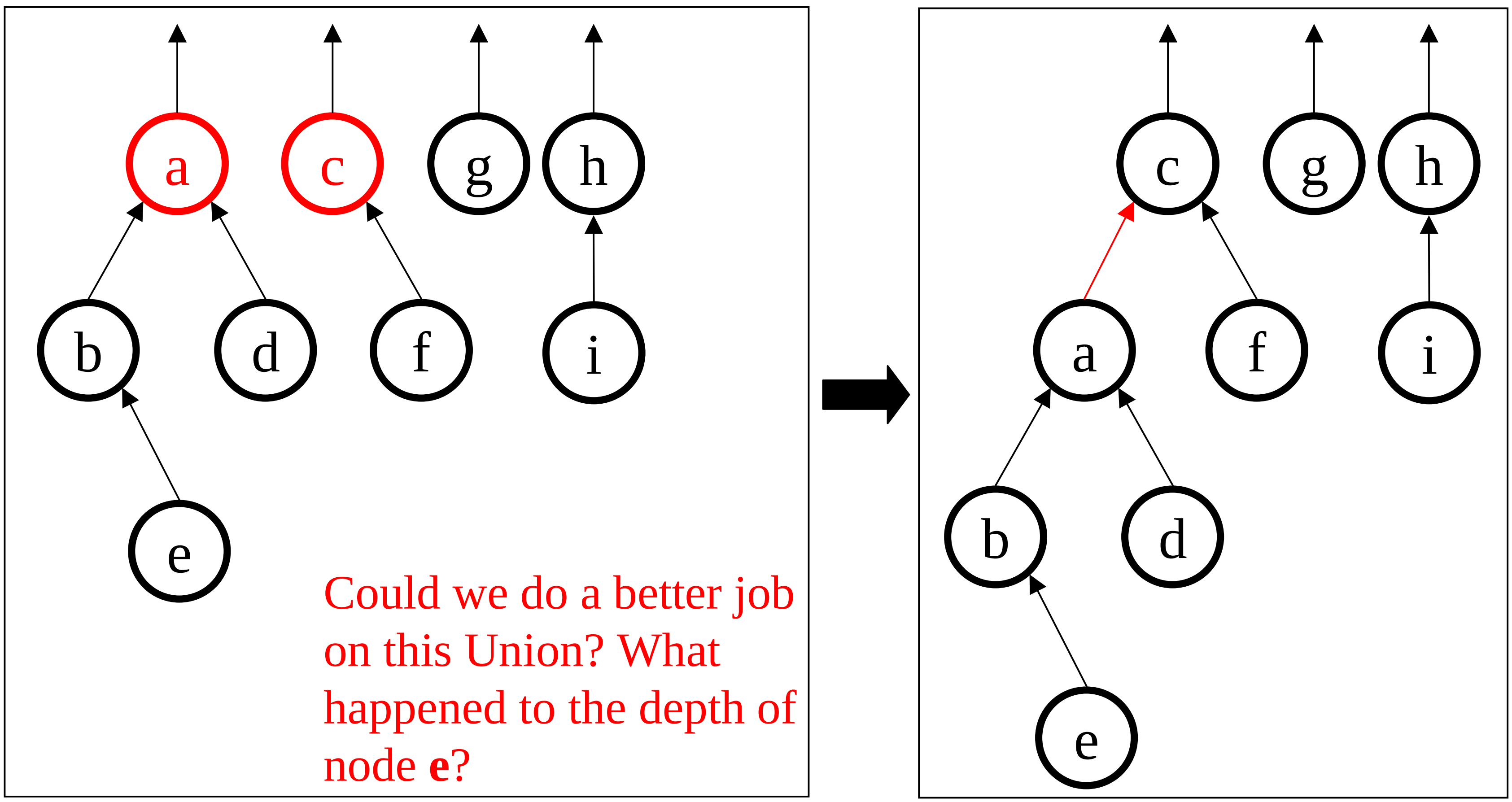
An Implementation of Union

```
void Union(DisjSet up, int X, int Y) {
   //Make sure X, Y are roots
  assert(up[X] == 0);
  assert(up[Y] == 0);
   up[Y] = X;
 }
```

Runtime of Union: O(1)

Issue with Union(c,a)

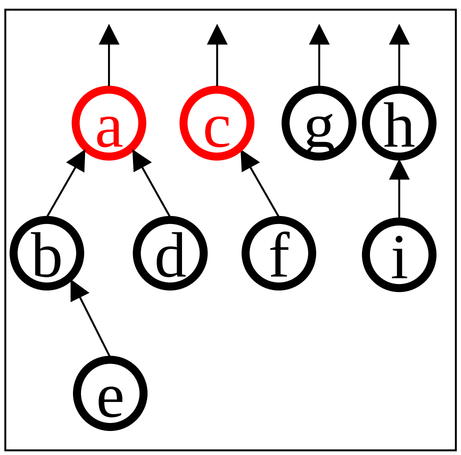
Union(c,a)



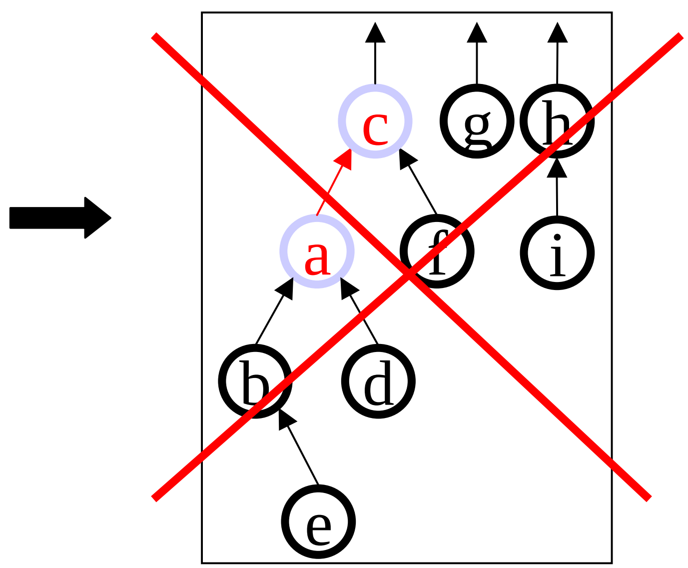
Could we do a better job on this Union? What happened to the depth of node **e**?

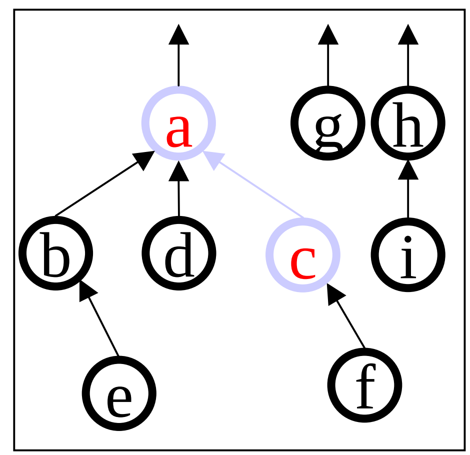
Speeding Up : Union-by-Size

- Can we speed things up by being clever about growing our up-trees?
  - › Always make root of *larger* tree the new root
  - › Why? Minimizes height of the new up-tree



Union(c,a)

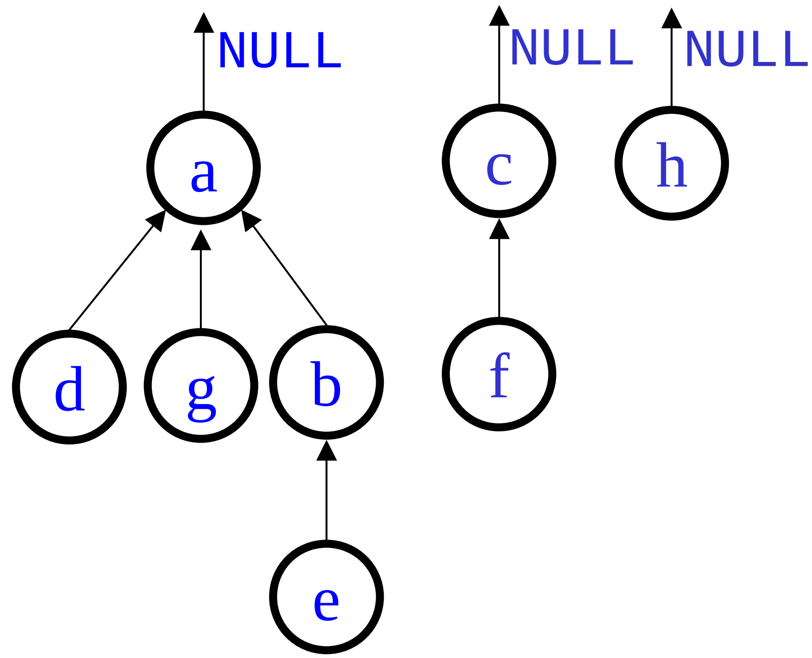




Union-by-Size

Storing Size Information

- Instead of storing 0 in root, store up-tree size as negative value in root node



Array up:

0	1/a	2/b	3/c	4/d	5/e	6/f	7/g	8/h
-	-5	1	-2	1	2	3	1	-1

Union-by-Size Code

```
void Union(DisjSet up, int X, int Y) {
   //X, Y are roots containing (-size) of up-trees
  assert(up[X] < 0);
  assert(up[Y] < 0);

  if (-up[X] > -up[Y]) {// X is bigger than Y
      up[X] += up[Y];   // so X is new root
      up[Y] = X;        // and Y points to X
  }
  else {                // size of X ≤ size of Y
      up[Y] += up[X];   // so Y is new root
      up[X] = Y;        // and X points to Y
  }
}
```

Union-by-Size: Analysis

- Finds are O(max up-tree height) for a forest of up-trees containing N nodes
- Number of nodes in an up-tree of height h using union-by-size is $\geq 2^h$

Base case: $h = 0$, tree has $2^0 = 1$ node

Induction hypothesis: Assume true for $h < h'$

Induction Step: New tree of height h' was formed via union of two trees of height h'-1
Each tree then has $\geq 2^{h'-1}$ nodes by the induction hypothesis

So, total nodes $\geq 2^{h'-1} + 2^{h'-1} = 2^{h'}$

→ True for all h

- Pick up-tree with max height
- Then, $2^{\text{max height}} \leq N$
- max height $\leq \log N$
- Find takes **O(log N)**

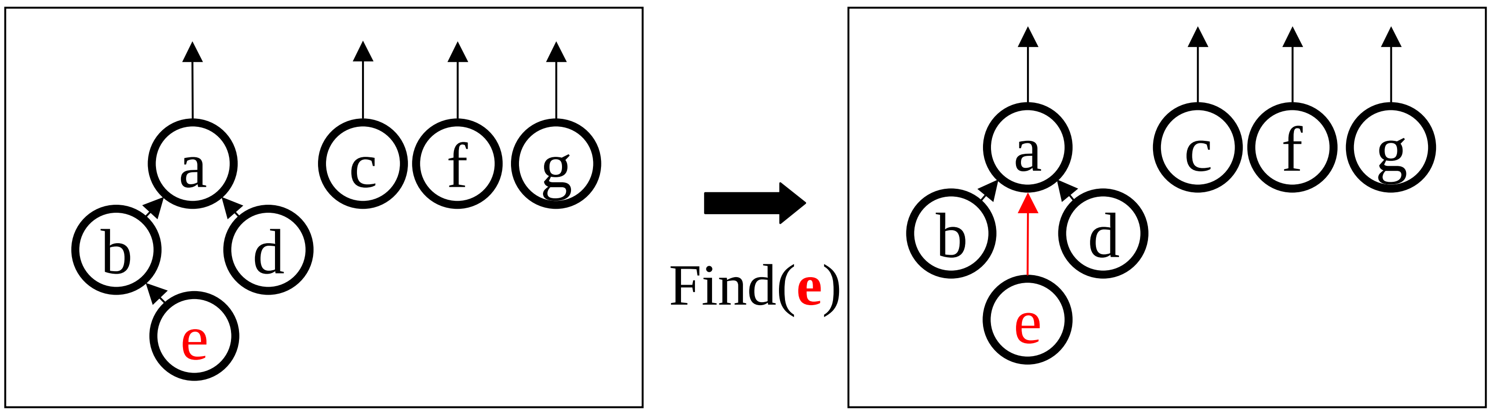
Union-by-Height

- Textbook describes alternative strategy of Union-by-height
  - › Keep track of height of each up-tree in the root nodes
  - › Union makes root of up-tree with greater height the new root
- Same results and similar implementation as Union-by-Size
  - › Find is O(log N) and Union is O(1)

Find and Path Compression

- M Finds on same element take O(M log N) time
  - Can we modify Find to have *side-effects* so that next Find will be faster?
- Path Compression
  - When we do a Find, we follow a path in the tree from the given element X all the way up to the root
  - The tree does not have to be a binary tree
  - So we can reroot the nodes on the path so that they are all direct children of the root of their tree

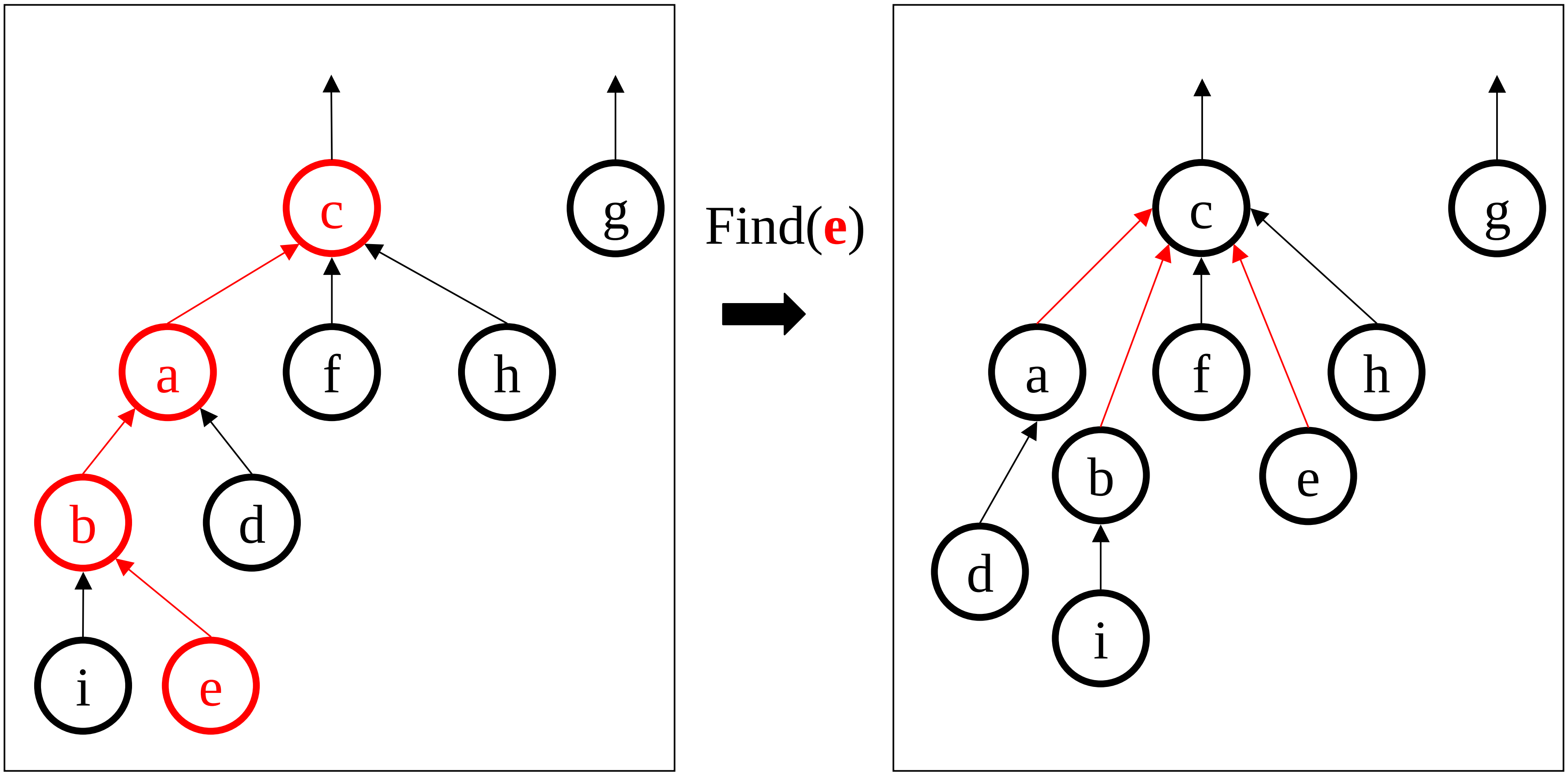
Example: Path Compression



Path compression! The next Find(e) will run faster.

Remember splay trees? Similar idea … self adjust to improve future performance based on actual usage.

Another Path Compression Example



Path Compression Code

```
int Find(int X, DisjSet up) {
// Assumes X = Hash(X_Element)
// X_Element could be str/char etc.
  if (up[X] <= 0)   // Parent is flag value
       return X;    // so X is root
  else              // else find root recursively
       return up[X] = Find(up[X],up);
}
```

Make all nodes along access path point to root

New running time of Find?

- Find still takes O(max up-tree height) worst case
- But what happens to the tree heights over time?
  - › we are collapsing the tree by having each node point to its root
- What is the *amortized* run time of Find if we do M Finds?

```
int Find(int X, DisjSet up) {
// Assumes X = Hash(X_Element)
// X_Element could be str/char etc.
  if (up[X] <= 0)      // Parent is flag value
       return X;       // so X is root
  else                 // else find root recursively
       return up[X] = Find(up[X],up);
}
```

Find Run Time Analysis

- What is the *amortized* run time of Find if we do M Finds?
  - › (one or more) operations that take O(max height)
  - › M-(one or more) operations that take O(1) constant time
  - › amortized total cost is O(1) constant time

Slow-growing functions

- How fast does log N grow? log N = 4 for N = 16 = 2^4
  - › Grows *quite* slowly
- Let $\log^{(k)}$ N = log (log (log … (log N))) (*k* logs)
- Let log* N = minimum *k* such that $\log^{(k)}$ N $\le$ 1
- How fast does $\log^*$N grow? $\log^*$N = 4 for N = 65536 = $2^{2^{2^2}}$
  - › Grows *very* slowly
- Ackermann created a <u>really</u> explosive function A(i, j) and its inverse α(M, N)
- How fast does α(M, N) grow? α(M, N) = 4 for M ($\ge$ N) **far** larger than the number of atoms in the universe (2^{300})!!
  - › grows *very*, *very* slowly (slower than $\log^*$ N)

Find and Union Run Time Analysis

- When both path compression and Union-by-Size are used, the worst case run time for a sequence of M operations (Unions or Finds)
  - › Textbook proves $O(M \log^* N)$ time
  - › R. E. Tarjan showed $\Theta(M\ \alpha(M,N))$
    - $\alpha(M, N) \leq 4$ for all practical choices of M and N
- Amortized run time per operation
  - › = total time/(# operations)
  - › $= \Theta(M\ \alpha(M,N))/M = \Theta(\alpha(M,N))$
  - › for all practical purposes: O(1) constant time

Disjoint Set and Union/Find

- Disjoint Set data structure arises in many applications where objects of interest fall into different equivalence classes or sets
  - › Cities on a map, electrical components on chip, computers in a network, people related to each other by blood, etc.
- Two main operations: Union of two classes and Find class name for a given element

Disjoint Set and Union/Find

- Up-Tree data structure allows efficient array implementation
  - › Unions take O(1) worst case time, Finds can take O(N)
  - › Union-by-Size reduces worst case time for Find to O(log N)
  - › Union-by-Size plus Path Compression allows further speedup
    - Any sequence of M Union/Find operations results in O(1) amortized time per operation (for all practical purposes)