Shell Sort

CSE 373 - Data Structures

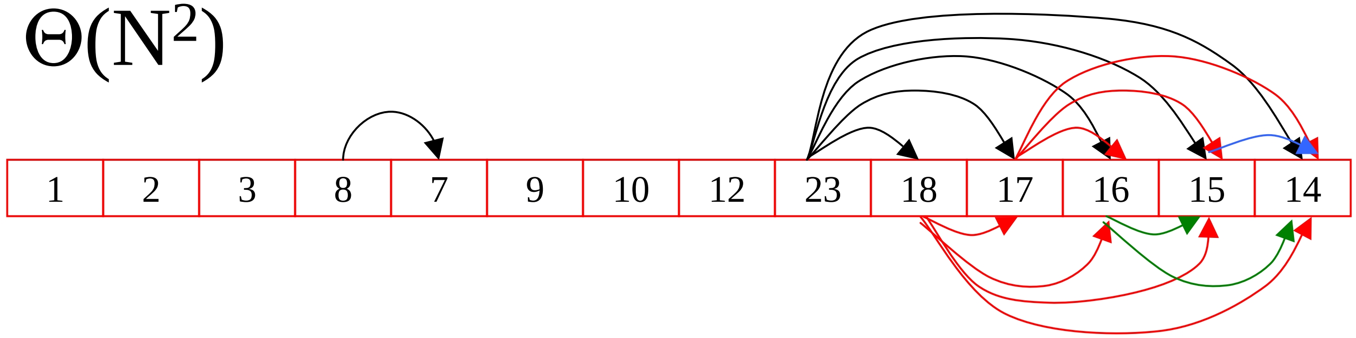
May 8, 2002

Readings and References

- Reading
  - › Sections 7.4, *Data Structures and Algorithm Analysis in C*, Weiss
- Other References

Swapping adjacent elements

- An "average" list will contain half the max number of inversions = $\frac{(n-1)(n)}{4}$
  - › So the average running time of Insertion sort is $\Theta(N^2)$



- <u>Any</u> sorting algorithm that only swaps *adjacent elements* requires $\Omega(N^2)$ time because each swap removes only one inversion

The search for speed

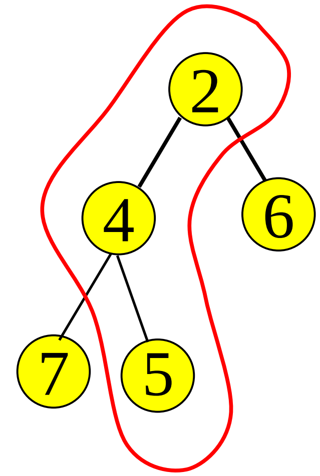
- If we are going to do better than $O(N^2)$, we are going to have to fix more than one inversion at a time
- How can we fix more than one inversion?
  - › Move the elements further with each swap

Shellsort: Better than Quadratic

- Named after Donald Shell – inventor of the first algorithm to achieve $o(N^2)$
  - Running time is $O(N^x)$ where x = 3/2, 5/4, 4/3, …, or 2 depending on "increment sequence"
- Shell sort uses repeated insertion sorts on selected subarrays of the larger array being sorted
- Multiple passes with changing subarrays

Subarrays (or subsequences)

- Remember that in the discussion of binary heaps I showed how we could sort a *path* through the tree



- Similarly, we can sort a *subarray* contained in a larger array

1	2	3	8	7	9	10	12	23	18	15	16	17	14
0	1	2	3	4	5	6	7	8	9	10	11	12	13

Subarrays: increment = 5

subarray 1

1	2	3	8	7	**9**	10	12	23	18	**15**	16	17	14
0	1	2	3	4	5	6	7	8	9	10	11	12	13

subarray 2

1	**2**	3	8	7	9	**10**	12	23	18	15	**16**	17	14
0	1	2	3	4	5	6	7	8	9	10	11	12	13

subarray 3

1	2	**3**	8	7	9	10	**12**	23	18	15	16	**17**	14
0	1	2	3	4	5	6	7	8	9	10	11	12	13

subarray 4

1	2	3	**8**	7	9	10	12	**23**	18	15	16	17	**14**
0	1	2	3	4	5	6	7	8	9	10	11	12	13

subarray 5

1	2	3	8	**7**	9	10	12	23	**18**	15	16	17	14
0	1	2	3	4	5	6	7	8	9	10	11	12	13

Subarrays: increment = 2

subarray 1

1	2	3	8	7	9	10	12	23	18	15	16	17	14
0	1	2	3	4	5	6	7	8	9	10	11	12	13

subarray 2

1	2	3	8	7	9	10	12	23	18	15	16	17	14
0	1	2	3	4	5	6	7	8	9	10	11	12	13

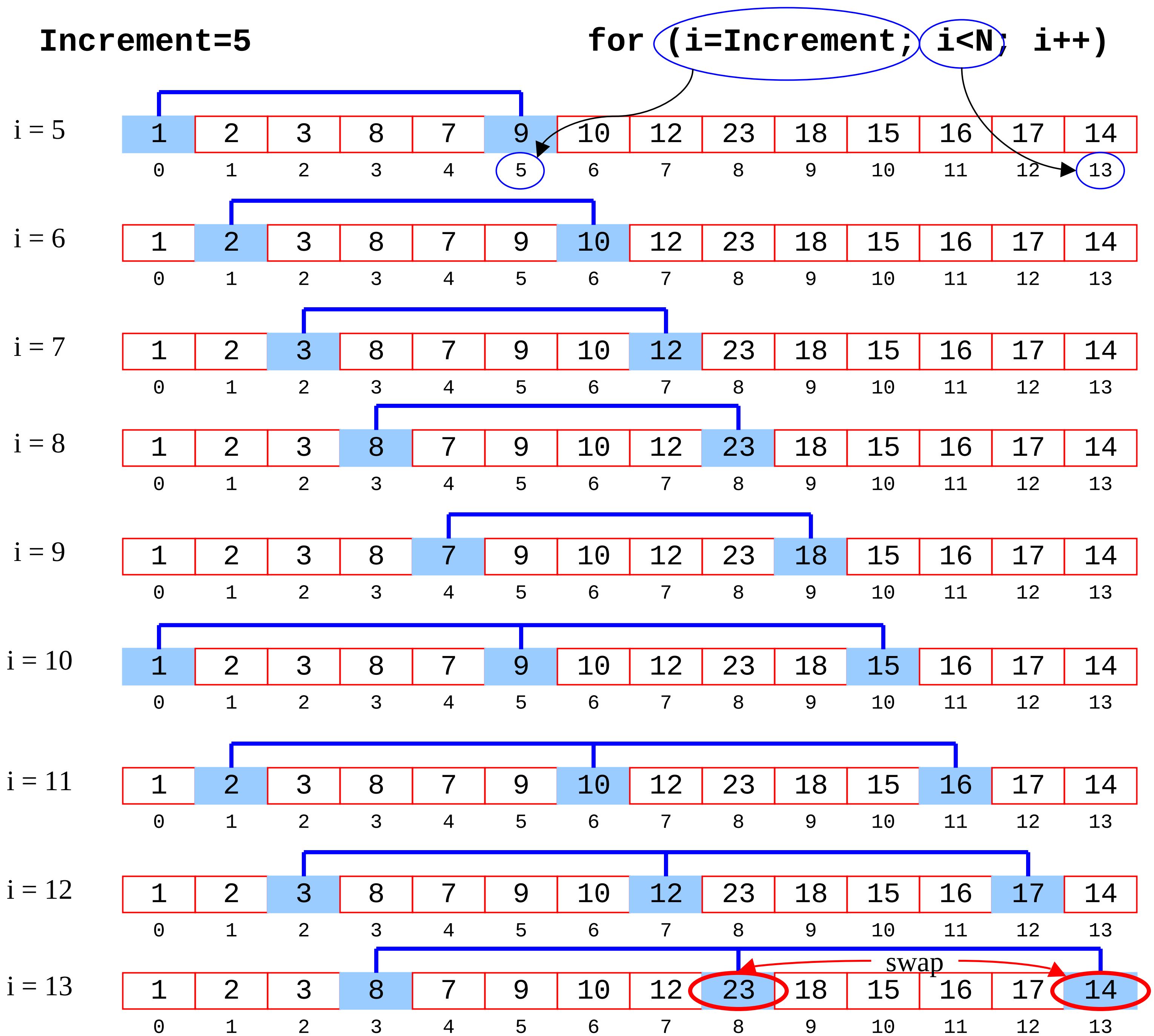
Shell Sort: diminishing increments

- Uses an *increment sequence* $h_1 < h_2 < \ldots < h_t$
  - › Start sorting with the largest increment h_t
  - › Sort all subarrays of elements that are h_k apart so that $A[i] \le A[i+h_k]$ for all i → known as an *h_k-sort*
  - › Go to next smaller increment h_{k-1} and repeat
- Stop sorting after h_1 (=1)
- Choice of increments is important
  - › and hard to analyze

Shellsort

```
void Shellsort( ElementType A[ ], int N ){
  int i, j, Increment; ElementType Tmp;
  for( Increment = N/2; Increment > 0; Increment /= 2 )
    for( i = Increment; i < N; i++ )                {
      Tmp = A[ i ];
      for( j = i; j >= Increment; j -= Increment )
             if( Tmp < A[ j - Increment ] )
                    A[ j ] = A[ j - Increment ];
             else
                    break;
      A[ j ] = Tmp;
    }
}
```

Note: the actual sorting is done by insertion sort: "copy down and insert the value in the right place" on each subarray in the innermost loop and A[j]=Tmp



Increment=5
for (i=Increment; i<N; i++)
i = 5
1 2 3 8 7 9 10 12 23 18 15 16 17 14
0 1 2 3 4 5 6 7 8 9 10 11 12 13
i = 6
1 2 3 8 7 9 10 12 23 18 15 16 17 14
0 1 2 3 4 5 6 7 8 9 10 11 12 13
i = 7
1 2 3 8 7 9 10 12 23 18 15 16 17 14
0 1 2 3 4 5 6 7 8 9 10 11 12 13
i = 8
1 2 3 8 7 9 10 12 23 18 15 16 17 14
0 1 2 3 4 5 6 7 8 9 10 11 12 13
i = 9
1 2 3 8 7 9 10 12 23 18 15 16 17 14
0 1 2 3 4 5 6 7 8 9 10 11 12 13
i = 10
1 2 3 8 7 9 10 12 23 18 15 16 17 14
0 1 2 3 4 5 6 7 8 9 10 11 12 13
i = 11
1 2 3 8 7 9 10 12 23 18 15 16 17 14
0 1 2 3 4 5 6 7 8 9 10 11 12 13
i = 12
1 2 3 8 7 9 10 12 23 18 15 16 17 14
0 1 2 3 4 5 6 7 8 9 10 11 12 13
i = 13
swap
1 2 3 8 7 9 10 12 23 18 15 16 17 14
0 1 2 3 4 5 6 7 8 9 10 11 12 13

Shellsort: Basic Insight

- Insertion sort runs fast on nearly sorted sequences
  - immediate termination when proper spot is found
- do *several passes of Insertion sort* on different subsequences of elements
- note that the subsequences stay sorted from pass to pass

Example

- Sort 19, 5, 2, 1 with increment sequence 1,2
  - Insertion sort on subsequences of elements spaced apart by 2: 1st and 3rd, 2nd and 4th
    - 19, *5*, 2, *1* → 2, *1*, 19, *5*
  - Do Insertion sort on subsequence of elements spaced apart by 1:
    - 2, 1, 19, 5 → 1, 2, 19, 5 → 1, 2, 19, 5 → 1, 2, 5, 19
- Fewer shifts than plain Insertion sort
  - 4 versus 6 for this example

Some increment sequences

- Some increments that have been studied
  - › Shell's increments $h_1 = \left\lfloor \frac{N}{2} \right\rfloor, h_k = \left\lfloor \frac{h_{k+1}}{2} \right\rfloor$
    - bad choice since the subarrays can coincide and so you end up re-sorting something that is already sorted, and not mixing other elements that need it
  - › Hibbard's increments
    - relatively prime values: 1, 3, 7, 15, 2^k-1
  - › Sedgewick
    - {1, 5, 19, 41, 109, …} = $9 \cdot 4^i - 9 \cdot 2^i + 1$ or $4^i - 3 \cdot 2^i + 1$

Example using Shell's Increments

- Example: Shell's original sequence: $h_t = N/2$ and $h_k = h_{k+1}/2$
  - › Sort 21, *33*, 7, *25*   (N = 4, increment sequence = 2, 1)
  - › 7, *25*, 21, *33*   (after 2-sort)
  - › 7, 21, 25, 33   (after 1-sort)

Shellsort: Run time

```
void Shellsort( ElementType A[ ], int N ){
   int i, j, Increment; ElementType Tmp;
   for( Increment = N/2; Increment > 0; Increment /= 2 )
     for( i = Increment; i < N; i++ )
     {
        Tmp = A[ i ];
        for( j = i; j >= Increment; j -= Increment )
                if( Tmp < A[ j - Increment ] )
                        A[ j ] = A[ j - Increment ];
                else
                break;
        A[ j ] = Tmp;
     }
}
```

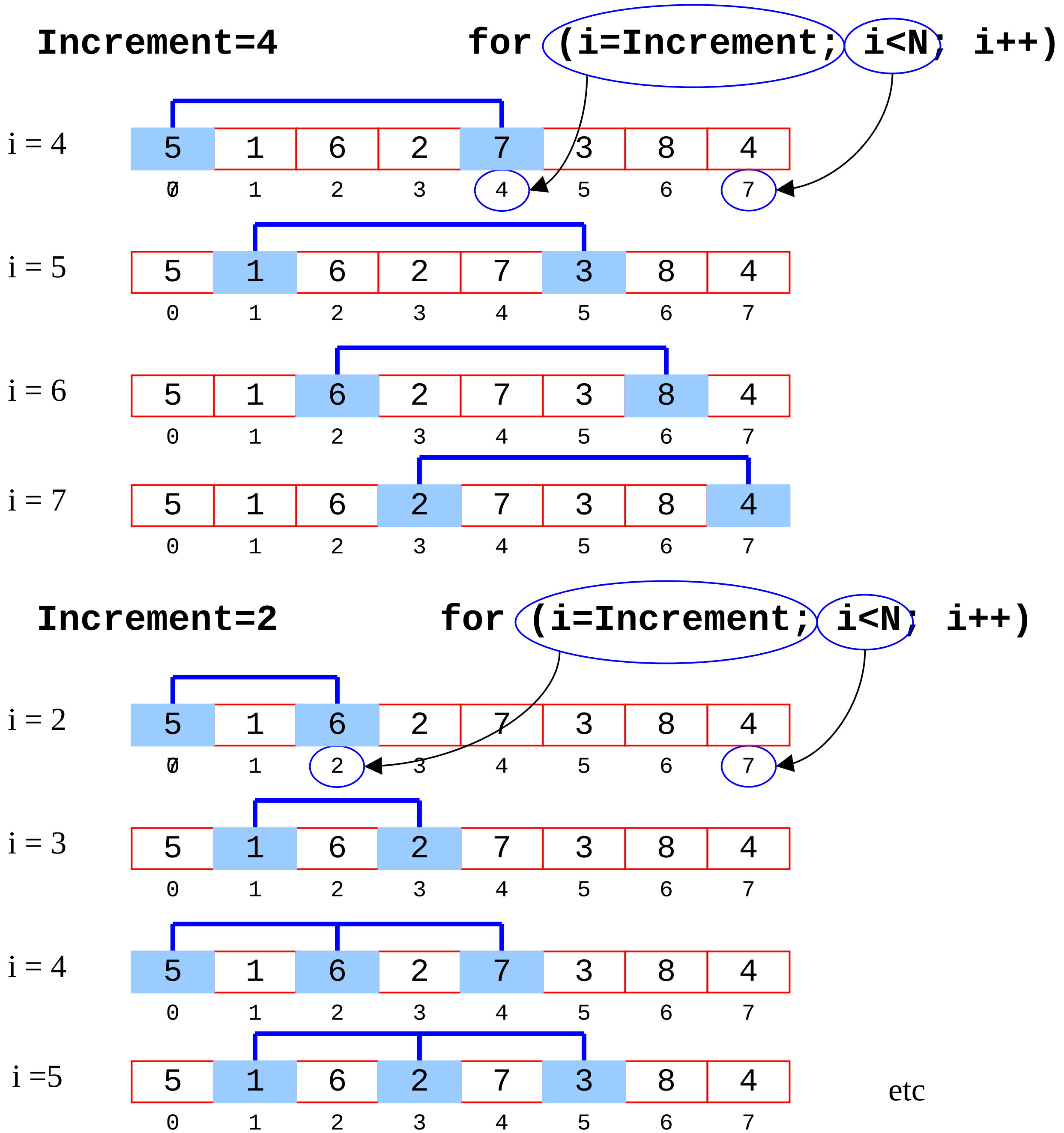
Shellsort: Shell's Increments

- Algorithm is simple to code but hard to analyze
  - › depends on increment sequence
- Shell's increment sequence 1, 2, 4, … , N/4, N/2
  - › What is the Upper bound?
  - › Shellsort does h_k insertion sorts with N/h_k elements for k = 1 to t
  - › Running time = $O(\Sigma_{k=1\ldots t}\, h_k\, (N/h_k)^2) = O(N^2\, \Sigma_{k=1\ldots t}\, 1/h_k) = \mathbf{O(N^2)}$

Shellsort: Shell's Increments

- What is the lower bound?
  - › Worst case is: smallest elements in odd positions, largest in even positions
    - 2, 11, 4, 12, 6, 13, 8, 14
  - › None of the passes N/2, N/4, …, 2 do anything!
  - › Last pass ($h_1 = 1$) must shift N/2 smallest elements to first half and N/2 largest elements to second half
  - › at least N^2 steps = $\mathbf{\Omega(N^2)}$

Shell's increments



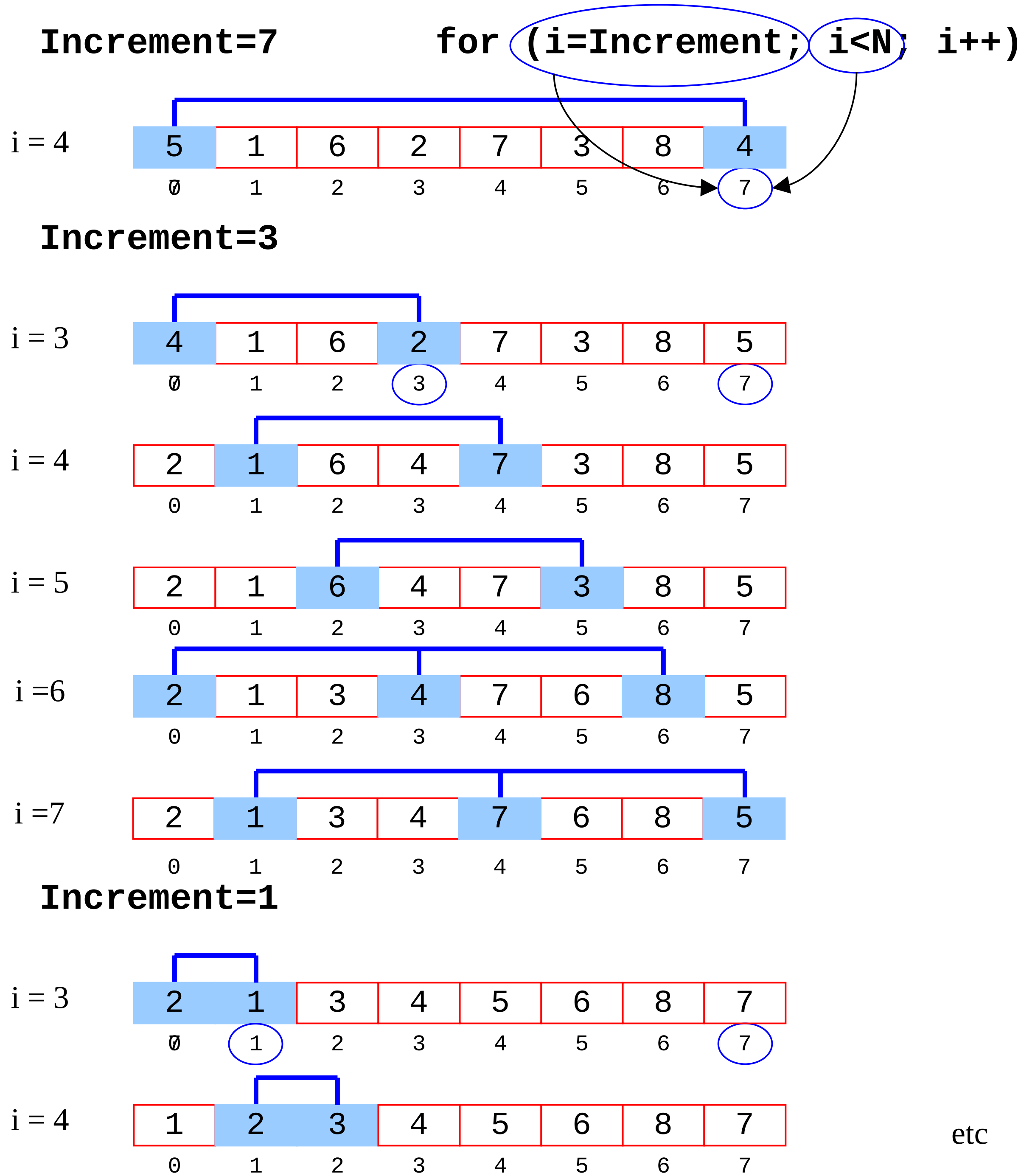
Shell's Increments: $\Omega(N^2)$

- The reason we got **$\Omega(N^2)$** was because of increment sequence
  - › Adjacent increments have common factors (e.g. 8, 4, 2, 1)
  - › We keep comparing same elements over and over again
  - › Need increments such that different elements are in different passes

Hibbard's Increments

- Hibbard's increment sequence:
  - $2^k - 1, 2^{k-1} - 1, \ldots, 7, 3, 1$
  - Adjacent increments have no common factors
  - Worst case running time of Shellsort with Hibbard's increments = **$\Theta(N^{1.5})$** (Theorem 7.4 in text)
  - Average case running time for Hibbard's = **$O(N^{1.25})$** in simulations but nobody has been able to prove it!

Hibbard's increments



General performance

- Insertion sort good for small input sizes
  - ~20
  - often incorporated in other procedures where the list to be sorted is short and is likely to be sorted already
- Shellsort better for moderately large inputs
  - ~10,000