Analysis of Algorithms

CSE 373 - Data Structures

April 10, 2002

Readings and References

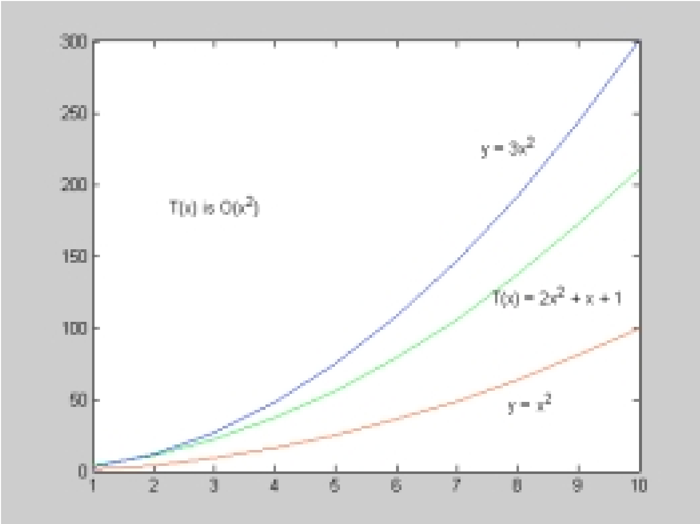
- Reading
  - › Chapter 2, *Data Structures and Algorithm Analysis in C,* Weiss
- Other References

Asymptotic Behavior

- The "asymptotic" performance as $N \rightarrow \infty$, regardless of what happens for small input sizes N, is generally most important
- Performance for small input sizes may matter in practice, if you are <u>sure</u> that <u>small</u> N will be common <u>forever</u>
- We will compare algorithms based on how they scale for large values of N

Big-Oh Notation

- The growth rate of the time or space required in relation to the size of the input N is generally the critical issue
- T(N) is said to be O(f(N)) if
  - › there are positive constants c and n_0 such that $T(N) \leq cf(N)$ for $N \geq n_0$.
  - › ie, f(N) is an upper bound on T(N) for $N \geq n_0$
- T(N) is "big-oh" of f(N) or "order" f(N)




```
x = 1:10;
y1 = x.*x;
y2 = 2*(x.*x)+x+1;
y3 = 3*(x.*x);
plot(x,y1,'r')
hold on
plot(x,y2,'g')
plot(x,y3,'b')
```

$T(x) = 2x^2 + x + 1$ is $O(x^2)$

Big-Oh Notation

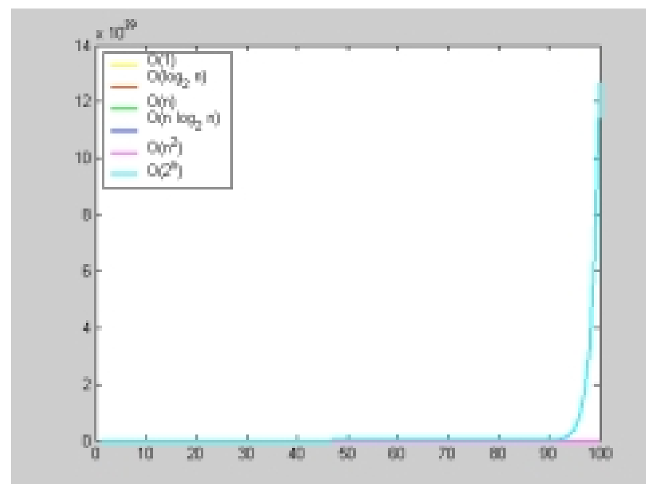
- Suppose $T(N) = 50N$
  - › $T(N) = O(N)$
  - › Take $c = 50$ and $n_0 = 1$
- Suppose $T(N) = 50N+11$
  - › $T(N) = O(N)$
  - › $T(N) \leq 50N+11N = 61N$ for $N \geq 1$. So, $c = 61$ and $n_0 = 1$ works

The common comparisons

Increasing cost

Name	Big-Oh
Constant	$O(1)$
Log log	$O(\log \log N)$
Logarithmic	$O(\log N)$
Log squared	$O((\log N)^2)$
Linear	$O(N)$
N log N	$O(N \log N)$
Quadratic	$O(N^2)$
Cubic	$O(N^3)$
Exponential	$O(2^N)$

Polynomial time



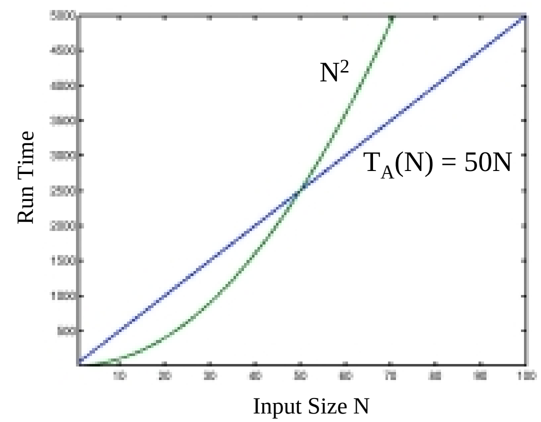

```
n = 1:100;
y1 = n-n+1;
y2 = log2(n);
y3 = n;
y4 = n.*log2(n);
y5 = n.^2;
y6 = 2.^n;

from bigo.m
```

Exponential Growth swamps everything else

Bounds

- Upper bound (O) is not the only bound of interest
- Big-Oh (O), Little-Oh (o), Omega (Ω), and Theta (Θ):
  (Fraternities of functions…)
  - Examples of time and space efficiency analysis

Big-Oh Notation



$T_A(N) = O(N^2)$

$T_A(N)$ is $O(N^2)$ because $50N \le N^2$ for $N \ge 50$.

So N^2 is an upper bound. But it's not a very tight upper bound.

Big-Oh and Omega

- $T(N) = O(f(N))$ if there are positive constants c and n_0 such that $T(N) \le cf(N)$ for $N \ge n_0$.
  - $O(f(N))$ is an upper bound for $T(N)$
  - $100 \log N$, $N^{0.9}$, $0.0001\ N$, $2^{100} N + \log N$ are $O(N)$
- $T(N) = \Omega(f(N))$ if there are positive constants c and n_0 such that $T(N) \ge cf(N)$ for $N \ge n_0$.
  - $\Omega(f(N))$ is a lower bound for $T(N)$
  - 2^N, $N^{\log N}$, $N^{1.2}$, $0.0001\ N$, $N + \log N$ are $\Omega(N)$

Theta and Little-Oh

- $T(N) = \Theta(f(N))$ iff $T(N) = O(f(N))$ and $T(N) = \Omega(f(N))$
  - $\Theta(f(N))$ is a tight bound, upper and lower
  - $0.0001\ N$, $2^{100} N + \log N$ are all $= \Theta(N)$
- $T(N) = o(f(N))$ iff $T(N) = O(f(N))$ and $T(N) \ne \Theta(f(N))$
  - $f(N)$ grows faster than $T(N)$
  - $100 \log N$, $N^{0.9}$, sqrt(N), 17 are all $= o(N)$

For large N and ignoring constant factors

- $T(N) = O(f(N))$
  - › means $T(N)$ is less than or equal to $f(N)$
  - › Upper bound
- $T(N) = \Omega(f(N))$
  - › means $T(N)$ is greater than or equal to $f(N)$
  - › Lower bound
- $T(N) = \Theta(f(N))$
  - › means $T(N)$ is equal to $f(N)$
  - › "Tight" bound, same growth rate
- $T(N) = o(f(N))$
  - › means $T(N)$ is strictly less than $f(N)$
  - › Strict upper bound: $f(N)$ grows faster than $T(N)$

Big-Oh Analysis of iterative sum function

Find the sum of the first num integers stored in array v. Assume num ≤ size of v.

```
int sum ( int v[ ], int num) {
   int temp_sum = 0, i;                          //1
   for (  i = 0; i < num; i++  )                 //2
         temp_sum = temp_sum + v[i] ;            //3
   return temp_sum;                              //4
   }
```

- lines 1, 3, and 4 take fixed (constant) amount of time
- line 2: i goes from 0 to **num**-1= **num** iterations
- Running time = constant + (num)*constant = O(num)
- Actually, Θ(num) because there are no fast cases

Big-Oh Analysis of recursive sum function

Recursive function to find the sum of first **num** integers in v:

```
int sum ( int v[ ], int num){
      if (num == 0) return 0;                // constant time here
      else return v[num-1] + sum(v,num-1);   // constant + T(num-1)
   }
```

- Let T(num) be the running time of sum
- Then, T(num) = constant + T(num-1)
- = 2*constant + T(num-2) =…= num*constant + constant
- = Θ(num) (same as iterative algorithm!)

Common Recurrence Relations

- Common recurrence relations in analysis of algorithms:
  - › $T(N) = T(N-1) + \Theta(1) \Rightarrow T(N) = O(N)$
  - › $T(N) = T(N-1) + \Theta(N) \Rightarrow T(N) = O(N^2)$
  - › $T(N) = T(N/2) + \Theta(1) \Rightarrow T(N) = O(\log N)$
  - › $T(N) = 2T(N/2) + \Theta(N) \Rightarrow T(N) = O(N \log N)$
  - › $T(N) = 4T(N/2) + \Theta(N) \Rightarrow T(N) = O(N^2)$

Big-Oh Analysis of Recursive Algorithms

- To derive, expand the right side and count
- Note: Multiplicative constants matter in recurrence relations:
  - $T(N) = 4T(N/2) + \Theta(N)$ is $O(N^2)$, not $O(N \log N)$!
- You will see these again later
  - you will only need to know a few specific relations and their big-oh answers

Recursion

- Recall the example using Fibonacci numbers

1, 1, 2, 3, 5, 8, 13, 21, 34, …

Leonardo Pisano Fibonacci (1170-1250)

  - First two are defined to be 1
  - Rest are sum of preceding two
  - $F_n = F_{n-1} + F_{n-2}$ $(n > 1)$

Recursive Fibonacci Function

```
int fib(int N) {
  if (N < 0) return 0;  //invalid input
  if (N == 0 || N == 1) return 1;  //base cases
  else return fib(N-1)+fib(N-2);
  }
```

- Running time T(N) = ?

Recursive Fibonacci Analysis

```
int fib(int N) {
  if (N < 0) return 0;  // time = 1 for (N < 0)
  if (N == 0 || N == 1) return 1;  // time = 3
  else return fib(N-1)+fib(N-2); //T(N-1)+T(N-2)+1
  }
```

- Running time $T(N) = T(N-1) + T(N-2) + 5$
- Using $F_n = F_{n-1} + F_{n-2}$ we can show by induction that $T(N) \geq F_N$. We can also show by induction that $F_N \geq (3/2)^N$
- Therefore, $T(N) \geq (3/2)^N$
  - Exponential running time!

Iterative Fibonacci Function

```
int fib_iter(int N) {
  int fib0 = 1, fib1 = 1, fibj = 1;
  if (N < 0) return 0;  //invalid input
  for (int j = 2; j <= N; j++) { //all fib nos. up to N
      fibj = fib0 + fib1;
      fib0 = fib1;
      fib1 = fibj;
      }
  return fibj;
  }
```

- Running time = ?

Iterative Fibonacci Analysis

```
int fib_iter(int N) {
  int fib0 = 1, fib1 = 1, fibj = 1;
  if (N < 0) return 0;  //invalid input
  for (int j = 2; j <= N; j++) { //all fib nos. up to N
      fibj = fib0 + fib1;
      fib0 = fib1;
      fib1 = fibj;
      }
  return fibj;
  }
```

- Running time $T(N) = \text{constant} + (N-1) \cdot \text{constant}$
- $T(N) = \Theta(N)$
  - › Exponentially faster than recursive Fibonacci

Appendix

Matlab - bigo.m

```
% bigO functions
% calculate and plot several functions used in comparing growth rates

figure

n = 1:100;                      % the x axis
y1 = n-n+1;                     % O(1)
y2 = log2(n);                   % O(log2(n))
y3 = n;                         % O(n)
y4 = n.*log2(n);                % O(nlog2(n))
y5 = n.^2;                      % O(n^2)
y6 = 2.^n;                      % O(2^n)

plot(n,y1,'y','LineWidth',2)
hold on
plot(n,y2,'r','LineWidth',2)
plot(n,y3,'g','LineWidth',2)
plot(n,y4,'b','LineWidth',2)
plot(n,y5,'m','LineWidth',2)
plot(n,y6,'c','LineWidth',2)

legend('O(1)','O(log_2 n)','O(n)','O(n log_2 n)','O(n^2)','O(2^n)',2)

hold off
```