# Graph Introduction

CSE 373
Data Structures
Lecture 18

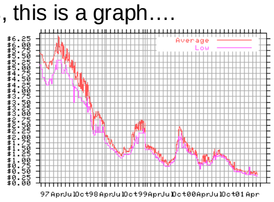
## Reading

- Reading
  - › Section 9.1



## What are graphs?

- Yes, this is a graph….

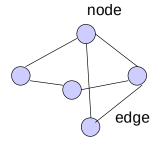


- But we are interested in a different kind of “graph”



## Graphs

- Graphs are composed of
  - › Nodes (vertices)
  - › Edges





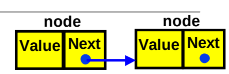

## Varieties

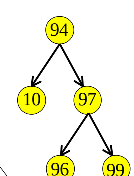
- Nodes
  - › Labeled or unlabeled
- Edges
  - › Directed or undirected
  - › Labeled or unlabeled

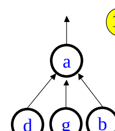


## Motivation for Graphs

- Consider the data structures we have looked at so far…
- Linked list: nodes with 1 incoming edge + 1 outgoing edge
- Binary trees/heaps: nodes with 1 incoming edge + 2 outgoing edges
- Binomial trees/B-trees: nodes with 1 incoming edge + multiple outgoing edges
- Up-trees: nodes with multiple incoming edges + 1 outgoing edge



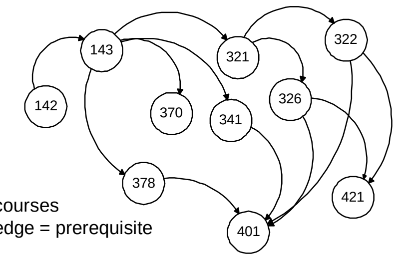



## Motivation for Graphs

- What is common among these data structures?
- How can you generalize them?
- Consider data structures for representing the following problems…



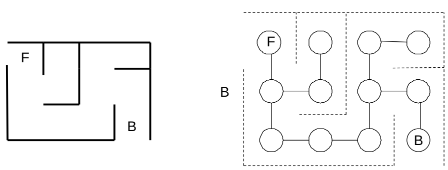
## CSE Course Prerequisites at UW




Nodes = courses
Directed edge = prerequisite



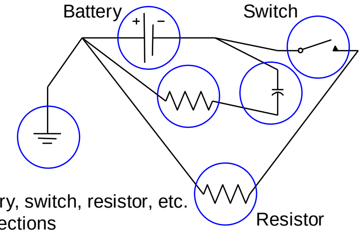
## Representing a Maze




Nodes = rooms
Edge = door or passage



## Representing Electrical Circuits




Nodes = battery, switch, resistor, etc.
Edges = connections



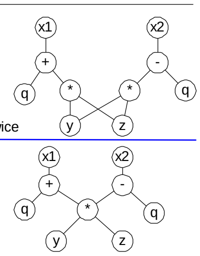
## Program statements

```
x1=q+y*z
x2=y*z-q
```

Naive:

y*z calculated twice

common subexpression eliminated:

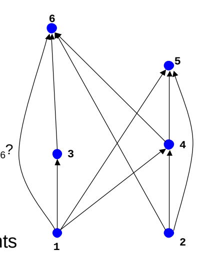



Nodes = symbols/operators
Edges = relationships



## Precedence

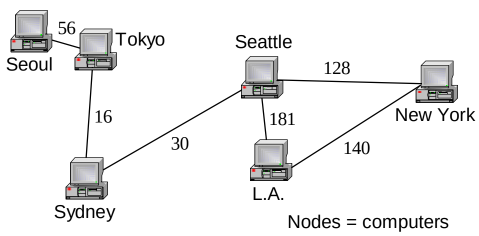
| | |
|---|---|
| $S_1$ | `a=0;` |
| $S_2$ | `b=1;` |
| $S_3$ | `c=a+1` |
| $S_4$ | `d=b+a;` |
| $S_5$ | `e=d+1;` |
| $S_6$ | `e=c+d;` |




Which statements must execute before $S_6$?
$S_1$, $S_2$, $S_3$, $S_4$

Nodes = statements
Edges = precedence requirements

## Information Transmission in a Computer Network




Nodes = computers
Edges = transmission rates

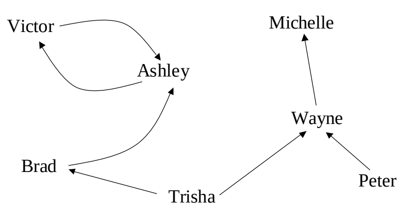


## Traffic Flow on Highways


Nodes = cities
Edges = # vehicles on connecting highway

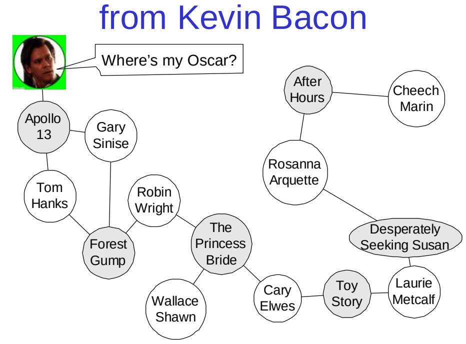


## Soap Opera Relationships

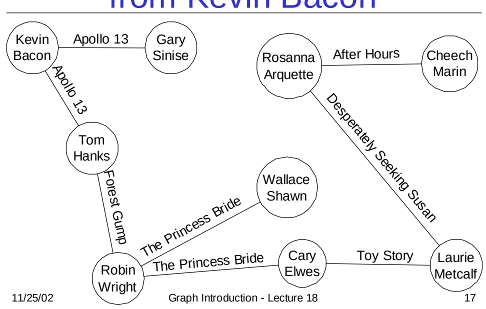





## Six Degrees of Separation from Kevin Bacon

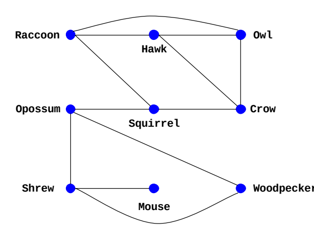



## Six Degrees of Separation from Kevin Bacon






## Niche overlaps



## Graph Definition

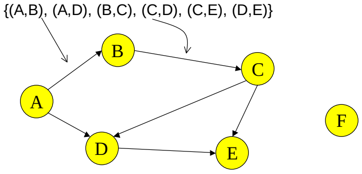
- A graph is simply a collection of nodes plus edges
  - › Linked lists, trees, and heaps are all special cases of graphs
- The nodes are known as vertices (node = “vertex”)
- Formal Definition: A graph *G* is a pair (*V*, *E*) where
  - › *V* is a set of vertices or nodes
  - › *E* is a set of edges that connect vertices



## Graph Example

- Here is a directed graph $G = (V, E)$
  - › Each edge is a pair $(v_1, v_2)$, where $v_1, v_2$ are vertices in $V$

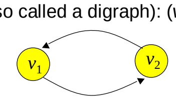
$V$ = {A, B, C, D, E, F}
$E$ = {(A,B), (A,D), (B,C), (C,D), (C,E), (D,E)}






## Directed vs Undirected Graphs

- If the order of edge pairs $(v_1, v_2)$ matters, the graph is directed (also called a digraph): $(v_1, v_2) \neq (v_2, v_1)$




- If the order of edge pairs $(v_1, v_2)$ does not matter, the graph is called an undirected graph: in this case, $(v_1, v_2) = (v_2, v_1)$

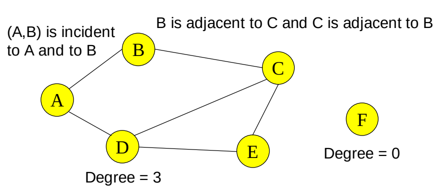



## Undirected Terminology

- Two vertices u and v are adjacent in an undirected graph G if {u,v} is an edge in G
  - › edge e = {u,v} is incident with vertex u and vertex v
- The degree of a vertex in an undirected graph is the number of edges incident with it
  - › a self-loop counts twice (both ends count)
  - › denoted with deg(v)



## Undirected Terminology



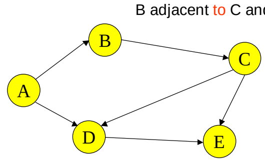



## Directed Terminology

- Vertex u is adjacent to vertex v in a directed graph G if (u,v) is an edge in G
  - › vertex u is the initial vertex of (u,v)
- Vertex v is adjacent from vertex u
  - › vertex v is the terminal (or end) vertex of (u,v)
- Degree
  - › in-degree is the number of edges with the vertex as the terminal vertex
  - › out-degree is the number of edges with the vertex as the initial vertex

## Directed Terminology

B adjacent to C and C adjacent from B





In-degree = 2
Out-degree = 1

In-degree = 0
Out-degree = 1



## Handshaking Theorem

- Let G=(V,E) be an undirected graph with |E|=e edges
- Then $2e = \sum_{v \in V} \deg(v)$
- Every edge contributes +1 to the degree of each of the two vertices it is incident with
  - › number of edges is exactly half the sum of deg(v)
  - › the sum of the deg(v) values must be even

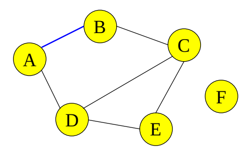


## Graph Representations

- Space and time are analyzed in terms of:
  - Number of vertices = $|V|$ and
  - Number of edges = $|E|$
- There are at least two ways of representing graphs:
  - The *adjacency matrix* representation
  - The *adjacency list* representation



## Adjacency Matrix



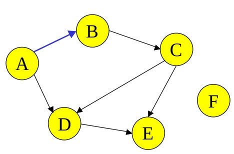

$$\begin{array}{c|cccccc} & A & B & C & D & E & F \\ \hline A & 0 & 1 & 0 & 1 & 0 & 0 \\ B & 1 & 0 & 1 & 0 & 0 & 0 \\ C & 0 & 1 & 0 & 1 & 1 & 0 \\ D & 1 & 0 & 1 & 0 & 1 & 0 \\ E & 0 & 0 & 1 & 1 & 0 & 0 \\ F & 0 & 0 & 0 & 0 & 0 & 0 \end{array}$$

$$M(v, w) = \begin{cases} 1 \text{ if } (v, w) \text{ is in E} \\ 0 \text{ otherwise} \end{cases}$$

Space = $|V|^2$



## Adjacency Matrix for a Digraph




$$\begin{array}{c|cccccc} & A & B & C & D & E & F \\ \hline A & 0 & 1 & 0 & 1 & 0 & 0 \\ B & 0 & 0 & 1 & 0 & 0 & 0 \\ C & 0 & 0 & 0 & 1 & 1 & 0 \\ D & 0 & 0 & 0 & 0 & 1 & 0 \\ E & 0 & 0 & 0 & 0 & 0 & 0 \\ F & 0 & 0 & 0 & 0 & 0 & 0 \end{array}$$

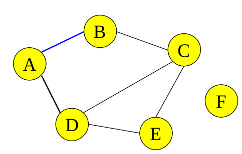
$$M(v, w) = \begin{cases} 1 \text{ if } (v, w) \text{ is in E} \\ 0 \text{ otherwise} \end{cases}$$

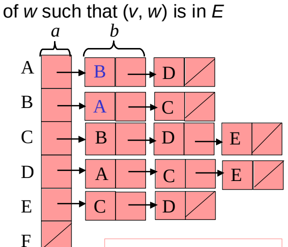
Space = $|V|^2$



## Adjacency List

For each $v$ in $V$, $L(v)$ = list of $w$ such that $(v, w)$ is in $E$

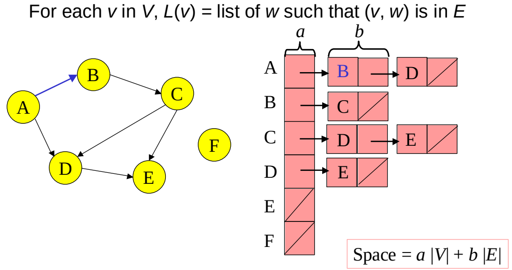



Space = $a\,|V| + 2\,b\,|E|$

## Adjacency List for a Digraph

For each $v$ in $V$, $L(v)$ = list of $w$ such that $(v, w)$ is in $E$



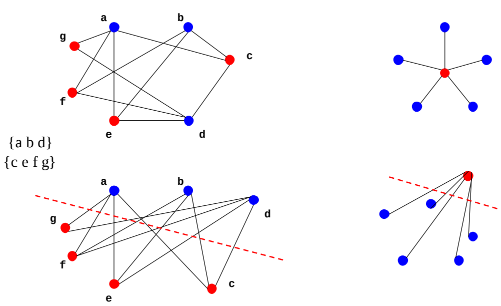



## Bipartite

- A simple graph is bipartite if:
  - › its vertex set V can be partitioned into two disjoint non-empty sets such that
    - every edge in the graph connects a vertex in one set to a vertex in the other set
    - which also means that no edge connects a vertex in one set to another vertex in the same set
  - › no triangular or other odd length cycles

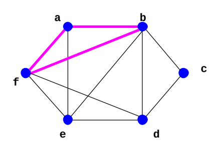


## Bipartite examples






## Bipartite example - not

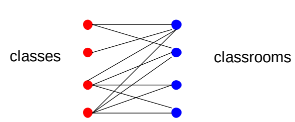


*a* says that *b* and *f* should be in $S_2$,
but *b* says *a* and *f* should be in $S_1$.
TILT!



## Bipartite Graph Application

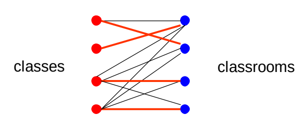
- Classroom scheduling
  - › Nodes are Classrooms and Classes
  - › Edge between a classroom and class if the class will fit in the classroom and has the right technology.






## Matching Problem

- Find an assignment of classes to classrooms so that every class fits and has the right technology.



## Steps in Solving the Problem

- Abstract the problem as a graph problem.
- Find an algorithm for solving the graph problem.
- Design data structures and algorithms to implement the graph solution.
- Write code

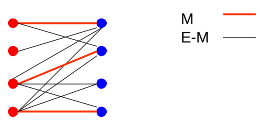


## Alternating Path

- Let G = (U,V,E) be a bipartite graph where (u,v) in E only if u in U and v in V.
- A partial matching M is subset of E such that if (u,v) and (u',v') in M then either (u = u' and v = v') or (u $\neq$ u' or v $\neq$ v' )
- An alternating path is $x_1,x_2,\ldots,x_{2n}$ such that
  - › $(x_i,x_{i+1})$ in E – M if i is odd
  - › $(x_i,x_{i+1})$ in M if i is even
  - › $x_1$ and $x_{2n}$ are not matched in the partial matching

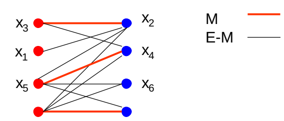


## Partial Matching






## Alternating Path





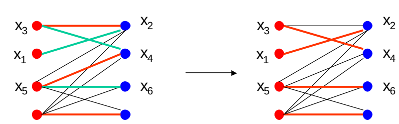

## Matching Algorithm

set M to be the empty set initially
repeat
  find an alternating path $x_1,x_2,\ldots,x_{2n}$
  // $(x_i,x_{i+1})$ in E – M if i is odd and $(x_i,x_{i+1})$ in M if i is even
    neither $x_1$ nor $x_{2n}$ matched //
  delete $(x_i,x_{i+1})$ from M if i is even
  add $(x_i,x_{i+1})$ to M if i is odd
until no alternating path can be found

if M has every vertex of U then M is a matching
if M does not have some vertex then there is complete matching, but M is a maximum size matching



## One step in the Algorithm



## Maximum Matching

- Prove that M is maximum size if and only if there is no alternating path.
- Design data structures algorithms to find alternating paths or determine they don't exist.
  - › Goal: fast data structures and algorithms