Divide and Conquer Sorting

CSE 373

Data Structures

Lecture 14

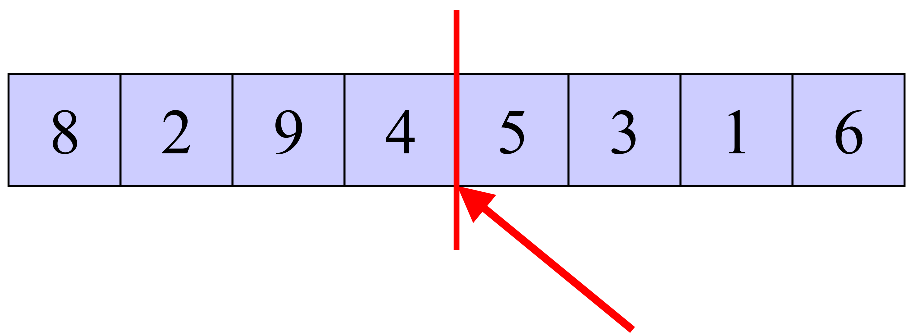
Readings

- Reading
  - › Section 7.6, Mergesort
  - › Section 7.7, Quicksort

"Divide and Conquer"

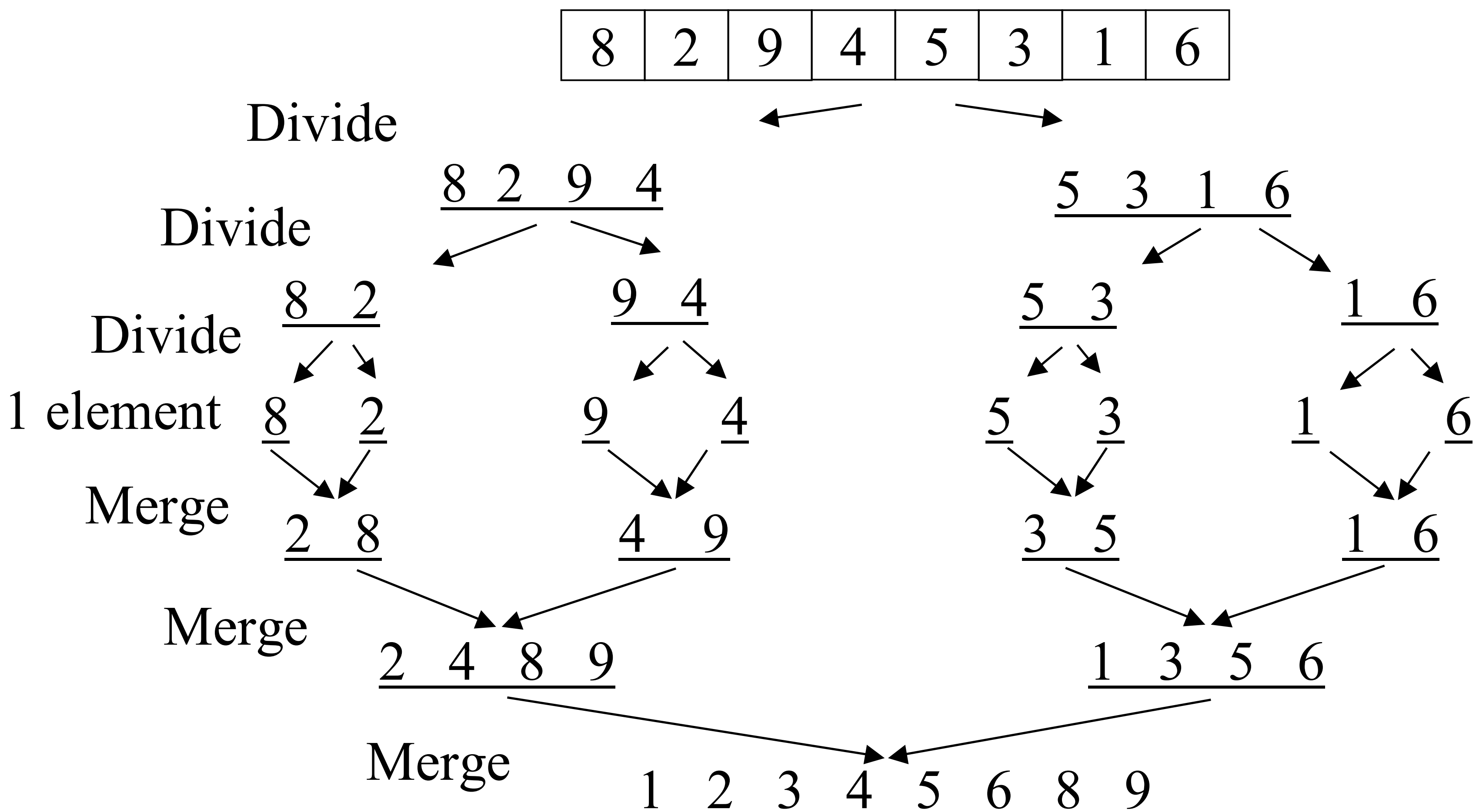
- Very important strategy in computer science:
  - Divide problem into smaller parts
  - Independently solve the parts
  - Combine these solutions to get overall solution
- **Idea 1**: Divide array into two halves, *recursively* sort left and right halves, then *merge* two halves → known as Mergesort
- **Idea 2 :** Partition array into small items and large items, then recursively sort the two sets → known as Quicksort

Mergesort



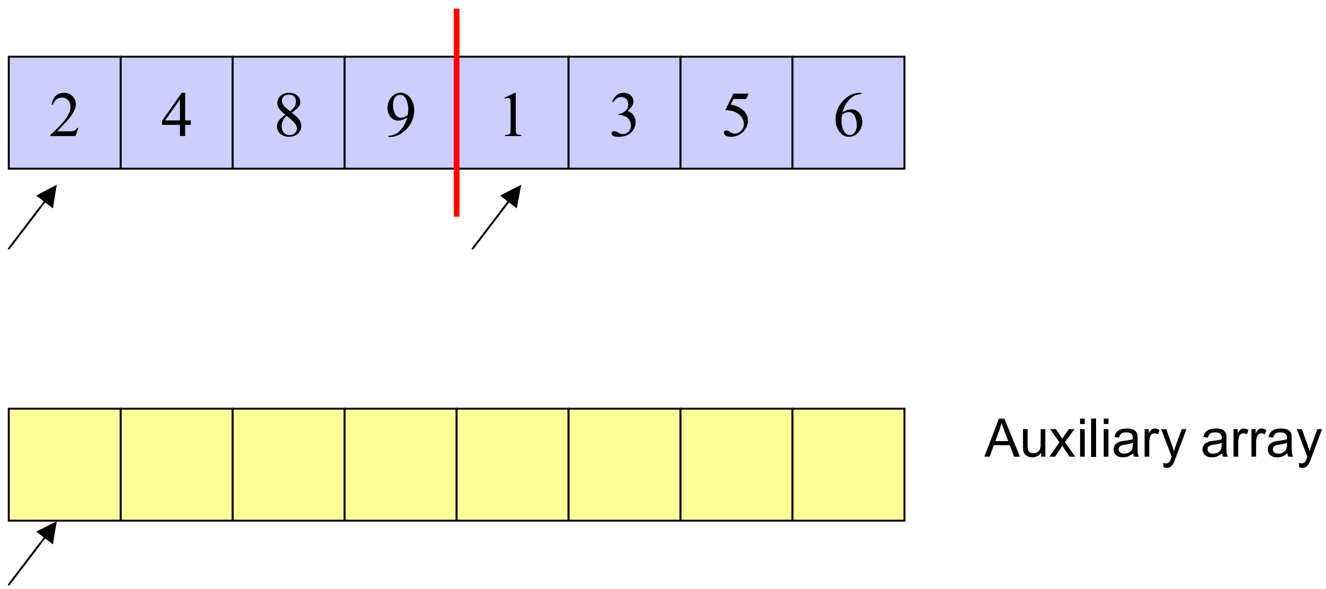
- Divide it in two at the midpoint
- Conquer each side in turn (by recursively sorting)
- Merge two halve together

Mergesort Example



Auxiliary Array

- The merging requires an auxiliary array.



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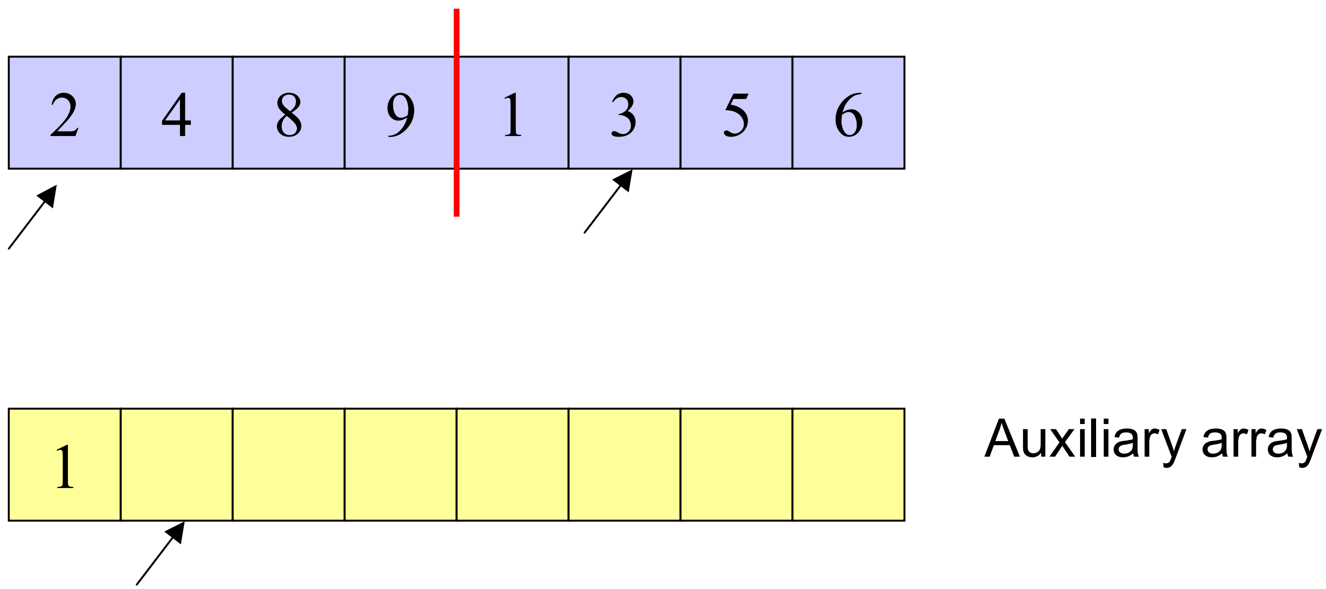

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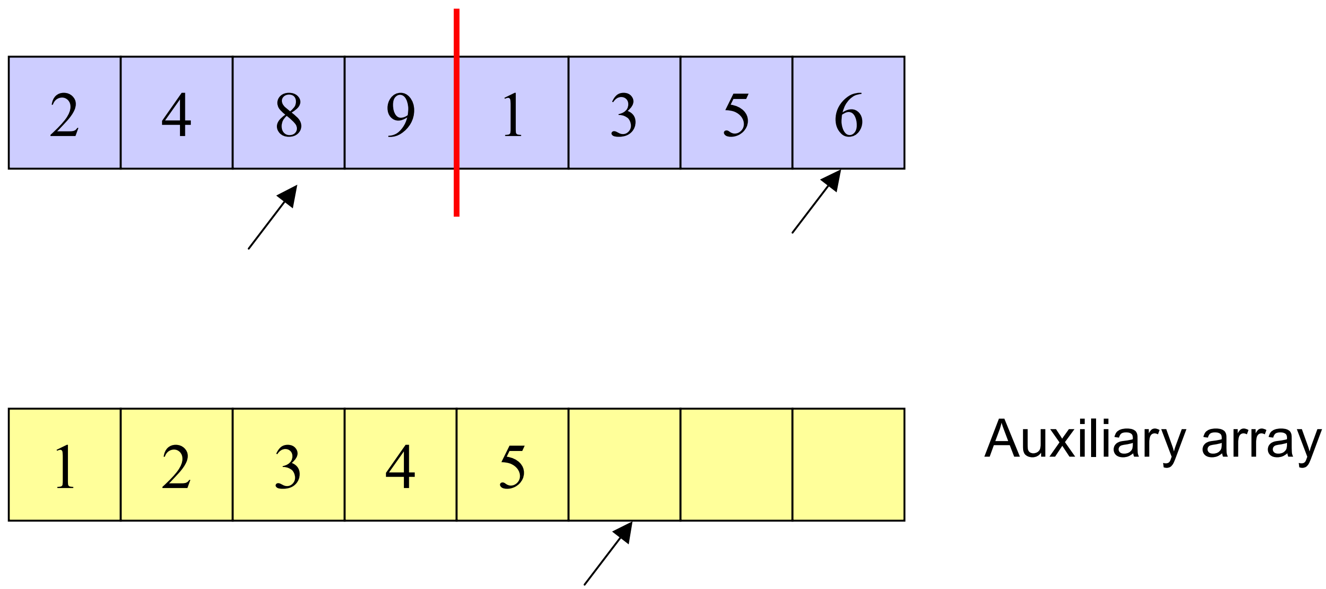
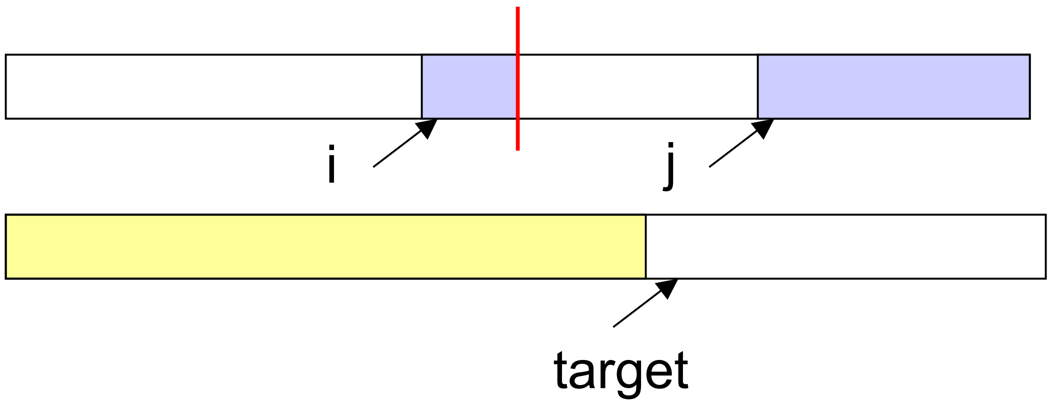
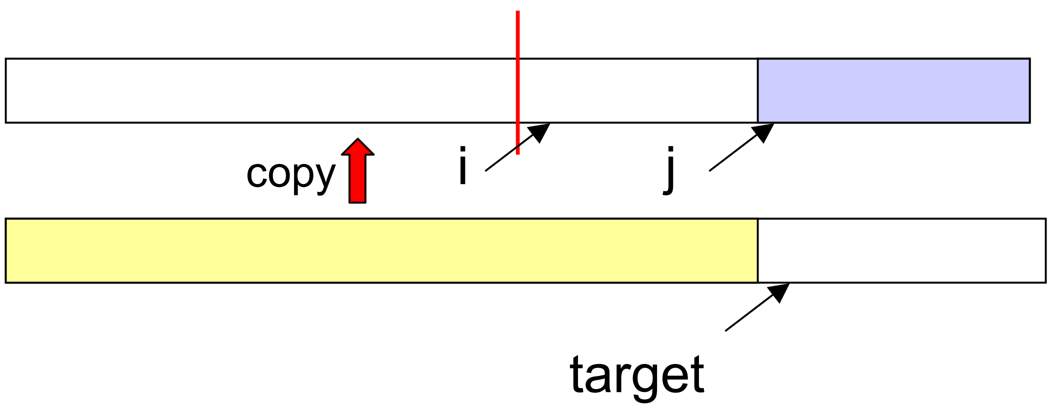


Merging

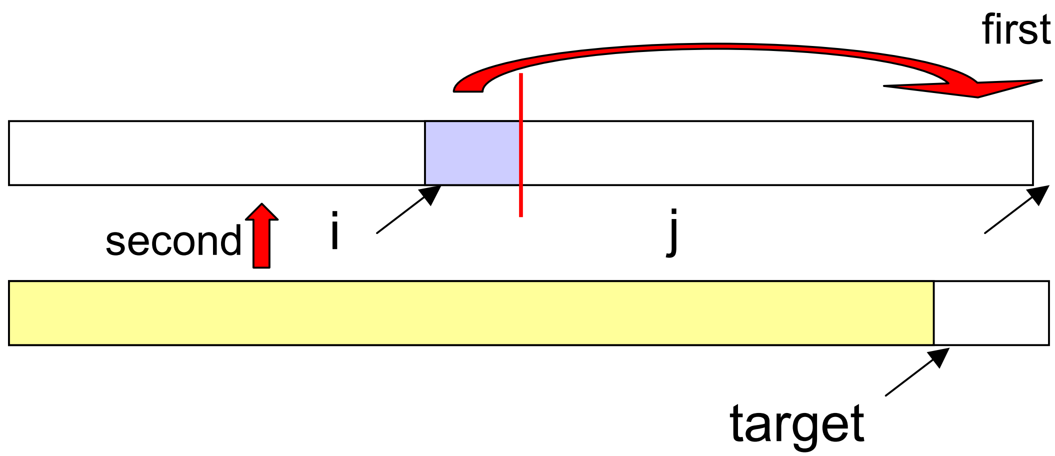


normal



Left completed first

Merging



Right completed first

Merging

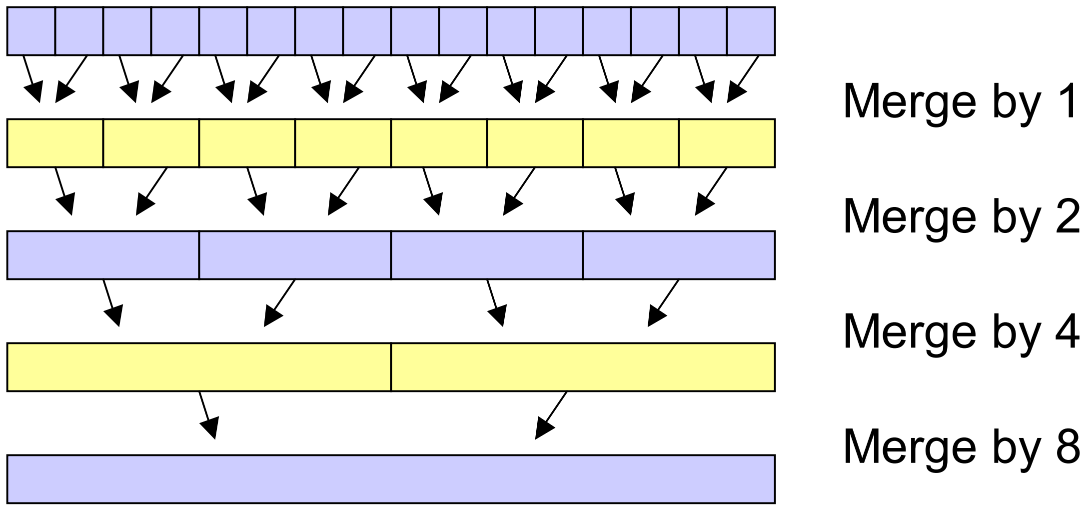
```
Merge(A[], T[] : integer array, left, right : integer) : {
  mid, i, j, k, l, target : integer;
  mid := (right + left)/2;
  i := left; j := mid + 1; target := left;
  while i ≤ mid and j ≤ right do
    if A[i] ≤ A[j] then T[target] := A[i] ; i:= i + 1;
      else T[target] := A[j]; j := j + 1;
    target := target + 1;
  if i > mid then //left completed//
    for k := left to target-1 do A[k] := T[k];
  if j > right then //right completed//
    k : = mid; l := right;
    while k ≥ i do A[l] := A[k]; k := k-1; l := l-1;
    for k := left to target-1 do A[k] := T[k];
}
```

Recursive Mergesort

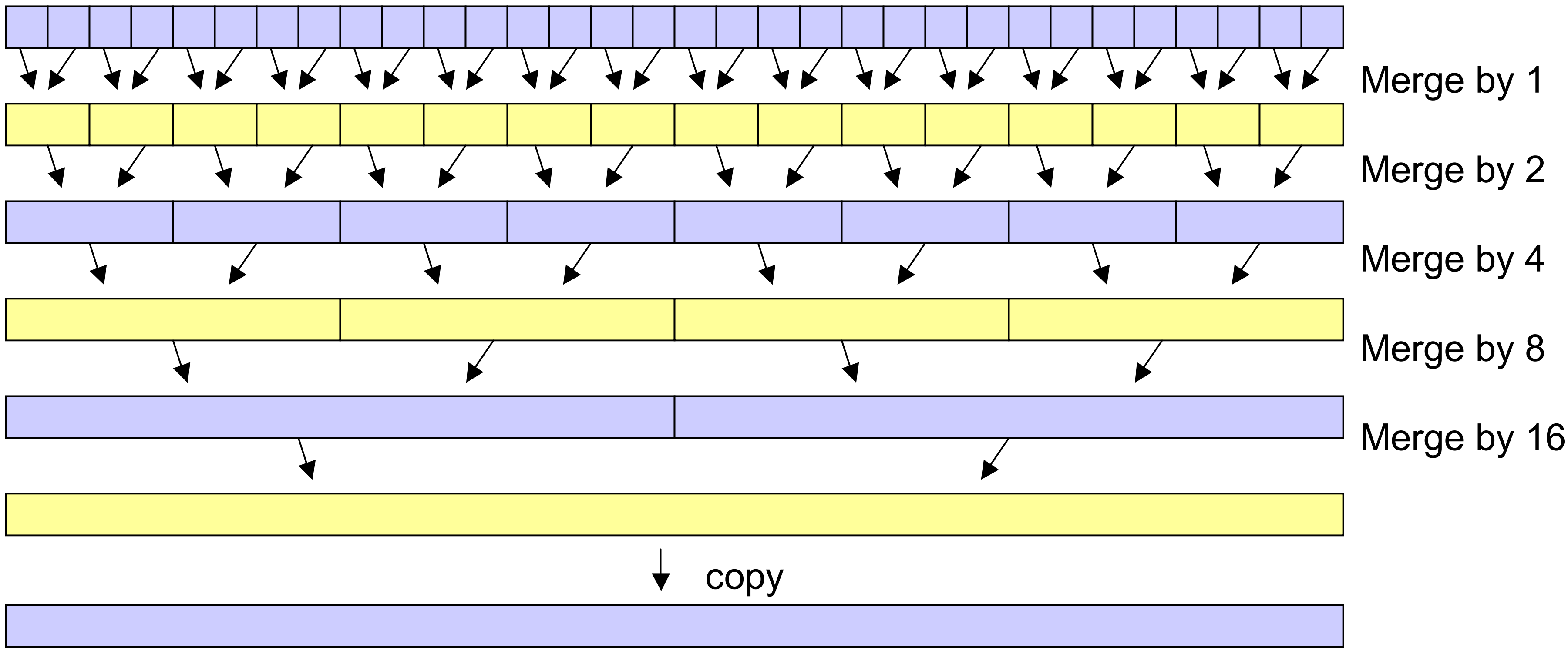
```
Mergesort(A[], T[] : integer array, left, right : integer) : {
  if left < right then
    mid := (left + right)/2;
    Mergesort(A,T,left,mid);
    Mergesort(A,T,mid+1,right);
    Merge(A,T,left,right);
}

MainMergesort(A[1..n]: integer array, n : integer) : {
  T[1..n]: integer array;
  Mergesort[A,T,1,n];
}
```

Iterative Mergesort



Iterative Mergesort



Iterative Mergesort

```
IterativeMergesort(A[1..n]: integer array, n : integer) : {
//precondition: n is a power of 2//
  i, m, parity : integer;
  T[1..n]: integer array;
  m := 2; parity := 0;
  while m ≤ n do
    for i = 1 to n - m + 1 by m do
       if parity = 0 then Merge(A,T,i,i+m-1);
         else Merge(T,A,i,i+m-1);
    parity := 1 - parity;
    m := 2*m;
  if parity = 1 then
    for i = 1 to n do A[i] := T[i];
}
```

How do you handle non-powers of 2?
How can the final copy be avoided?

Mergesort Analysis

- Let T(N) be the running time for an array of N elements
- Mergesort divides array in half and calls itself on the two halves. After returning, it merges both halves using a temporary array
- Each recursive call takes T(N/2) and merging takes O(N)

Mergesort Recurrence Relation

- The recurrence relation for T(N) is:
  - › $T(1) \leq a$
    - base case: 1 element array → constant time
  - › $T(N) \leq 2T(N/2) + bN$
    - Sorting N elements takes
      - – the time to sort the left half
      - – plus the time to sort the right half
      - – plus an O(N) time to merge the two halves
- $T(N) = O(n \log n)$

Properties of Mergesort

- Not in-place
  - › Requires an auxiliary array
- Stable
  - › Make sure that left is sent to target on equal values.
- Very few comparisons
- Iterative Mergesort reduces copying.

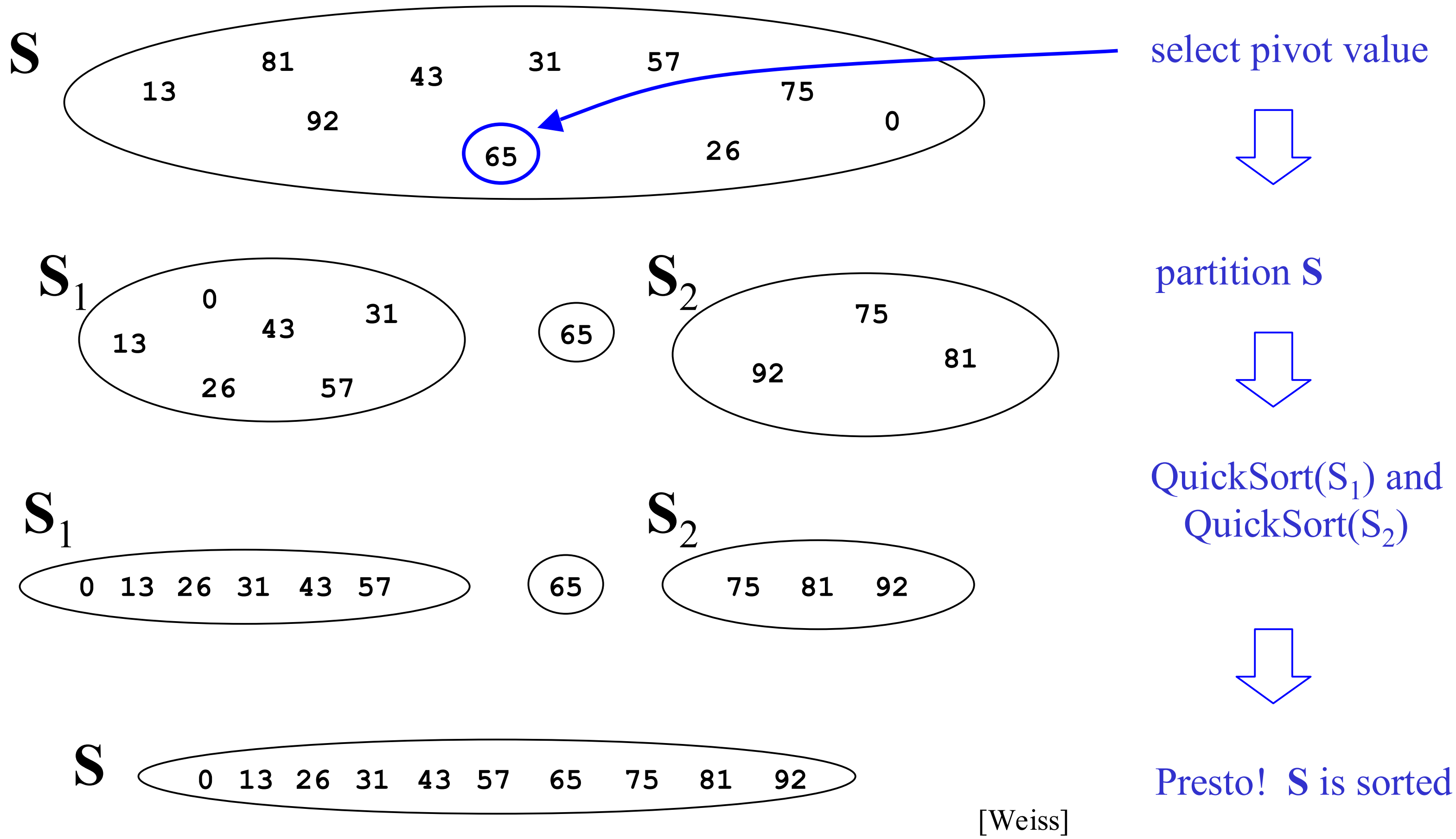
Quicksort

- Quicksort uses a divide and conquer strategy, but does not require the O(N) extra space that MergeSort does
  - › Partition array into left and right sub-arrays
    - the elements in left sub-array are all less than pivot
    - elements in right sub-array are all greater than pivot
  - › Recursively sort left and right sub-arrays
  - › Concatenate left and right sub-arrays in O(1) time

"Four easy steps"

- To sort an array **S**
  - › If the number of elements in **S** is 0 or 1, then return. The array is sorted.
  - › Pick an element *v* in **S**. This is the *pivot* value.
  - › Partition **S**-{*v*} into two disjoint subsets, $\mathbf{S}_1$ = {all values $x \leq v$}, and $\mathbf{S}_2$ = {all values $x \geq v$}.
  - › Return QuickSort($\mathbf{S}_1$), *v*, QuickSort($\mathbf{S}_2$)

The steps of QuickSort



Details, details

- "The algorithm so far lacks quite a few of the details"
- Implementing the actual partitioning
- Picking the pivot
  - › want a value that will cause $|S_1|$ and $|S_2|$ to be non-zero, and close to equal in size if possible
- Dealing with cases where the element equals the pivot

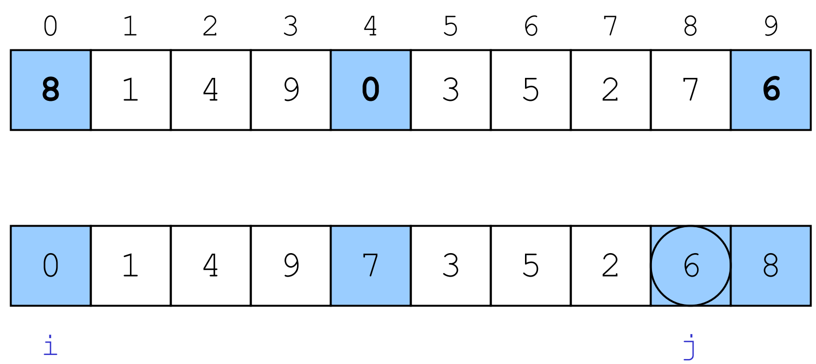
Quicksort Partitioning

- Need to partition the array into left and right sub-arrays
  - › the elements in left sub-array are ≤ pivot
  - › elements in right sub-array are ≥ pivot
- How do the elements get to the correct partition?
  - › Choose an element from the array as the pivot
  - › Make one pass through the rest of the array and swap as needed to put elements in partitions

Partitioning is done In-Place

- One implementation (there are others)
  - median3 finds pivot and sorts left, center, right
  - Swap pivot with next to last element
  - Set pointers i and j to start and end of array
  - Increment i until you hit element A[i] > pivot
  - Decrement j until you hit element A[j] < pivot
  - Swap A[i] and A[j]
  - Repeat until i and j cross
  - Swap pivot (= A[N-2]) with A[i]

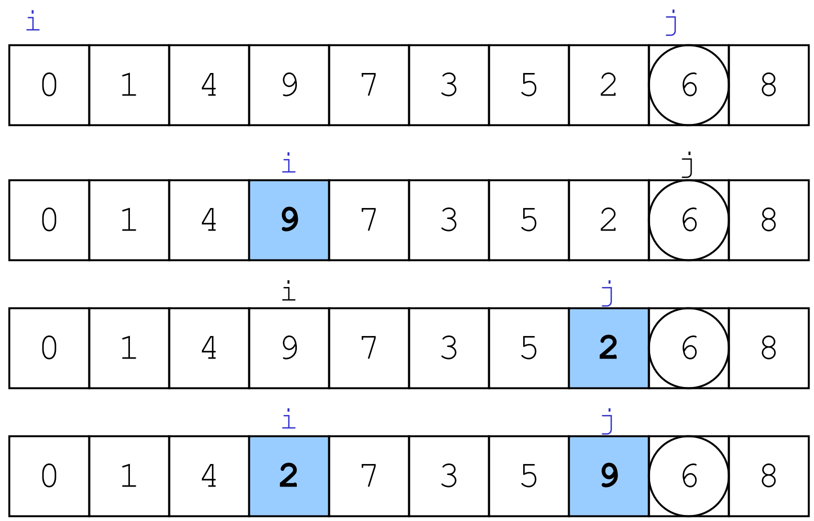
Example



Choose the pivot as the median of three.

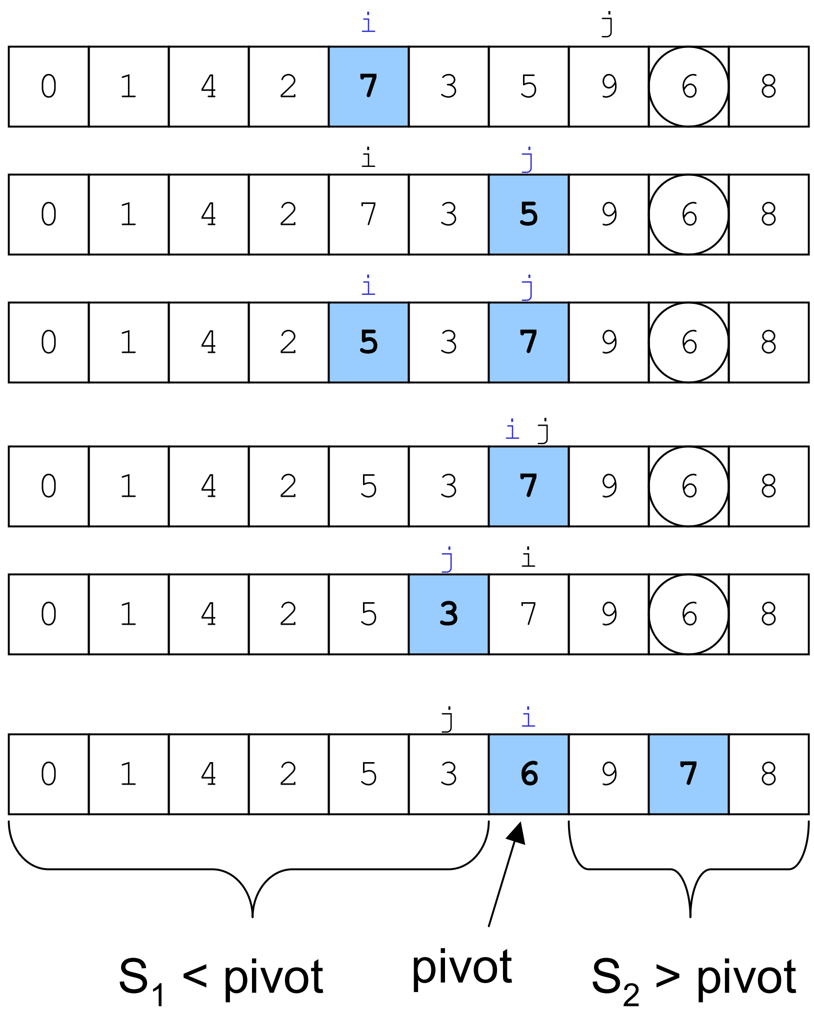
Place the pivot and the largest at the right and the smallest at the left

Example



Move i to the right to be larger than pivot.
Move j to the left to be smaller than pivot.
Swap

Example


i j
0 1 4 2 7 3 5 9 6 8
i j
0 1 4 2 7 3 5 9 6 8
i j
0 1 4 2 5 3 7 9 6 8
i j
0 1 4 2 5 3 7 9 6 8
j i
0 1 4 2 5 3 7 9 6 8
j i
0 1 4 2 5 3 6 9 7 8
S1 < pivot
pivot
S2 > pivot

Recursive Quicksort

```
Quicksort(A[]: integer array, left,right : integer): {
pivotindex : integer;
if left + CUTOFF ≤ right then
  pivot := median3(A,left,right);
  pivotindex := Partition(A,left,right-1,pivot);
  Quicksort(A, left, pivotindex – 1);
  Quicksort(A, pivotindex + 1, right);
else
  Insertionsort(A,left,right);
}
```

Don't use quicksort for small arrays.
CUTOFF = 10 is reasonable.

Alternative Pivot Rules

- Chose A[left]
  - Fast, but may be too biased
- Chose A[random], left $\le$ random $\le$ right
  - Completely unbiased
  - Will cause relatively even split, but slow
- Median of three, A[left], A[right], A[(left+right)/2]
  - The standard, tends to be unbiased, and does a little sorting on the side.

Quicksort Best Case Performance

- Algorithm always chooses best pivot and splits sub-arrays in half at each recursion
  - › T(0) = T(1) = O(1)
    - constant time if 0 or 1 element
  - › For N > 1, 2 recursive calls plus linear time for partitioning
  - › T(N) = 2T(N/2) + O(N)
    - Same recurrence relation as Mergesort
  - › T(N) = O(N log N)

Quicksort Worst Case Performance

- Algorithm always chooses the worst pivot – one sub-array is empty at each recursion
  - › $T(N) \leq a$ for $N \leq C$
  - › $T(N) \leq T(N-1) + bN$
  - › $\qquad \leq T(N-2) + b(N-1) + bN$
  - › $\qquad \leq T(C) + b(C+1) + \ldots + bN$
  - › $\qquad \leq a + b(C + C+1 + C+2 + \ldots + N)$
  - › $T(N) = O(N^2)$
- Fortunately, *average case performance* is O(N log N) (see text for proof)

Properties of Quicksort

- Not stable because of long distance swapping.
- No iterative version (without using a stack).
- Pure quicksort not good for small arrays.
- "In-place", but uses auxiliary storage because of recursive calls.
- O(n log n) average case performance, but $O(n^2)$ worst case performance.

Folklore

- "Quicksort is the best in-memory sorting algorithm."
- Truth
  - › Quicksort uses very few comparisons on average.
  - › Quicksort does have good performance in the memory hierarchy.
    - Small footprint
    - Good locality