Hashing

CSE 373

Data Structures

Lecture 10

Readings and References

- Reading
  - Chapter 5

The Need for Speed

- Data structures we have looked at so far
  - › Use comparison operations to find items
  - › Need O(log N) time for Find and Insert
- In real world applications, N is typically between 100 and 100,000 (or more)
  - › log N is between 6.6 and 16.6
- Hash tables are an abstract data type designed for O(1) Find and Inserts

Fewer Functions Faster

- compare lists and stacks
  - by reducing the flexibility of what we are allowed to do, we can increase the performance of the remaining operations
  - insert(L,X) into a list versus push(S,X) onto a stack
- compare trees and hash tables
  - trees provide for known ordering of all elements
  - hash tables just let you (quickly) find an element

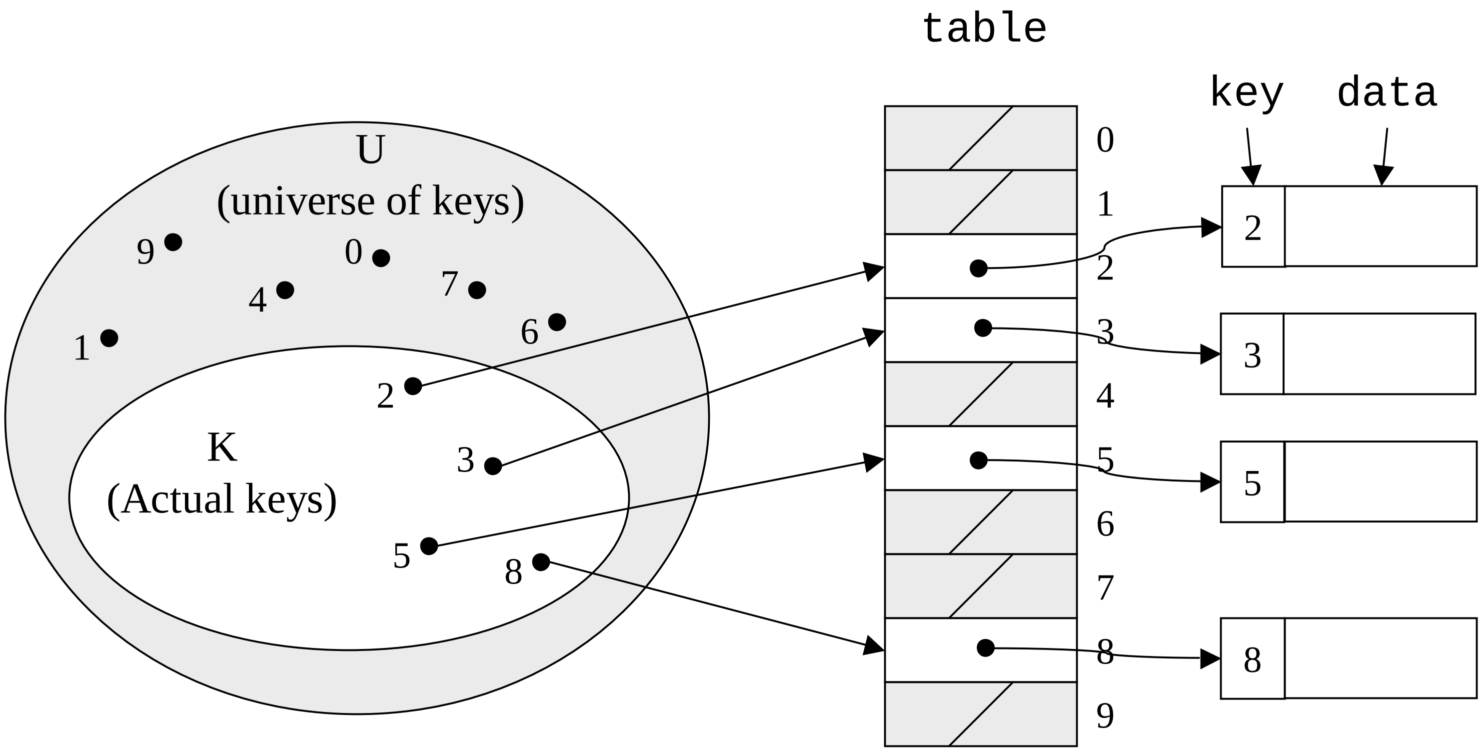
Limited Set of Hash Operations

- For many applications, a limited set of operations is all that is needed
  - › Insert, Find, and Delete
  - › Note that no ordering of elements is implied
- For example, a compiler needs to maintain information about the symbols in a program
  - › user defined
  - › language keywords

Direct Address Tables

- Direct addressing using an array is very fast
- Assume
  - keys are integers in the set U={0,1,…*m*-1}
  - *m* is small
  - no two elements have the same key
- Then just store each element at the array location array[key]
  - search, insert, and delete are trivial

Direct Access Table



[Cormen, et al]

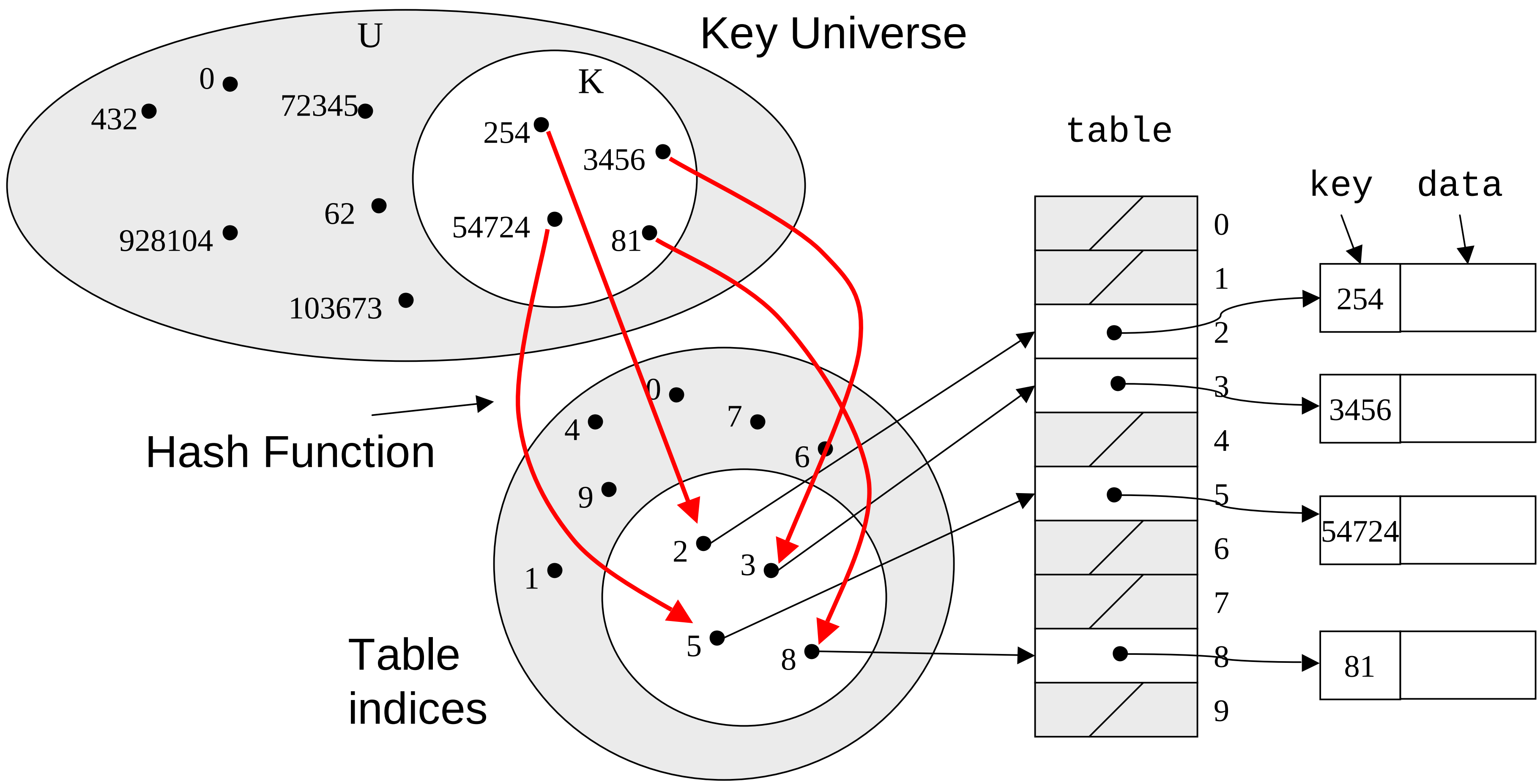
Direct Address Implementation

```
Delete(Table t, ElementType x)
   T[key[x]] = NULL

Insert(Table t, ElementType x)
   T[key[x]] = x

Find(Table t, Key k)
   return T[k]
```

An Issue

- The largest possible key in U may be much larger than the number of elements actually stored (|U| much greater than |K|)
  - › the table is very sparse and wastes space
  - › in worst case, table too large to have in memory
- If most keys in U are used
  - › direct addressing can work very well
- If most keys in U are not used
  - › need to map U to a smaller set closer in size to K

Mapping the Keys



Hashing Schemes

- We want to store N items in a table of size M, at a location computed from the key K
- Hash function
  - › Method for computing table index from key
- Collision resolution strategy
  - › How to handle two keys that hash to the same index

Looking for an Element

- Data records can be stored in arrays.
  - A[0] = {“CHEM 110”, Size 89}
  - A[3] = {“CSE 142”, Size 251}
  - A[17] = {“CSE 373”, Size 85}
- Class size for CSE 373?
  - Linear search the array – O(N) worst case time
  - Binary search - O(log N) worst case

Go Directly to the Element

- What if we could directly index into the array using the key?
  - › A[“CSE 373”] = {Size 85}
- Main idea behind hash tables
  - › Use a key based on some aspect of the data element to index directly into an array
  - › O(1) time to access records

Indexing into Hash Table

- Need a fast *hash function* to convert the element key (string or number) to an integer (the *hash value*) (ie, map from U to index)
  - › Then use this value to index into an array
  - › Hash(“CSE 373”) = 157, Hash(“CSE 143”) = 101
- Output of the hash function
  - › must always be less than size of array
  - › should be as evenly distributed as possible

Choosing the Hash Function

- What properties do we want from a hash function?
  - › Want universe of hash values to be distributed randomly to minimize collisions
  - › Don't want systematic nonrandom pattern in selection of keys to lead to systematic collisions
  - › Want hash value to depend on all values in entire key and their positions

The Key Values are Important

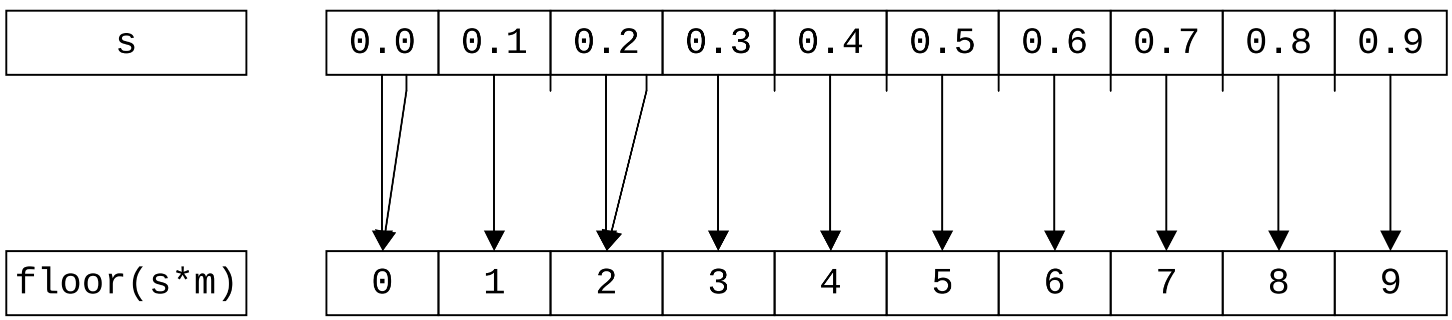
- Notice that one issue with all the hash functions is that the actual content of the key set matters
- The elements in K (the keys that are used) are quite possibly a restricted subset of U, not just a random collection
  - variable names, words in the English language, reserved keywords, telephone numbers, etc, etc

Simple Hashes

- It's possible to have very simple hash functions if you are certain of your keys
- For example,
  - › suppose we know that the keys *s* will be real numbers uniformly distributed over $0 \le s < 1$
  - › Then a very fast, very good hash function is
    - hash(s) = floor($s \cdot m$)
    - where *m* is the size of the table

Very Simple Mapping

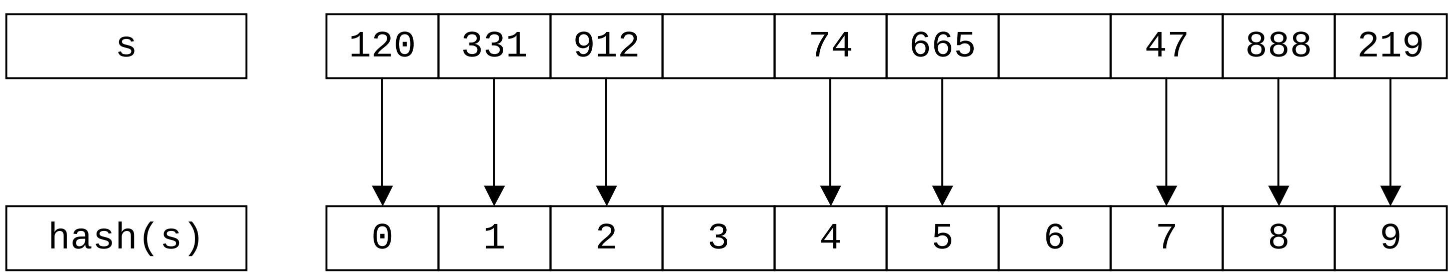
- hash(s) = floor($s \cdot m$) maps from $0 \leq s < 1$ to 0..m-1
  - › m = 10



Note the even distribution. There are collisions, but we will deal with them later.

Perfect Hashing

- In some cases it's possible to map a known set of keys uniquely to a set of index values
- You must know every single key beforehand and be able to derive a function that works *one-to-one* (not necessarily *onto*)

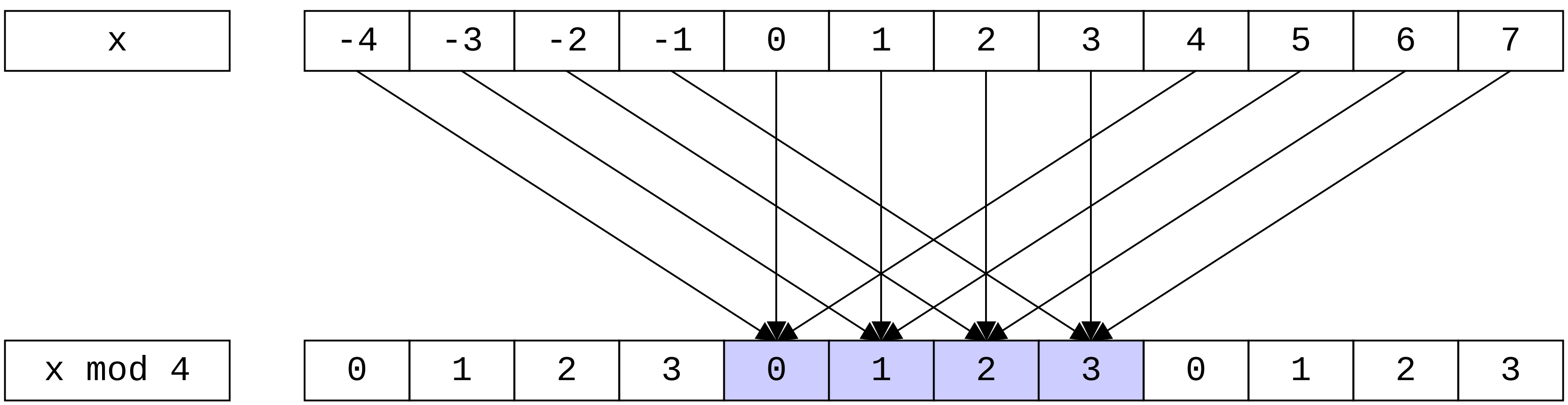


Mod Hash Function

- One solution for a less constrained key set
  - › modular arithmetic
- `a` **`mod`** `size`
  - › remainder when "a" is divided by "size"
  - › in C or Java this is written as **`r = a % size;`**
  - › If TableSize = 251
    - 408 mod 251 = 157
    - 352 mod 251 = 101

Modulo Mapping

- *a* mod *m* maps from integers to 0..m-1
  - › one to one? no
  - › onto? yes



Hashing Integers

- If keys are integers, we can use the hash function:
  - Hash(key) = key mod TableSize
- Problem 1: What if TableSize is 11 and all keys are 2 repeated digits? (eg, 22, 33, …)
  - all keys map to the same index
  - Need to pick TableSize carefully: often, a prime number

Nonnumerical Keys

- Many hash functions assume that the universe of keys is the natural numbers $\mathbf{N}=\{0,1,\ldots\}$
- Need to find a function to convert the actual key to a natural number quickly and effectively before or during the hash calculation
- Generally work with the ASCII character codes when converting strings to numbers

Characters to Integers

- If keys are strings can get an integer by adding up ASCII values of characters in *key*
- We are converting a very large string $c_0c_1c_2 \ldots c_n$ to a relatively small number $c_0+c_1+c_2+\ldots+c_n$ mod size.

character	C	S	E		3	7	3	<0>
ASCII value	67	83	69	32	51	55	51	0

Hash Must be Onto Table

- Problem 2: What if *TableSize* is 10,000 and all keys are 8 or less characters long?
  - › chars have values between 0 and 127
  - › Keys will hash only to positions 0 through 8*127 = 1016
- Need to distribute keys over the entire table or the extra space is wasted

Problems with Adding Characters

- Problems with adding up character values for string keys
  - › If string keys are short, will not hash evenly to all of the hash table
  - › Different character combinations hash to same value
    - “abc”, “bca”, and “cab” all add up to the same value

Characters as Integers

- An character string can be thought of as a base 256 number. The string $c_1c_2...c_n$ can be thought of as the number
  $c_n + 256c_{n-1} + 256^2c_{n-2} + \ldots + 256^{n-1}\, c_1$
- Use Horner's Rule to Hash!

```
r= 0;
for i = 1 to n do
r := (c[i] + 256*r) mod TableSize
```

Collisions

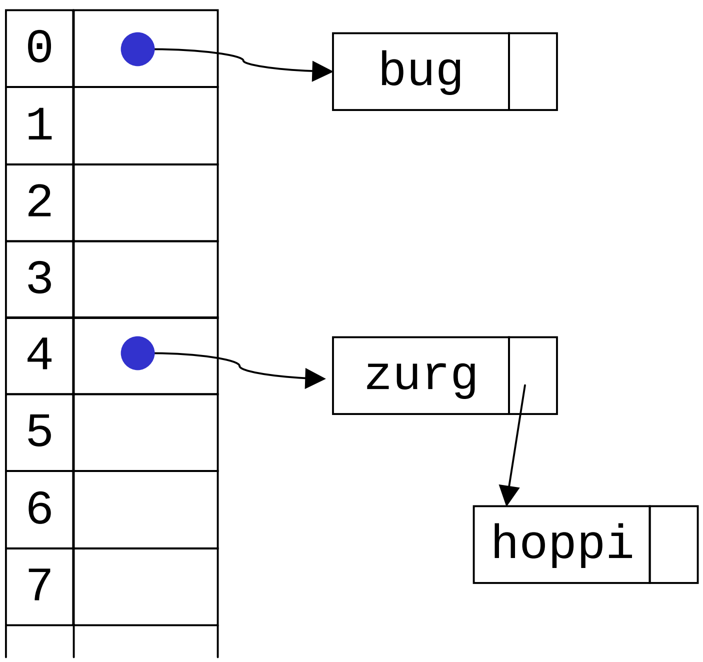
- A collision occurs when two different keys hash to the same value
  - › E.g. For *TableSize* = 17, the keys 18 and 35 hash to the same value
  - › 18 mod 17 = 1 and 35 mod 17 = 1
- Cannot store both data records in the same slot in array!

Collision Resolution

- Separate Chaining
  - › Use data structure (such as a linked list) to store multiple items that hash to the same slot
- Open addressing (or probing)
  - › search for empty slots using a second function and store item in first empty slot that is found

Resolution by Chaining

- Each hash table cell holds pointer to linked list of records with same hash value
- Collision: Insert item into linked list
- To Find an item: compute hash value, then do Find on linked list
- Note that there are potentially as many as TableSize lists



Why Lists?

- Can use List ADT for Find/Insert/Delete in linked list
  - › O(N) runtime where N is the number of elements in the particular chain
- Can also use Binary Search Trees
  - › O(log N) time instead of O(N)
  - › But the number of elements to search through should be small
  - › generally not worth the overhead of BSTs

Load Factor of a Hash Table

- Let N = number of items to be stored
- Load factor λ = N/TableSize
  - › TableSize = 101 and N =505, then $\lambda = 5$
  - › TableSize = 101 and N = 10, then $\lambda = 0.1$
- Average length of chained list = λ and so average time for accessing an item = $O(1) + O(\lambda)$
  - › Want λ to be close to 1 (i.e. TableSize $\approx$ N)
  - › But chaining continues to work for $\lambda > 1$

Resolution by Open Addressing

- No links, all keys are in the table
  - reduced overhead saves space
- When searching for **X**, check locations $h_1(X)$, $h_2(X)$, $h_3(X)$, … until either
  - **X** is found; or
  - we find an empty location (**X** not present)
- Various flavors of open addressing differ in which probe sequence they use

Cell Full? Keep Looking.

- $\mathbf{h_i(X)=(Hash(X)+F(i))\ mod\ TableSize}$
  - › Define $F(0) = 0$
- F is the collision resolution function. Some possibilities:
  - › Linear: $F(i) = i$
  - › Quadratic: $F(i) = i^2$
  - › Double Hashing: $F(i) = i \cdot Hash_2(X)$

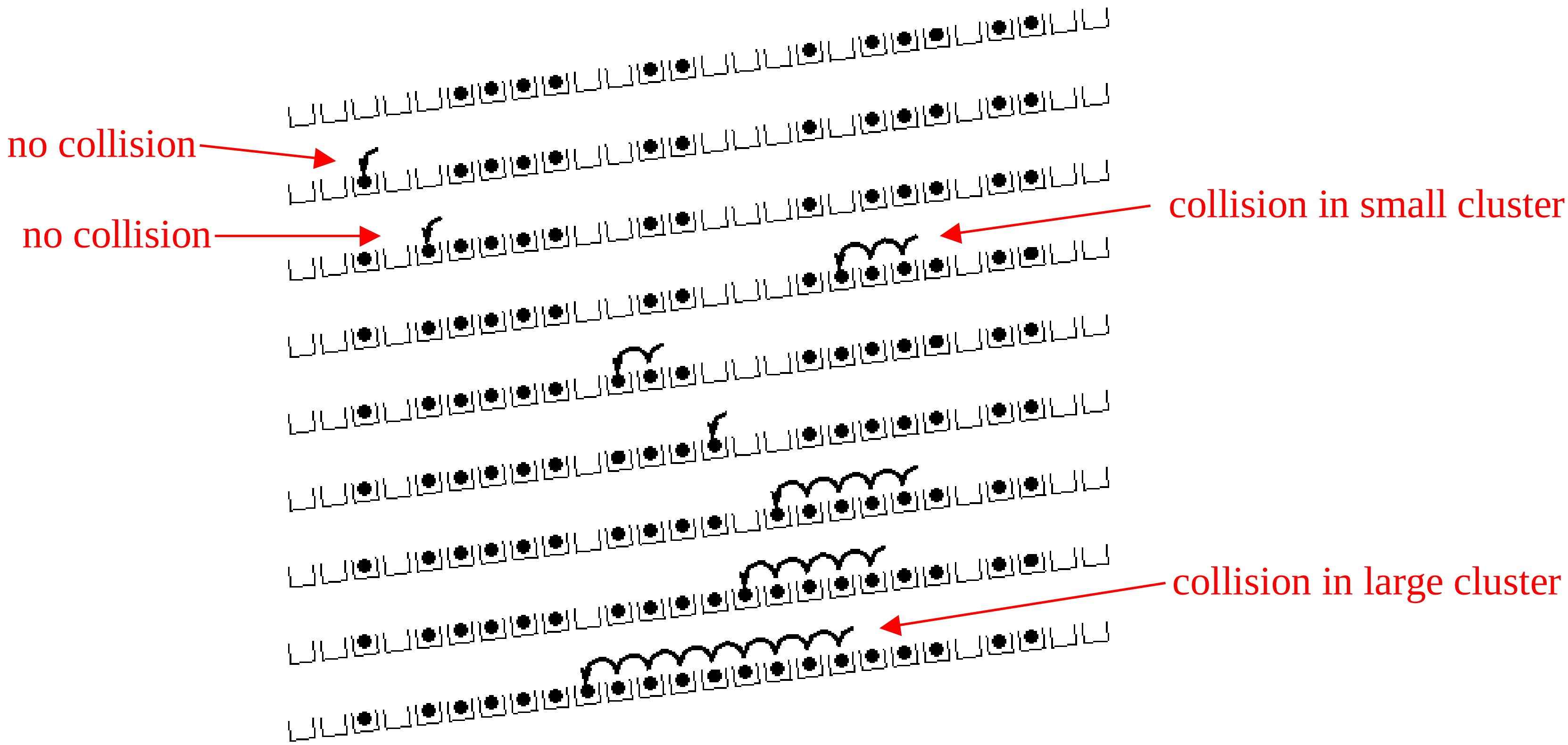
Linear Probing

- When searching for **`K`**, check locations **`h(K), h(K)+1, h(K)+2, …`** mod TableSize until either
  - **`K`** is found; or
  - we find an empty location (**`K`** not present)
- If table is very sparse, almost like separate chaining.
- When table starts filling, we get clustering but still constant average search time.
- Full table $\Rightarrow$ infinite loop.

Primary Clustering Problem

- Once a block of a few contiguous occupied positions emerges in table, it becomes a “target” for subsequent collisions
- As clusters grow, they also merge to form larger clusters.
- Primary clustering: elements that hash to different cells probe same alternative cells

Linear Probing – Clustering



[R. Sedgewick]

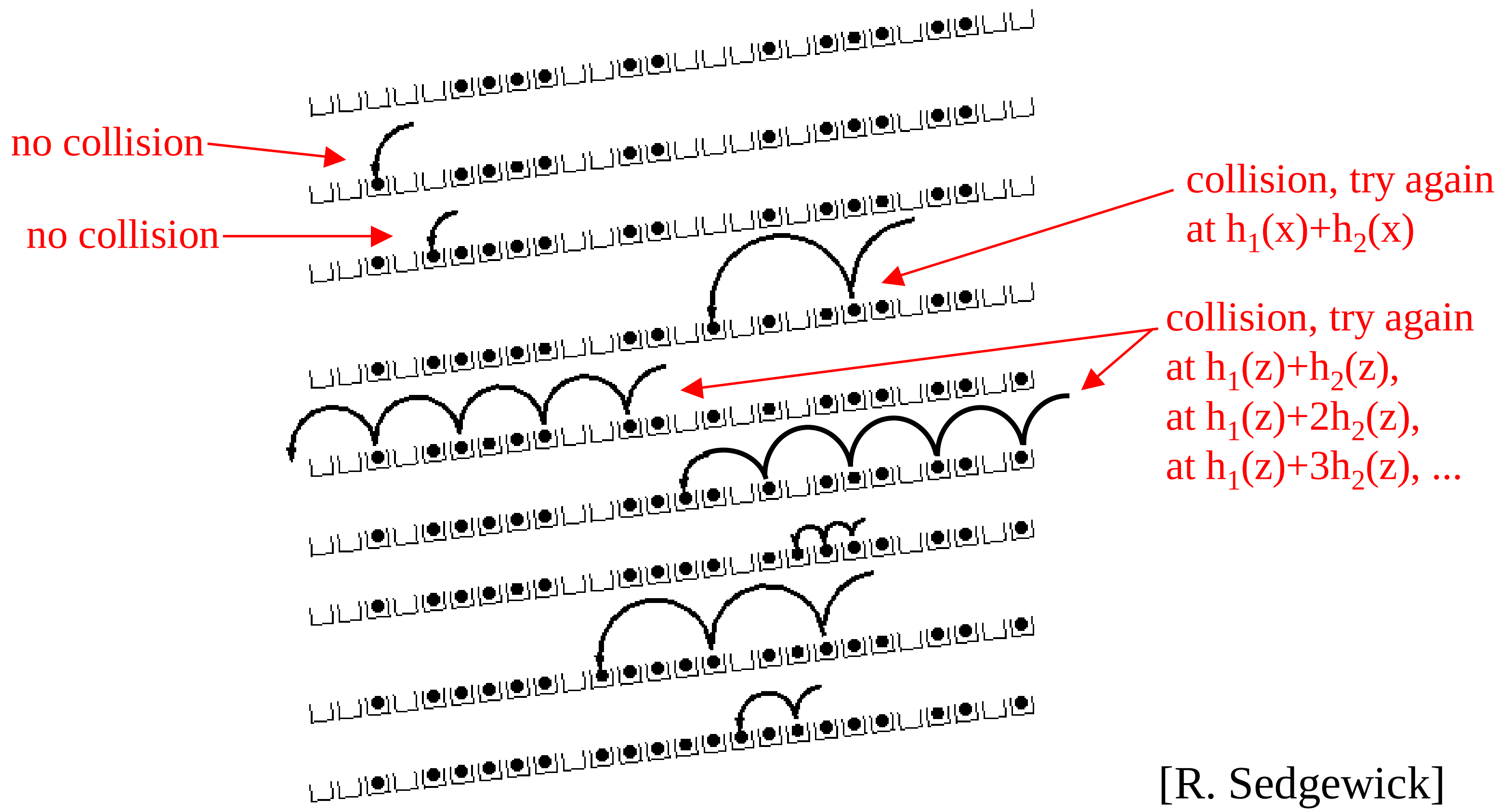
Quadratic Probing

- When searching for **`X`**, check locations $h_1(X)$, $h_1(X)+ i^2$, $h_1(X)+i^3$,… **`mod TableSize`** until either
  - › **`X`** is found; or
  - › we find an empty location (**`X`** not present)
- No primary clustering but secondary clustering possible

Double Hashing

- When searching for **X**, check locations $h_1(X)$, $h_1(X) + h_2(X)$, $h_1(X) + 2*h_2(X)$, … **mod Tablesize** until either
  - **X** is found; or
  - we find an empty location (**X** not present)
- Must be careful about $h_2(X)$
  - Not 0 and not a divisor of **M**
  - eg, $h_1(k) = k \bmod m_1$, $h_2(k) = 1 + (k \bmod m_2)$
  - where m_2 is slightly less than m_1

Double Hashing



[R. Sedgewick]

Rules of Thumb

- Separate chaining is simple but wastes space…
- Linear probing uses space better, is fast when tables are sparse, interacts well with paging and caching
- Double hashing is space efficient, fast (get initial hash and increment at the same time), needs careful implementation
- For average cost about t + O(1)
  - › Max load for Linear Probing is $1-1/\sqrt{t}$
  - › Max load for Double Hashing is $1-1/t$

Rehashing – Rebuild the Table

- Need to use lazy deletion if we use probing (why?)
  - Need to mark array slots as deleted after Delete
  - consequently, deleting doesn't make the table any less full than it was before the delete
- If table gets too full ($\lambda \approx 1$) or if many deletions have occurred, running time gets too long and Inserts may fail

Rehashing

- Build a bigger hash table of approximately twice the size when λ exceeds a particular value
  - Go through old hash table, ignoring items marked deleted
  - Recompute hash value for each non-deleted key and put the item in new position in new table
  - Cannot just copy data from old table because the bigger table has a new hash function
- Running time is O(N) but happens very infrequently
  - Not good for real-time safety critical applications

Rehashing Example

- Open hashing – $h_1(x) = x \bmod 5$ rehashes to $h_2(x) = x \bmod 11$.

$\lambda = 1$

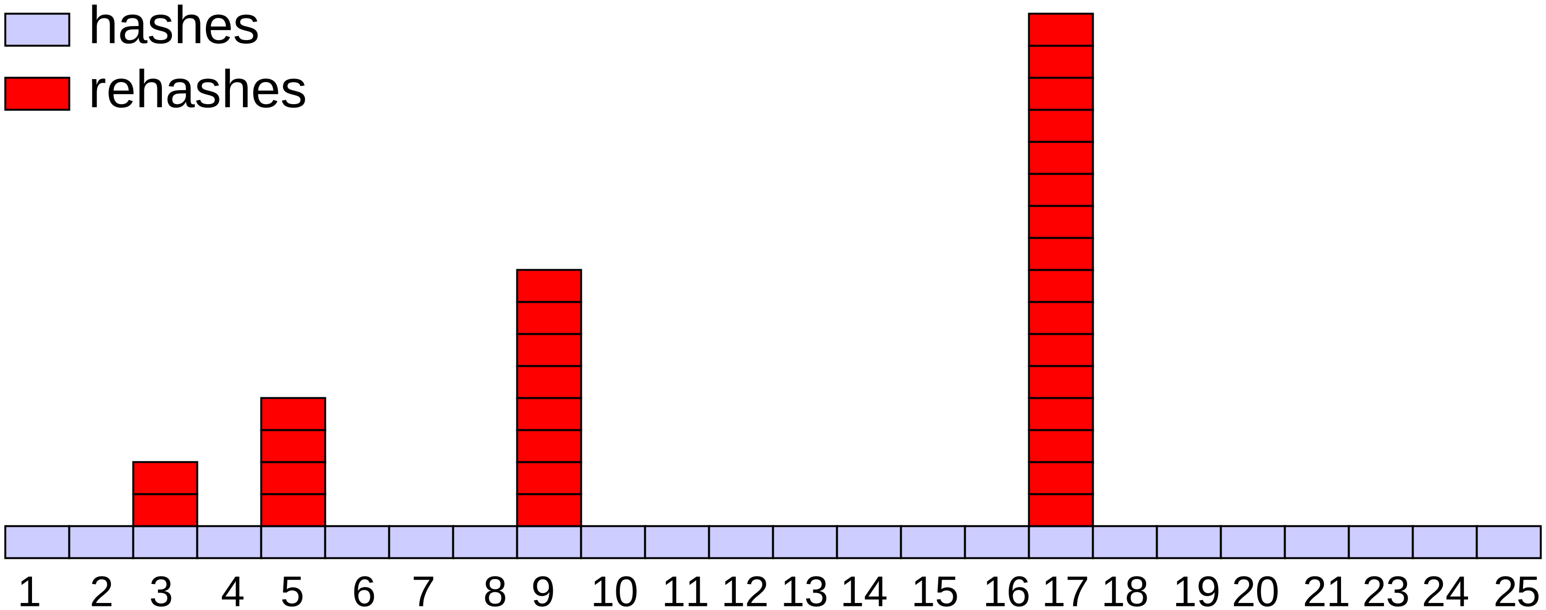
0	1	2	3	4
25		37	83	
		52	98	

$\lambda = 5/11$

0	1	2	3	4	5	6	7	8	9	10
			25	37		83		52		98

Rehashing Picture

- Starting with table of size 2, double when load factor > 1.



Amortized Analysis of Rehashing

- Cost of inserting n keys is < 3n
- $2^k + 1 \leq n \leq 2^{k+1}$
  - › Hashes = n
  - › Rehashes = $2 + 2^2 + \ldots + 2^k = 2^{k+1} - 2$
  - › Total = $n + 2^{k+1} - 2 < 3n$
- Example
  - › n = 33, Total = 33 + 64 –2 = 95 < 99

Caveats

- Hash functions are very often the cause of performance bugs.
- Hash functions often make the code not portable.
- Sometime a poor HF distribution-wise is faster overall.
- Always check where the time goes