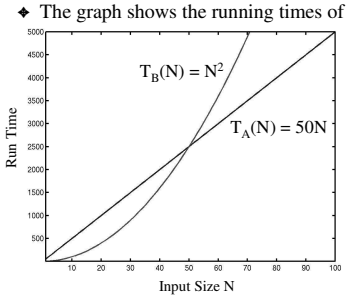
CSE 373 Lecture 3: Analysis of Algorithms

- Today, we will review:
  - Big-Oh, Little-Oh, Omega (Ω), and Theta (Θ): (Fraternities of functions…)
  - Examples of time and space efficiency analysis
- Covered in Chapter 2 of the text

Recall from Last Time: Big-Oh Notation

- The graph shows the running times of algorithms A and B:



$T_A(N) = O(T_B(N))$

$T_A(N)$ is "Big-Oh" of $T_B(N)$ because $50N \leq N^2$ for $N \geq 50$.

Actually, for $N \geq 51$, $50N < N^2$

Big-Oh, Omega, Theta, and Little-Oh

- $T(N) = O(f(N))$ if there are positive constants c and n_0 such that $T(N) \leq cf(N)$ for $N \geq n_0$.
  - 100 log N, $N^{0.9}$, 0.0001 N, 2^{100} N + log N are all = O(N)
- $T(N) = \Omega(f(N))$ if there are positive constants c and n_0 such that $T(N) \geq cf(N)$ for $N \geq n_0$.
  - 2^N, $N^{\log N}$, $N^{1.2}$, 0.0001 N, N + log N are all = Ω(N)
- $T(N) = \Theta(f(N))$ iff $T(N) = O(f(N))$ and $T(N) = \Omega(f(N))$
  - 0.0001 N, 2^{100} N + log N are all = Θ(N)
- $T(N) = o(f(N))$ iff $T(N) = O(f(N))$ and $T(N) \neq \Theta(f(N))$
  - 100 log N, $N^{0.9}$, sqrt(N), 17 are all = o(N)

Big-Oh, Omega, Theta, and Little-Oh

- Tips:
- Think of O(f(N)) as "less than or equal to" f(N)
  - Upper bound
- Think of Ω(f(N)) as "greater than or equal to" f(N)
  - Lower bound
- Think of Θ(f(N)) as "equal to" f(N)
  - "Tight" bound, same growth rate
- Think of o(f(N)) as "strictly less than" f(N)…
  - Strict upper bound
  - T(N) = o(f(N)) means f(N) has faster growth rate than T(N)
- (*for large N and ignoring constant factors*)

Big-Oh Analysis: Example 1

Problem: Find the sum of the first num integers stored in array v. Assume num ≤ size of v.

```
int sum ( int v[ ], int num)
  {
  int temp_sum = 0, i;
  for (  i = 0; i < num; i++  )
       temp_sum = temp_sum + v[i] ;
  return temp_sum;
  }
```

Running time = ?

Big-Oh Analysis: Example 1

Problem: Find the sum of the first num integers stored in array v. Assume num ≤ size of v.

```
int sum ( int v[ ], int num)
  {
  int temp_sum = 0, i;                    //1
  for (  i = 0; i < num; i++  )           //2
       temp_sum = temp_sum + v[i] ;       //3
  return temp_sum;                        //4
  }
```

- i goes from 0 to num-1= num iterations
- lines 1, 3, and 4 take fixed (constant) amount of time
- Running time = constant + (num)*constant = O(num)
- Actually, Θ(num)

Big-Oh Analysis: Example 1 (Recursion)

Recursive function to find the sum of the first num integers stored in array v:

```
int sum ( int v[ ], int num)
  {
    if (num == 0) return 0;
    else return v[num-1] + sum(v,num-1);
  }
```

- Running time = ?

Big-Oh Analysis: Example 1 (Recursion)

Recursive function to find the sum of first num integers in v:

```
int sum ( int v[ ], int num)
  {
    if (num == 0) return 0;  // constant time here
    else return v[num-1] + sum(v,num-1);
                // constant time + T(num-1)
  }
```

- Let T(num) be the running time of sum
- Then, T(num) = constant + T(num-1)
- = 2*constant + T(num-2) =…= num*constant + constant
- = Θ(num) (same as iterative algorithm!)

Big-Oh Analysis of Recursive Algorithms

- Common recurrence relations in analysis of algorithms:
  - $T(N) = T(N-1) + \Theta(1) \Rightarrow T(N) = O(N)$
  - $T(N) = T(N-1) + \Theta(N) \Rightarrow T(N) = O(N^2)$
  - $T(N) = T(N/2) + \Theta(1) \Rightarrow T(N) = O(\log N)$
  - $T(N) = 2T(N/2) + \Theta(N) \Rightarrow T(N) = O(N \log N)$
  - How do you get these? Just expand the right side and count!
- Note: Multiplicative constants matter in recurrence relations:
  - $T(N) = 4T(N/2) + \Theta(N)$ is $O(N^2)$, not $O(N \log N)$!
- You will see these again later – you will only need to know a few specific relations and their big-oh answers

Example 2: Fibonacci Numbers

- Recall the tale of Rabbits and Fibonacci numbers F_n

1, 1, 2, 3, 5, 8, 13, 21, 34, …

Leonardo Pisano
Fibonacci (1170-1250)

- First two are defined to be 1
- Rest are sum of preceding two
- $F_n = F_{n-1} + F_{n-2}$ $(n > 1)$

Example 2: Recursive Fibonacci

```
int fib(int N) {
  if (N < 0) return 0;  //invalid input
  if (N == 0 || N == 1) return 1;  //base cases
  else return fib(N-1)+fib(N-2);
  }
```

- Running time T(N) = ?

Example 2: Recursive Fibonacci

```
int fib(int N) {
  if (N < 0) return 0;  // time = 1 for (N < 0)
  if (N == 0 || N == 1) return 1;  // time = 3
  else return fib(N-1)+fib(N-2); //T(N-1)+T(N-2)+1
  }
```

- Running time $T(N) = T(N-1) + T(N-2) + 5$
- Using $F_n = F_{n-1} + F_{n-2}$ we can show by induction that $T(N) \geq F_N$. We can also show by induction that $F_N \geq (3/2)^N$
- Therefore, $T(N) \geq (3/2)^N$ or, $T(N) = \Omega((1.5)^N)$
  - Exponential running time!

Example 2: Iterative Fibonacci

```
int fib_iter(int N) {
        int fib0 = 1, fib1 = 1, fibj = 1;
        if (N < 0) return 0;  //invalid input
        for (int j = 2; j <= N; j++) { //all fib nos. up to N
                fibj = fib0 + fib1;
                fib0 = fib1;
                fib1 = fibj;
        }
        return fibj;
}
```

- Running time = ?

Example 2: Iterative Fibonacci

```
int fib_iter(int N) {
        int fib0 = 1, fib1 = 1, fibj = 1; //constant time
        if (N < 0) return 0;  //constant time
        for (int j = 2; j <= N; j++) { //N-1 iterations
                fibj = fib0 + fib1;
                fib0 = fib1;
                fib1 = fibj;
        }
        return fibj;
}
```

- Running time T(N) = constant+(N-1)•constant
- T(N) = Θ(N)
  - Exponentially faster than recursive Fibonacci.

Example 3: Time and Space Tradeoffs

- Problem DUP: Given an array A of n positive integers, are there any duplicates?
- For example, A: 34, 9, 40, 87, 223, 109, 58, 9, 71, 8
- An easy algorithm for DUP:

```
for (i = 0; i < N-1; i++)
    for (j = i+1; j < N; j++)
        if (A[i] == A[j]) {
            printf("Duplicates!\n"); return 0;}
printf("No Duplicates!\n");
```

- Space required = ?

Example 3: Time and Space Tradeoffs

- Problem DUP: Given an array A of n positive integers, are there any duplicates?
- An easy algorithm for DUP:

```
for (i = 0; i < N-1; i++)
    for (j = i+1; j < N; j++)
        if (A[i] == A[j]) {
            printf("Duplicates!\n"); return 0; }
printf("No Duplicates!\n");
```

- Space required = N + 2 = Θ(N)
  - *Does not depend on values stored in A*
- Running time: How many steps in the worst case?

Example 3: Time and Space Tradeoffs

- Analyze the running time of easy algorithm for DUP:

```
for (i = 0; i < N-1; i++) // N-1 iterations
    for (j = i+1; j < N; j++) // N-i-1 iterations
        if (A[i] == A[j]) { // constant time c
            printf("Duplicates!\n"); return 0; }
printf("No Duplicates!\n");
```

- How many steps in the worst case?
  - Worst case = no duplicates. Total time steps = ?

$$\sum_{i=1}^{N-1}\sum_{j=i+1}^{N} c = \sum_{i=1}^{N-1} c(N-i-1) = c\sum_{i=1}^{N-1} N - c\sum_{i=1}^{N-1} i - c(N-1)$$

$$= cN(N-1) - c\frac{(N-1)N}{2} - c(N-1) = \Theta(N^2)$$

Example 3: Trading more space for less time

- New Algorithm for DUP:
  - Idea: Use A[i] as index into new array B initialized to 0's

```
for (i = 0; i < N; i++)
   if (B[A[i]] == 1) {
     printf("Duplicates!\n"); return 0; }
   else B[A[i]] = 1;
printf("No Duplicates!\n");
```

  - Similar to detecting collisions in hashing (Chapter 5)
- Worst Case Running Time = ?
- Space Required = ?

Example 3: Trading more space for less time

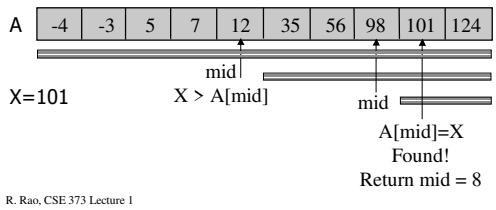
- New Algorithm for DUP:

```
for (i = 0; i < N; i++) // N iterations
   if (B[A[i]] == 1) {
     printf("Duplicates!\n"); return 0; }
   else B[A[i]] = 1;
printf("No Duplicates!\n");
```

- Worst Case Running Time = O(N)
- Space Required = $O(2^m)$ where m is the number of bits required to represent the largest value A[i] that can potentially occur in A. E.g. m = 8 if max A[i] = 255.
- Prev. algorithm: more time [$O(N^2)$] but less space [O(N)]
- Such tradeoffs between space and time are common…

Example 4: Searching for an Item

- Problem: Search for an item X in a sorted array A. Return index of item if found, otherwise return –1.
- Idea: Compare X with middle item A[mid], go to left half if X < A[mid] and right half if X > A[mid]. Repeat.



Example 4: Binary Search

A	-4	-1	5	7	12	35	56	98	101	124

```
int BinarySearch( const ElementType A[ ], ElementType X, int N )
{
    int Low, Mid, High;
        Low = 0; High = N - 1;
        while( Low <= High )
        {
                Mid = ( Low + High ) / 2; // Find middle of array index
                if( X > A[ Mid ] )  // Search second half of array
                    Low = Mid + 1;
                else if( X < A[ Mid ] )        // Search first half
                            High = Mid - 1;
                else return Mid;    // Found X!
        }
        return -1;
}
```


Example 4: Running Time of Binary Search

- Given an array A with N elements, what is the worst case running time of BinarySearch?
- Think about it over the weekend…
- We will discuss the answer in the next class.

For you to do:

Homework no. 1 (due next Friday)

Begin reading Chapters 3 and 4

Next Week:

1. Review of Lists, Stacks, and Queues
2. The wonderful world of Trees!

Have a great weekend!