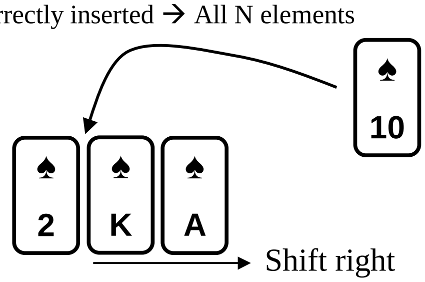
CSE 373 Lecture 16: Sorting Faster and Faster…

- What's on our plate today?
  - Faster sorting Algorithms:
    - Shellsort
    - Heapsort
    - Mergesort
- Covered in Chapter 7 of the textbook

Recall from Last Time: Insertion Sort

- Main Idea:
  - Start with 1^{st} element, insert 2^{nd} if < 1^{st} after shifting 1^{st} element → First 2 are now sorted…
  - Insert 3^{rd} after shifting 1^{st} and/or 2^{nd} as needed → First 3 sorted…
  - Repeat until last element is correctly inserted → All N elements sorted
- Example: Sort 19, 5, 2, 1
  - <u>5, 19,</u> 2, 1 (shifted 19)
  - <u>2, 5, 19</u>, 1 (shifted 5, 19)
  - <u>1, 2, 5, 19</u> (shifted 2, 5, 19)



- Running time:
  - Worst case → reverse order input = $\Theta(N^2)$
  - Best case → input already sorted = O(N)

Shellsort: Motivation

- Main Insight: Insertion sort runs fast on nearly sorted sequences → do *several passes of Insertion sort* on different subsequences of elements
- Example: Sort 19, 5, 2, 1
  1. Do Insertion sort on subsequences of elements spaced apart by 2: 1st and 3rd, 2nd and 4th
     - 19, *5*, 2, *1* → 2, *1*, 19, *5*
  2. Do Insertion sort on subsequence of elements spaced apart by 1:
     - 2, 1, 19, 5 → 1, 2, 19, 5 → 1, 2, 19, 5 → 1, 2, 5, 19
- Note: Fewer number of shifts than plain Insertion sort
  - 4 versus 6 for this example

Shellsort: Overview

- Named after Donald Shell – first algorithm to achieve $o(N^2)$
  - Running time is $O(N^x)$ where x = 3/2, 5/4, 4/3, …, or 2 depending on "increment sequence"
- In our example, we used the increment sequence: N/2, N/4, …, 1 = 2, 1 (for N = 4 elements)
  - This is Shell's original increment sequence
- Shellsort: Pick an *increment sequence* $h_t > h_{t-1} > \ldots > h_1$
  - Start with k = t
  - Insertion sort all subsequences of elements that are h_k apart so that $A[i] \leq A[i+h_k]$ for all i → known as an *h_k-sort*
  - Go to next smaller increment h_{k-1} and repeat until k = 1 (note: $h_1 = 1$)

Shellsort: Nuts and Bolts

```
void Shellsort( ElementType A[ ], int N ){
  int i, j, Increment; ElementType Tmp;
  for( Increment = N/2; Increment > 0; Increment /= 2 )
    for( i = Increment; i < N; i++ )                {
      Tmp = A[ i ];
      for( j = i; j >= Increment; j -= Increment )
             if( Tmp < A[ j - Increment ] )
                    A[ j ] = A[ j - Increment ];
             else
             break;
      A[ j ] = Tmp;
    }
}
```

- Note: The two inner `for` loops correspond almost exactly to the code for Insertion sort!
- Running time = ? (What is the worst case?)

Shellsort: Analysis

- Simple to code but hard to analyze → depends on increment sequence
- What about the increment sequence N/2, N/4, …, 2, 1?
  - Upper bound
    - Shellsort does h_k insertions sort with N/h_k elements for k = 1 to t
    - Running time = $O(\Sigma_{k=1\ldots t}\, h_k\, (N/h_k)^2) = O(N^2\, \Sigma_{k=1\ldots t}\, 1/h_k) = \mathbf{O(N^2)}$
  - Lower bound
    - What is the worst case?

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  - Lower bound
    - What is the worst case?
    - Smallest elements in odd positions, largest in even positions
      - 2, 11, 4, 12, 6, 13, 8, 14
    - None of the passes N/2, N/4, …, 2 do anything!
    - Last pass (h_1 = 1) must shift N/2 smallest elements to first half and N/2 largest elements to second half → 4 shifts 1 slot, 6 shifts 2, 8 shifts 3, … = 1 + 2 + 3 + … (N/2 terms)
    - at least N^2 steps = $\mathbf{\Omega(N^2)}$

Shellsort: Breaking the $O(N^2)$ Barrier

- The reason we got $\mathbf{\Omega(N^2)}$ was because of increment sequence
  - Adjacent increments have common factors (e.g. 8, 4, 2, 1)
  - We keep comparing same elements over and over again
  - Need to increment so that different elements are in different passes
- Hibbard's increment sequence: $2^k - 1, 2^{k-1} - 1, \ldots, 7, 3, 1$
  - Adjacent increments have no common factors
  - Worst case running time of Shellsort with Hibbard's increments = $\mathbf{\Theta(N^{1.5})}$ (Theorem 7.4 in text)
  - Average case running time for Hibbard's = $\mathbf{O(N^{1.25})}$ in simulations but *nobody has been able to prove it!* (next homework assignment?)
- Final Thoughts: Insertion sort good for small input sizes (~20); Shellsort better for moderately large inputs (~10,000)

Hey…How about using Binary Search Trees?

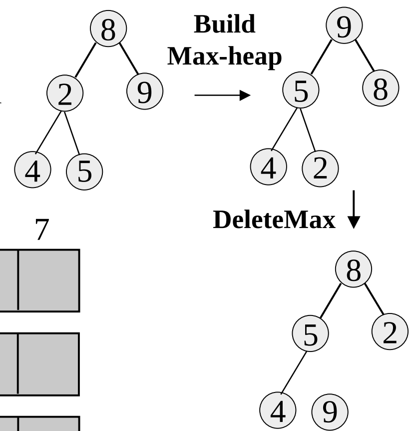
- Can we beat O($\mathbf{N}^{\mathbf{1.5}}$) using a BST to sort N elements?

Using Binary Search Trees for Sorting

- Can we beat O($\mathbf{N}^{\mathbf{1.5}}$) using a BST to sort N elements?
  - Yes!!
  - Insert each element into an initially empty BST
  - Do an In-Order traversal to get sorted output
- Running time: N Inserts, each takes O(log N) time, plus O(N) for In-Order traversal = **O(N log N)** = o($N^{1.5}$)
- Drawback – Extra Space: Need to allocate space for tree nodes and pointers → O(N) extra space, not *in place* sorting
- Waittaminute…what if the tree is complete, and we use an array representation – can we sort in place?
  - Recall your favorite data structure with the initials B. H.

Using Binary Heaps for Sorting

- Main Idea:
  - Build a max-heap
  - Do N DeleteMax operations and store each Max element in the unused end of array



0	1	2	3	4	5	6	7
8	2	9	4	5			
9	5	8	4	2			
8	5	2	4	9			

Largest element in correct place

Heapsort: Analysis

- Running time = time to build max-heap + time for N DeleteMax operations = ?

Heapsort: Analysis

- Running time = time to build max-heap + time for N DeleteMax operations = O(N) + N O(log N) = **O(N log N)**
- Can also show that running time is Ω(N log N) for some inputs, so *worst case* is **Θ(N log N)**
- *Average case* running time is also O(N log N) (see text for proof if you are interested)

How about a "Divide and Conquer" strategy?

- Very important strategy in computer science:
  1. Divide problem into smaller parts
  2. Independently solve the parts
  3. Combine these solutions to get overall solution
- **Idea**: Divide array into two halves, *recursively* sort left and right halves, then *merge* two halves → known as Mergesort
- Example: Mergesort the input array:

0	1	2	3	4	5	6	7
8	2	9	4	5	3	1	6

Questions to ponder over the Weekend

Is Mergesort an in place sorting algorithm?

What is the running time for Mergesort?

How can I find time to read Chapter 7?

What is the meaning of life? (extra credit)

Have a good weekend!