Lecture 3:
All Hail George Boole

CSE 370, Autumn 2007
Benjamin Ylvisaker

Where We Are

- Last lecture: Binary numbers & arithmetic
- This lecture: Boolean algebra
- Next lecture: Playing around w/ Boolean functions
- Homework 1 due Wednesday at the beginning of class
- Lab 1 this week. Read it before the session starts!

Boolean Logic/Algebra

- Notation for writing down precise logical statements (in propositional logic)
- Primitives: true, false, variables
- Connectives: NOT, AND, OR, IMPLIES, ...
- (Almost) all memoryless digital circuits can be seen as Boolean algebra expressions
Why Do We Care?

- Understanding Boolean logic helps us design “simpler” circuits, both by hand and automatically.
- \[((A \text{ AND } B) \text{ OR } (\text{NOT } A \text{ AND } B)) \text{ AND } A\]
- Equivalent to: \(A \text{ AND } B\)

Lots of Alternative Notations

- I will mostly use:
  - \(\neg A\) for NOT \(A\)
  - \(A + B\) for \(A \text{ OR } B\)
  - \(A \cdot B\) for \(A \text{ AND } B\)
- Book lists all of the common notations

From Expressions to Gates

- NOT \(A\) \quad \begin{tikzpicture}[baseline]
  \node (a) at (0,0) {$A$};
  \node (b) [right of=a, shape=circle, draw] {};
  \draw (a) -- (b);
\end{tikzpicture}

- A OR B \quad \begin{tikzpicture}[baseline]
  \node (a) at (0,0) {$A$};
  \node (b) [right of=a] {$B$};
  \node (c) [above of=a, shape=circle, draw] {};
  \draw (a) -- (c);
  \draw (b) -- (c);
\end{tikzpicture}

- A AND B \quad \begin{tikzpicture}[baseline]
  \node (a) at (0,0) {$A$};
  \node (b) [right of=a] {$B$};
  \node (c) [above of=a, shape=rectangle, draw] {};
  \draw (a) -- (c);
  \draw (b) -- (c);
\end{tikzpicture}
The Useful Theorems

• Several slides of statements of basic facts about Boolean algebra
• Every theorem comes with a "dual"

0 and 1

• \(X + 0 = X\)
• \(X \cdot 1 = X\)
• \(X + 1 = 1\)
• \(X \cdot 0 = 0\)

Idempotence

• \(X + X = X\)
• \(X \cdot X = X\)
Involution

• \( \neg \neg X = X \)

Complementarity

• \( X + \neg X = 1 \)
• \( X \cdot \neg X = 0 \)

Commutativity

• \( X + Y = Y + X \)
• \( X \cdot Y = Y \cdot X \)
Associativity

- \((X+Y)+Z = X+(Y+Z)\)
- \((X\cdot Y)\cdot Z = X\cdot(Y\cdot Z)\)
- \(X+Y+Z = X\cdot Y\cdot Z\)

Distributivity

- \(X\cdot(Y+Z) = (X\cdot Y) + (X\cdot Z)\)
- \(X+(Y\cdot Z) = (X+Y)\cdot(X+Z)\)

Some Simplifications

- \((X\cdot Y)+(X\cdot Y) = X\)
- \((X\cdot Y)\cdot(X\cdot Y) = X\)
- \(X\cdot(X\cdot Y) = X\cdot Y\)
- \((X\cdot Y)\cdot Y = X\cdot Y\)
Prove Simplification 1

- \((X \cdot Y) \cdot (X \cdot \neg Y) = X\)
  - By distributivity
- \(X \cdot (Y \cdot \neg Y) = X\)
  - By complementarity
- \(X \cdot 1 = X\)
  - By identity
- \(X = X\)
  - By identity

Prove Simplification 2

- \(X \cdot (X \cdot Y) = X\)
  - By identity
- \((X \cdot 1) \cdot (X \cdot Y) = X\)
  - By distributivity
- \(X \cdot (X \cdot 1) = X\)
  - By identity
- \(X \cdot (X \cdot 1) = X\)
  - By identity
- \(X = X\)
  - By identity

Prove Simplification 3

- \((X \cdot Y) \cdot (X \cdot Y) = X \cdot Y\)
  - By simplification 2
- \((X \cdot Y) \cdot (X \cdot Y) = X \cdot Y\)
  - By associativity
- \((X \cdot Y) \cdot (X \cdot Y) = X \cdot Y\)
  - By distributivity
- \((X \cdot Y) \cdot (X \cdot Y) = X \cdot Y\)
  - By distributivity
- \((X \cdot Y) \cdot (X \cdot Y) = X \cdot Y\)
  - By associativity, idempotence and complementarity
- \((X \cdot Y) \cdot (X \cdot Y) = X \cdot Y\)
  - By operations with 1 and 0
- \(X \cdot Y = X \cdot Y\)
  - By operations with 1 and 0
- \(X \cdot Y = X \cdot Y\)
  - By operations with 1 and 0
- \(X \cdot Y = X \cdot Y\)
  - By operations with 1 and 0
DeMorgan’s law (or theorem)

- \( \neg(X+Y) = \neg X \cdot \neg Y \)
- \( \neg(X \cdot Y) = \neg X + \neg Y \)

Duality

- A Boolean function is just an expression with a name and a "parameter list" of variables used in the expression
- \( f(A,B,C) = (A\cdot B) + C \)
- The dual of a function (written \( f(A,B,C)^D \)) is the function with \( \cdot \)'s and \(+\)'s swapped and \( 1 \)'s and \( 0 \)'s swapped
- \( f(A,B,C)^D = (A+B) \cdot C \)

A Bigger Circuit Diagram

- \( (X\cdot Y\cdot Y) + (-Y\cdot Y) \)
Real Circuits Can Hurt You

- Current flows from higher voltages to lower voltages
- $I = V_{CC} - V_{GND}$
- Must always hook logic chips up to power and ground
- Never connect the outputs of logic gates together!

Thank You for Your Attention

- Read the lab assignment before you show up for your session!
- Continue reading the book
- Continue homework 1