Finite state machine optimization

- State minimization
  - fewer states require fewer state bits
  - fewer bits require fewer logic equations

- Encodings: state, inputs, outputs
  - state encoding with fewer bits has fewer equations to implement
    - however, each may be more complex
  - state encoding with more bits (e.g., one-hot) has simpler equations
    - complexity directly related to complexity of state diagram
  - input/output encoding may or may not be under designer control

Algorithmic approach to state minimization

- Goal – identify and combine states that have equivalent behavior

- Equivalent states:
  - same output
  - for all input combinations, states transition to same or equivalent states

- Algorithm sketch
  1. place all states in one set
  2. initially partition set based on output behavior
  3. successively partition resulting subsets based on next state transitions
  4. repeat (3) until no further partitioning is required
    - states left in the same set are equivalent
  - polynomial time procedure
State minimization example

- Sequence detector for 010 or 110

<table>
<thead>
<tr>
<th>Input Sequence</th>
<th>Present State</th>
<th>Next State</th>
<th>Output</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>X=0</td>
<td>X=1</td>
</tr>
<tr>
<td>Reset</td>
<td>S0</td>
<td>S1</td>
<td>S2</td>
</tr>
<tr>
<td>0</td>
<td>S1</td>
<td>S3</td>
<td>S4</td>
</tr>
<tr>
<td>1</td>
<td>S2</td>
<td>S5</td>
<td>S6</td>
</tr>
<tr>
<td>00</td>
<td>S3</td>
<td>S0</td>
<td>S0</td>
</tr>
<tr>
<td>01</td>
<td>S4</td>
<td>S0</td>
<td>S0</td>
</tr>
<tr>
<td>10</td>
<td>S5</td>
<td>S0</td>
<td>S0</td>
</tr>
<tr>
<td>11</td>
<td>S6</td>
<td>S0</td>
<td>S0</td>
</tr>
</tbody>
</table>

S1 is equivalent to S2
S3 is equivalent to S5
S4 is equivalent to S6

Method of successive partitions

<table>
<thead>
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</tr>
<tr>
<td>0</td>
<td>S1</td>
<td>S3</td>
<td>S4</td>
</tr>
<tr>
<td>1</td>
<td>S2</td>
<td>S5</td>
<td>S6</td>
</tr>
<tr>
<td>00</td>
<td>S3</td>
<td>S0</td>
<td>S0</td>
</tr>
<tr>
<td>01</td>
<td>S4</td>
<td>S0</td>
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</tr>
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<td>S0</td>
</tr>
<tr>
<td>11</td>
<td>S6</td>
<td>S0</td>
<td>S0</td>
</tr>
</tbody>
</table>

(S0 S1 S2 S3 S4 S5 S6) S1 is equivalent to S2
(S0 S1 S2 S3 S5) (S4 S6) S3 is equivalent to S5
(S0 S3 S5) (S1 S2) (S4 S6) S4 is equivalent to S6
(S0) (S3 S5) (S1 S2) (S4 S6)
Minimized FSM

State minimized sequence detector for 010 or 110

<table>
<thead>
<tr>
<th>Input Sequence</th>
<th>Present State</th>
<th>Next State ( X=0 )</th>
<th>Output ( X=0 )</th>
<th>Next State ( X=1 )</th>
<th>Output ( X=1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 + 1</td>
<td>S0</td>
<td>S1'</td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>0 + 1</td>
<td>S1'</td>
<td>S3'</td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>X0</td>
<td>S3'</td>
<td>S0</td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>X1</td>
<td>S4'</td>
<td>S0</td>
<td>1</td>
<td>0</td>
<td></td>
</tr>
</tbody>
</table>

More complex state minimization

Multiple input example

Inputs here

Present state transition table
Minimized FSM

- Implication chart method
  - Cross out incompatible states based on outputs
  - Then cross out more cells if indexed chart entries are already crossed out

<table>
<thead>
<tr>
<th>State</th>
<th>Present State</th>
<th>Next State</th>
<th>Output</th>
</tr>
</thead>
<tbody>
<tr>
<td>S0'</td>
<td>S0' S1 S2 S3'</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>S1</td>
<td>S0' S3' S1 S3'</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>S2</td>
<td>S1 S3' S2 S0'</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>S3'</td>
<td>S1 S0' S0' S3'</td>
<td>0</td>
<td></td>
</tr>
</tbody>
</table>

Minimized state table
(S0==S4) (S3==S5)

Minimizing incompletely specified FSMs

- Equivalence of states is transitive when machine is fully specified
- But it's not transitive when don't cares are present
  - e.g., state output
  - S0 0 S1 is compatible with both S0 and S2
  - S1 1 but S0 and S2 are incompatible
  - S2 - 1

- No polynomial time algorithm exists for determining best grouping of states into equivalent sets that will yield the smallest number of final states
Minimizing states may not yield best circuit

Example: edge detector - outputs 1 when last two input changes from 0 to 1

\[
\begin{array}{c|c|c|c}
X & Q_0 & Q_1 & Q_0^+ \ Q_1^+ \\
\hline
0 & 0 & 0 & 0 \ 0 \\
0 & 0 & 1 & 0 \ 0 \\
0 & 1 & 1 & 0 \ 0 \\
1 & 0 & 0 & 0 \ 1 \\
1 & 0 & 1 & 1 \ 1 \\
1 & 1 & 1 & 1 \ 1 \\
- & 1 & 0 & 0 \ 0 \\
\end{array}
\]

\[Q_1^+ = X \ (Q_1 \ xor \ Q_0)\]

\[Q_0^+ = X \ Q_0^{'},\]

Another implementation of edge detector

"Ad hoc" solution - not minimal but cheap and fast
**State assignment**

- Choose bit vectors to assign to each "symbolic" state
  - with $n$ state bits for $m$ states there are $2^n! / (2^n - m)!$
  
  \[ \log n \leq m \leq 2^n \]
  - $2^n$ codes possible for 1st state, $2^n-1$ for 2nd, $2^n-2$ for 3rd, ...
  - huge number even for small values of $n$ and $m$
    - intractable for state machines of any size
    - heuristics are necessary for practical solutions
  - optimize some metric for the combinational logic
    - size (amount of logic and number of FFs)
    - speed (depth of logic and fanout)
    - dependencies (decomposition)

**State assignment strategies**

- Possible strategies
  - sequential – just number states as they appear in the state table
  - random – pick random codes
  - one-hot – use as many state bits as there are states (bit=1 -> state)
  - output – use outputs to help encode states
  - heuristic – rules of thumb that seem to work in most cases
- No guarantee of optimality – another intractable problem
One-hot state assignment

- Simple
  - easy to encode
  - easy to debug
- Small logic functions
  - each state function requires only predecessor state bits as input
- Good for programmable devices
  - lots of flip-flops readily available
  - simple functions with small support (signals its dependent upon)
- Impractical for large machines
  - too many states require too many flip-flops
  - decompose FSMs into smaller pieces that can be one-hot encoded
- Many slight variations to one-hot
  - one-hot + all-0

Heuristics for state assignment

- Adjacent codes to states that share a common next state
  - group 1's in next state map
    - group 1's in next state map
      - $i = i \cdot a + i \cdot b$
      - $c = i \cdot a + i \cdot b$
    - Adjacent codes to states that share a common ancestor state
      - group 1's in next state map
        - $i = i \cdot a + i \cdot b$
        - $b = i \cdot a$
        - $c = k \cdot a$
    - Adjacent codes to states that have a common output behavior
      - group 1's in output map
        - $i = i \cdot a + i \cdot c$
        - $j = i \cdot a + i \cdot c$
        - $b = i \cdot a$
        - $d = i \cdot c$
General approach to heuristic state assignment

- All current methods are variants of this:
  1) determine which states "attract" each other (weighted pairs)
  2) generate constraints on codes (which should be in same cube)
  3) place codes on Boolean cube so as to maximize constraints satisfied
     (weighted sum)
- Different weights make sense depending on whether we are optimizing for
  two-level or multi-level forms
- Can't consider all possible embeddings of state clusters in Boolean cube
  - heuristics for ordering embedding
  - to prune search for best embedding
  - expand cube (more state bits) to satisfy more constraints

Output-based encoding

- Reuse outputs as state bits - use outputs to help distinguish states
  - why create new functions for state bits when output can serve as well
  - fits in nicely with synchronous Mealy implementations

<table>
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<th>Next State</th>
<th>Outputs</th>
</tr>
</thead>
<tbody>
<tr>
<td>C</td>
<td>T</td>
<td>LS</td>
<td>ST</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>HG</td>
<td>H</td>
</tr>
<tr>
<td>-</td>
<td>1</td>
<td>HG</td>
<td>G</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>HG</td>
<td>G</td>
</tr>
<tr>
<td>-</td>
<td>-</td>
<td>HY</td>
<td>Y</td>
</tr>
<tr>
<td>1</td>
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<td>Y</td>
</tr>
<tr>
<td>-</td>
<td>0</td>
<td>FY</td>
<td>Y</td>
</tr>
<tr>
<td>-</td>
<td>1</td>
<td>FY</td>
<td>H</td>
</tr>
</tbody>
</table>

HG = ST' H1' H0' F1' F0' + ST H0 F1 F0
HY = ST H1' H0' F1' F0' + ST' H1 H0 F1 F0
FG = ST H1' H0 F1 F0' + ST' H1 H0' F1' F0'
HG = ST H1 H0' F1 F0' + ST H1' H0 F1 F0
HG = ST' H1' H0' F1' F0' + ST H1 H0 F1 F0
HG = ST H1 H0' F1' F0' + ST H1' H0 F1 F0
HG = ST H1 H0' F1 F0' + ST H1' H0 F1 F0
HG = ST H1 H0' F1' F0' + ST H1 H0 F1 F0
Output patterns are unique to states, we do not need any state bits - implement 5 functions
(one for each output) instead of 7 (outputs plus 2 state bits)
Current state assignment approaches

- For tight encodings using close to the minimum number of state bits
  - best of 10 random seems to be adequate (averages as well as heuristics)
  - heuristic approaches are not even close to optimality
  - used in custom chip design
- One-hot encoding
  - easy for small state machines
  - generates small equations with easy to estimate complexity
  - common in FPGAs and other programmable logic
- Output-based encoding
  - ad hoc - no tools
  - most common approach taken by human designers
  - yields very small circuits for most FSMs

Sequential logic optimization summary

- State minimization
  - straightforward in fully-specified machines
  - computationally intractable, in general (with don't cares)
- State assignment
  - many heuristics
  - best-of-10-random just as good or better for most machines
  - output encoding can be attractive (especially for PAL implementations)