Sum-of-Product & Product-of-Sum

- Sum of Product
  - see entries with value 1.
  - Ensure output 1 for those inputs. So **OR**
  - A'B+AB'
- Product of Sum
  - see entries with value 0
  - Ensure output 0 for those input. So **AND**
  - (A+B)(A'+B')

A	B	A **XOR** B
0	0	0
0	1	1
1	0	1
1	1	0

DeMorgan's Law

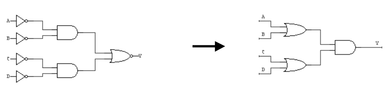
$$\overline{X+Y} = \overline{X} \cdot \overline{Y}$$

- Thus, the inversion doesn't distribute directly: it also changes the "or" to an "and"
- Example:

$$\overline{X \cdot Y} = \overline{X} + \overline{Y}$$

DeMorgan's law on circuits

- You can do DeMorgan's law directly on the circuit:



Simplification

- Some important rules for simplification (how do you prove these?):
  - AB + AB' = A
  - A + AB = A
- Note that you can use the rules in either direction, to remove terms, or to add terms. Indeed, sometimes you need to add some terms in order to get to the simples solution.

Examples

- Simplify: ab'c + abc + a'bc

  ab'c + abc + a'bc

  = ab'c + abc + abc + a'bc = ac + bc
- Show that X + X'Y = X + Y

  X + X'Y

  = X(1 + Y) + X'Y

  = X + XY + X'Y

  = X + Y

Examples (cont'd)

- Simplify:WX + XY + X'Z' + WY'Z'

  WX + XY + X'Z' + WY'Z'

  = WX + XY + X'Z' + WY'Z'X + WY'Z'X'

  = WX(1 + Y'Z') + XY + X'Z'(1 + WY')

  = WX + XY + X'Z'

Examples (cont'd)

- Prove the consensus theorem, which says:
  XY + X'Z + YZ = XY + X'Z
- Solution:
  XY + X'Z + YZ
  = XY + X'Z + (X + X')YZ
  = XY + X'Z + XYZ + X'YZ
  = XY + XYZ + X'Z + X'ZY
  = XY(1 + Z) + X'Z(1 + Y)
  = XY + X'Z

Long example

- Simplify:

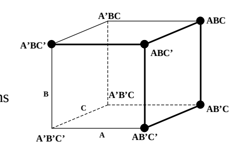
A'B'C'D' + A'BC'D' + A'BD + A'BC'D + ABCD + ACD' + B'CD'
= A'C'D'(B' + B) + A'BD(1 + C') + ABCD + ACD' + B'CD'
= A'C'D' + A'BD + ABCD + ACD' + B'CD'
= A'C'D' + BD(A' + AC) + ACD' + B'CD'
= A'C'D' + BD(A' + C) + ACD' + B'CD' (Since X + X'Y = X + Y)
= A'C'D' + A'BD + (BCD + ACD') + B'CD'
= A'C'D' + A'BD + (BCD + ACD' + ABC) + B'CD'
(Added ABC by consensus)
= A'C'D' + (A'BD + ABC + BCD) + (ABC + B'CD' + ACD')
= AC'D' + A'BD + ABC + B'CD'

Karnaugh Maps

- What was the idea in doing simplication? Well, one of the ideas was to try to apply the unification theorem (AB + AB' = A).
- What we're looking for then are terms that differ only in one variable.
- This can be difficult to do when there are many terms and many variables. Let's try to see if we there is a graphical method that makes it easier.

Cube Representation.

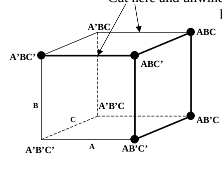
- Let's draw a cube. Each vertex is a possible term, AND two adjacent vertices only differ in one variable.
- Now, draw a dot for each term from our boolean expression, and group dots that are connected.
- An edge that connects two dots means that we can apply the unification theorem to merge those two terms. The variable that differs is dropped.
- By applying the unification theorem twice, we can merge 4 vertices that are fully connected.

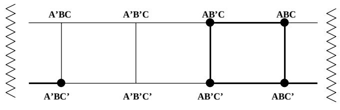


The above cube shows the expression A'BC + ABC' + ABC + AB'C + AB'C'.
It simplifies to:
A + BC'

Karnaugh Maps (cont'd)

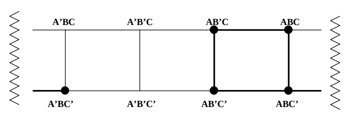
Cut here and unwind the cube to get the bottom diagram





Note how, because we are now in 2d, the lines kinda wrap around.

Karnaugh Maps (cont'd)



A simpler drawing of this is given below. We call this kind of table a Karnaugh map.

AB / C	01	00	10	11
1	0	0	1	1
0	1	0	1	1

Implicants

- Implicant: A product term whose “oneness” *implies* the functions “oneness”.
- Prime Implicant: Implicant that cannot be combined with another implicant.
- Essential Prime Implicant: Implicant that covers an element of the on-set which is not covered by any other implicant.

Example of Implicants

CD \ AB	00	01	11	10
00	0	1	1	0
01	1	1	1	0
11	1	0	1	1
10	0	0	1	1

- Implicants
- Six Prime Implicants: A’B’D, BC’, AC, A’C’D, AB, B’CD
- Essential PI: AC,BC’
- F=A’B’D+BC’+AC

K-map: SoP and PoS

CD \ AB	00	01	11	10
00	0	1	0	0
01	1	0	1	1
11	1	0	0	1
10	0	0	0	0

- SoP:

A’BC’D’+A’B’C’D+ABC’D+AB’C’D+A’B’CD+AB’CD

- Minimized Exp

A’BC’D’+B’D+AC’D

- PoS:

(A+B+C+D)(A’+B’+C+D)(A’+B+C+D) (A+B’+C+D’) ….

- Minimized PoS

(B+D)(A’+D)(B’+C’)(A+B’+D’)

Proof of Duality

- Duality says: If you have an equation that holds, and you change all the ANDs to ORs, the ORs to ANDs, the 0’s to 1’s, and the 1’s to 0’s, then you get another equation that holds
- Example:
  - A + 1 = 1 That’s certainly true
  - Dual is: A . 0 = 0. Wow, This holds aswell!
- Alright, let’s prove that this is true.

Proof of Duality (cont’d)

- Definitions
  - If E is an expression, let E^D be the dual of D.
    - ie: $f(A, B, C, \ldots, 0, 1, +, \bullet)^D = f(A, B, C, \ldots, 1, 0, \bullet, +)$
  - If E is an expression, let E^{ID} be E^D, but with all the variables inverted.
    - ie: $f(A, B, C, \ldots, 0, 1, +, \bullet)^{ID} = f(A', B', C', \ldots, 1, 0, \bullet, +)$
- Example:
  - $(A + 1)^D = A . 0$, however $(A + 1)^{ID} = A' . 0$
  - $(A + B)^D = A . B$, however $(A + B)^{ID} = A' . B'$
- What is E^{ID}?
  - Let’s take an example: $(A + B)^{ID} = A' . B' = (A + B)'$ (By DeMorgan’s)
  - So, it would look like E^{ID} is just E’… Well, that’s true, and this is called the generalized version of DeMorgan’s (proof is by induction, we won’t do it):
    - $f(A, B, C, \ldots, 0, 1, +, \bullet)' = f(A', B', C', \ldots, 1, 0, \bullet, +)$

Proof of Duality (cont’d)

- Now, we are equipped to prove the duality theorem.
- Let’s say that the original equation is:
  - S = T, where S and T are expressions.
  - Let’s now invert both sides of our equation:
    - S’ = T’
  - But inverting is the same as taking the ID, so
    - $S^{ID} = T^{ID}$
  - Now, let $S = f(A, B, C, \ldots, 0, 1, +, \bullet)$, and $T = g(A, B, C, \ldots, 0, 1, +, \bullet)$
  - Thus:
    - $f(A, B, C, \ldots, 0, 1, +, \bullet)^{ID} = g(A, B, C, \ldots, 0, 1, +, \bullet)^{ID}$
  - Or, by the definition of ID
    - $f(A', B', C', \ldots, 1, 0, \bullet, +) = g(A', B', C', \ldots, 0, 1, \bullet, +)$

Proof of Duality (cont'd)

- Now, do variable replacement: a = A', b = B', etc. Thus:
  - f(a, b, c, …, 1, 0, •, +) = g(a, b, c, …, 0, 1, •, +)
- Now, do another variable replacement: A = a, B = b, etc. Thus:
  - f(A, B, C, …, 1, 0, •, +) = g(A, B, C, …, 0, 1, •, +)
- By the definition of dual, this is:
  - f(A, B, C, …, 0, 1, +, •)^D = g(A, B, C, …, 0, 1, +, •)^D
- Or:
  - $S^D = T^D$