The Hardware/Software Interface

Floating Point

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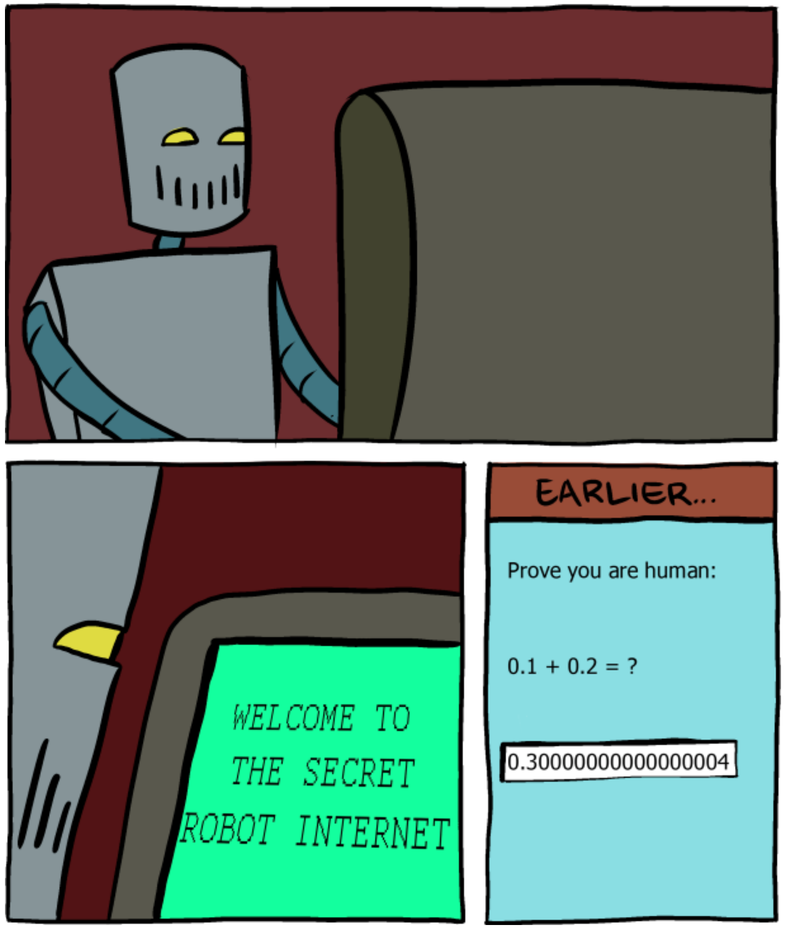

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Relevant Course Information

- Lecture polls are graded on *completion*
  - Don't change your answer afterward; misrepresents your understanding
- Early Course Reflection available on Canvas now, due Friday
- Lab 1a due tonight at 11:59 pm
  - Submit `pointer.c` and `lab1Asynthesis.txt`
    - Make sure there are no lingering `printf` statements in your code!
  - Make sure you submit *something* to Gradescope before the deadline and that the file names are correct
  - Can use late days to submit up until Wed 11:59 pm
- Lab 1b due next Monday (10/13)
  - Submit `aisle_manager.c`, `store_client.c`, and `lab1Bsynthesis.txt`

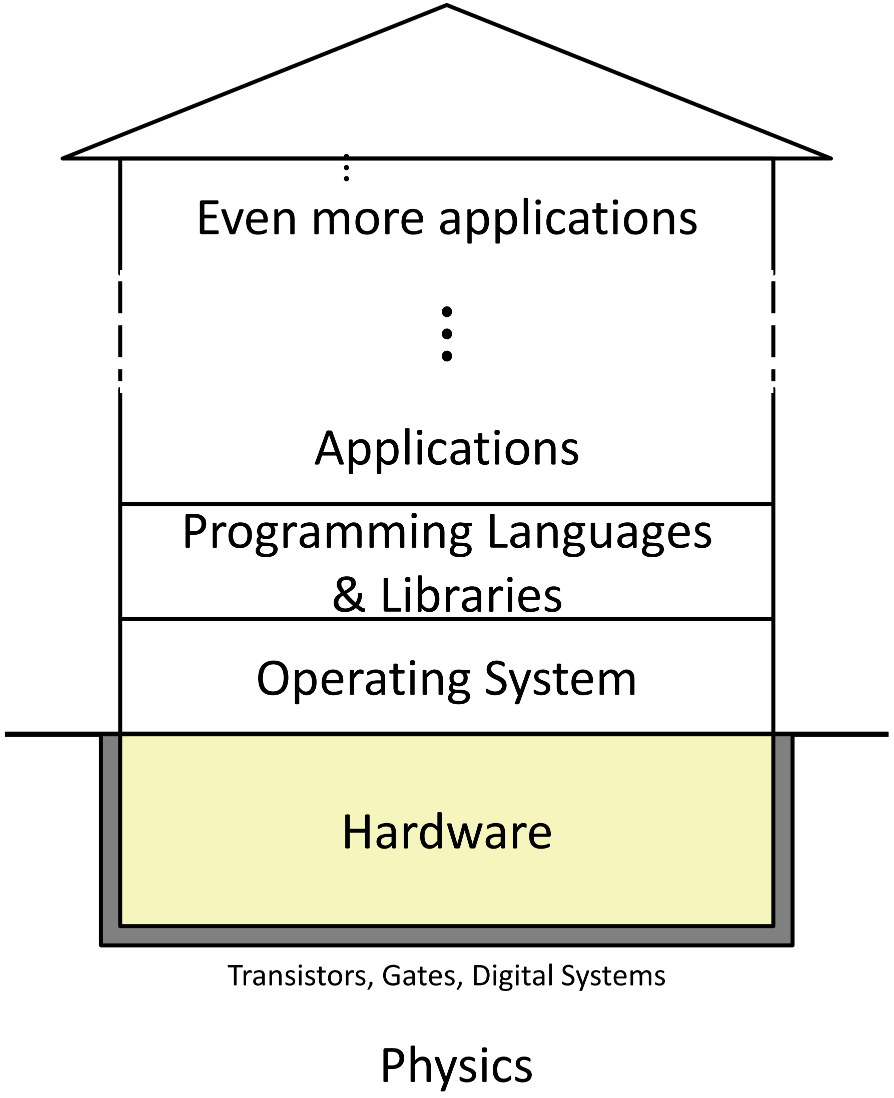
Lab 1b Aside: C Macros

- C macros basics:
  - Basic syntax is of the form: `#define NAME expression`
  - Allows you to use "`NAME`" instead of "`expression`" in code
    - Does naïve copy and replace *before* compilation – everywhere the characters "NAME" appear in the code, the characters "expression" will now appear instead
    - NOT the same as a Java constant
  - Useful to help with readability/factoring in code

- You'll use C macros in Lab 1b for defining bit masks
  - See Lab 1b starter code and Lecture 04 (card operations) for examples

House of Computing Check-In

- Topic Group 1: **Data**
  - Memory, Data, Integers, **Floating Point**, Arrays, Structs
- How do we store information for other parts of the house of computing to access?
  - How do we represent data and what limitations exist?
  - What design decisions and priorities went into these encodings?



Number Representation Revisited

- What can we represent in one word?
  - Addresses
  - Characters and Strings (ASCII)
  - Signed and Unsigned Integers

- How do we encode the following:
  - Real numbers (*e.g.*, 3.14159)
  - Very large numbers (*e.g.*, 6.02×10^{23})
  - Very small numbers (*e.g.*, 6.626×10^{-34})
  - Special numbers (*e.g.*, ∞, NaN)

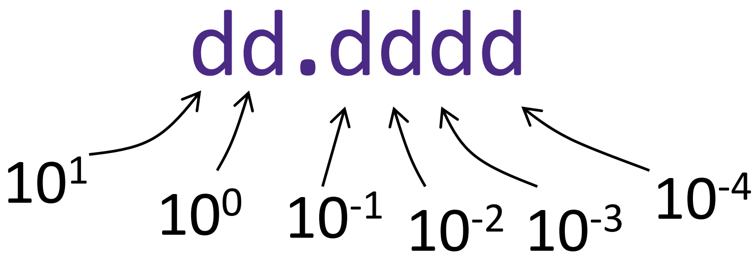
Floating Point

Lecture Outline (1/5)

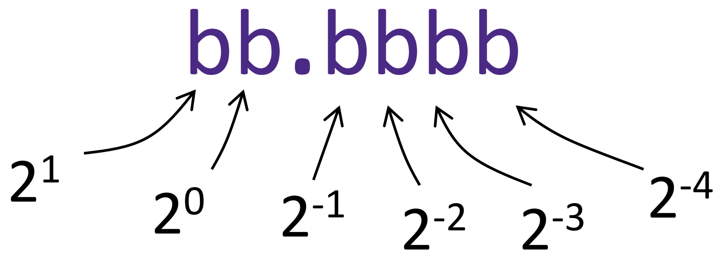
- **Scientific Notation**
- IEEE 754 Floating Point Encoding
- Floating Point Special Cases
- Floating Point Limitations and Dangers
- Floating Point in Real Life

Representation of Fractions (Review)

- In decimal, the *decimal point* signifies the boundary between integer and fractional parts:
  - Like leading zeros, can now have *trailing zeros* to the right of the point



- Same ideas apply in binary with the *binary point*:



Limits of Representation of Fractions

- Limitations:
  - Given a *fixed* number of (consecutive) digits, you are limited in range, based on where you place the point
    - *e.g.*, $\texttt{bb.bbbb}_2$ ranges from 0 – 3.9375
  - Even given an *arbitrary* number of digits, can only **exactly** represent numbers of the form $\sum_p (d_p \times b^p)$
    - b is the base, p is digit position (which can be negative), and d_p is the value of that digit's symbol
  - Plenty of real and rational numbers **cannot** be exactly represented using digits:

Value	Decimal	Binary
1/3	$0.3333[3]\ldots_{10}$	$0.010101[01]\ldots_2$
1/5	0.2_{10}	$0.0011[0011]\ldots_2$
π	$3.14159\ldots_{10}$	$11.001001000011111\ldots_2$

Scientific Notation (Review)

- General form:

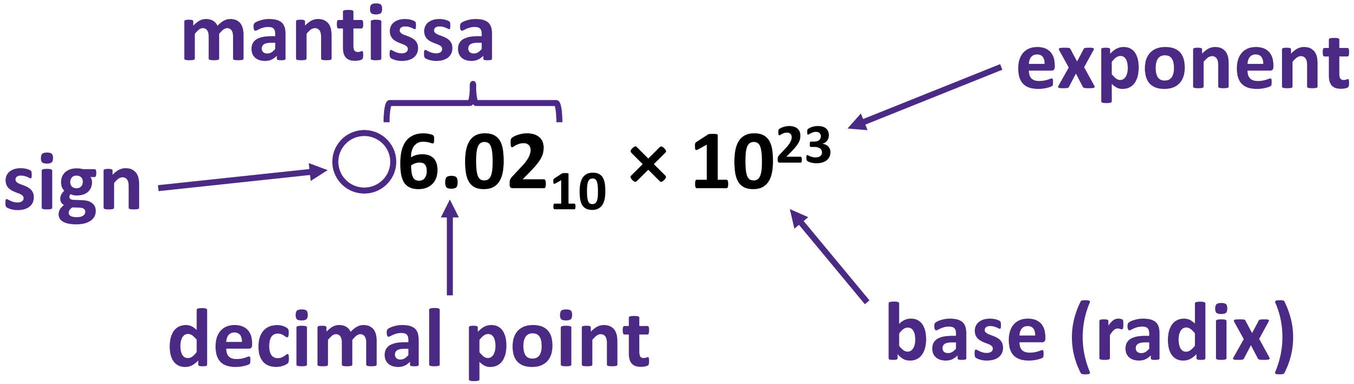
$$\textbf{numeral} \times \textbf{base}^{\textbf{power}}$$

  - Changing power allows us to “shift” the point in the numeral

- *Normalized form*: exactly one digit (non-zero) to left of point

Decimal Scientific Notation

- Terminology:

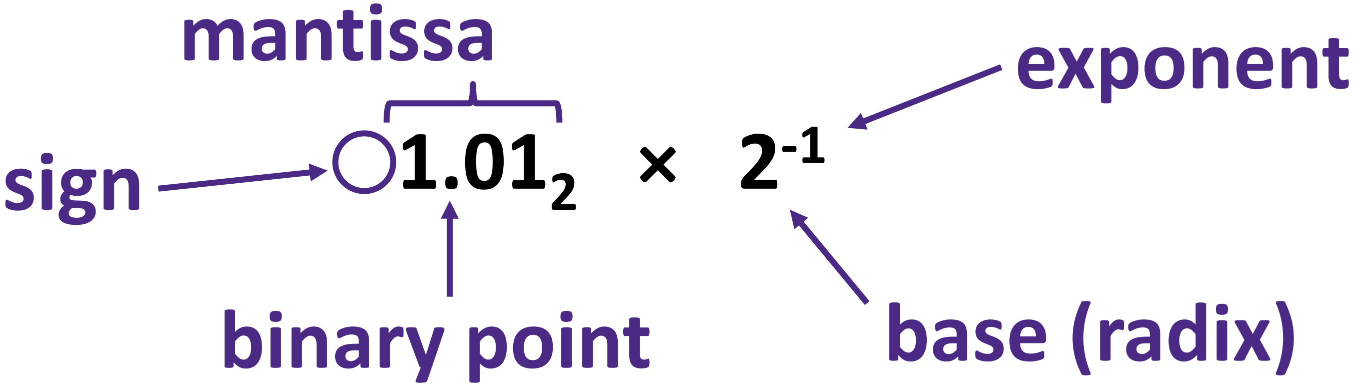


  - Changing power allows us to “shift” the point in the numeral
    - Example: $3.51 \times 10^1 = 0.351 \times 10^2 = 35.1 \times 10^0$

- *Normalized form*: exactly one digit (non-zero) to left of point

Binary Scientific Notation

- Terminology:



  - Changing power allows us to "shift" the point in the numeral
    - Example: $1.01_2 \times 2^{-1} = 0.101_2 \times 2^{0} = 10.1_2 \times 2^{-2}$

- *Normalized form*: exactly one digit (non-zero) to left of point

Lecture Outline (2/5)

- Scientific Notation
- **IEEE 754 Floating Point Encoding**
- Floating Point Special Cases
- Floating Point Limitations and Dangers
- Floating Point in Real Life

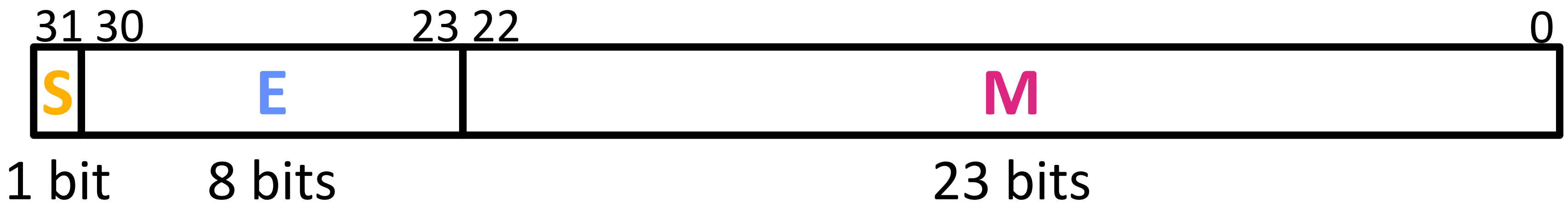
IEEE Floating Point

- IEEE 754 (established in 1985)
  - Standard to make numerically-sensitive programs portable
  - Specifies two things: *representation scheme* and result of *floating point operations*
  - Supported by all major CPUs

- Driven by numerical concerns
  - Users (*e.g.*, scientists, numerical analysts) want them to be as **real** as possible
  - Builders want them to be **easy to implement** and **fast**
  - Users mostly won out:
    - Nice standards for rounding, overflow, underflow, but... complex for hardware
    - Float operations can be an order of magnitude slower than integer ops → so slow that they are used as a performance gauge! (*e.g.*, FLOPS/s)

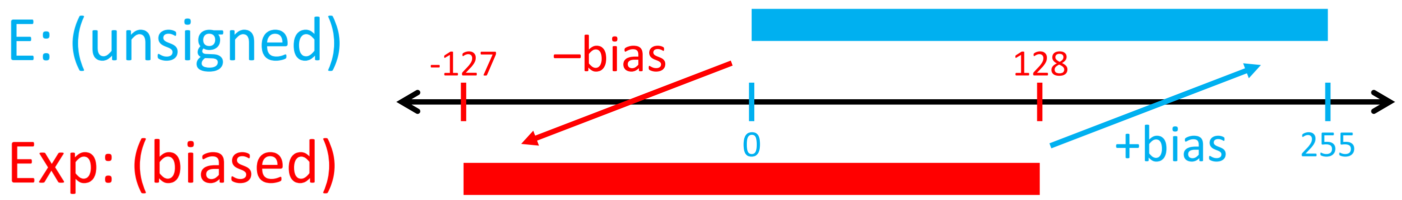
Floating Point Encoding (Review)

- C variable declared as `float`
- Use normalized, base 2 scientific notation:
  - Value: $\pm 1 \times \text{Mantissa} \times 2^{\text{Exponent}}$
  - Bit Fields: $(-1)^{S} \times 1.M \times 2^{(E-\text{bias})}$
- Representation Scheme:
  - Sign (bit **S** is 0 if positive, 1 if negative)
  - Mantissa is the fractional part of the normalized number; encoded in bit vector **M**
  - Exponent weights the value by a power of 2; encoded in the bit vector **E**



The Exponent Field (Review)

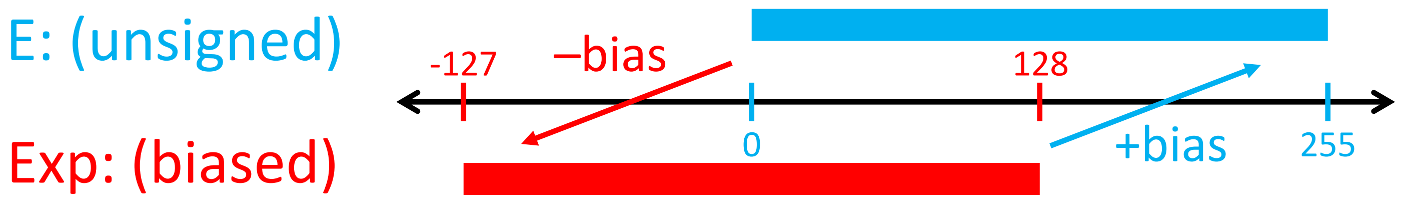
- Use biased notation
  - Read exponent as unsigned, but with *bias* of $2^{w-1}-1 = 127$
  - Representable exponents roughly half positive and half negative
  - E = Exp + bias ↔ Exp = E – bias



- Examples:
  - If value has Exp = 1, then *encode* 1 + 127 in unsigned, storing E = 0b 1000 0000
  - If float has E = 0b 0100 0000, then we read out 64 as unsigned, shift this value to get Exp = 64 − 127 = −63

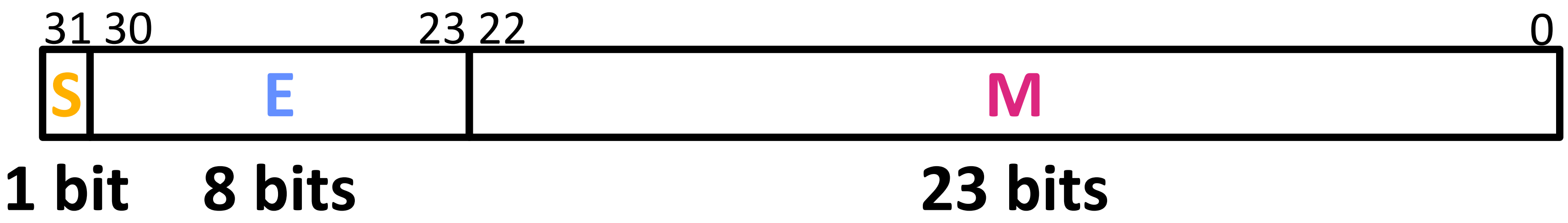
The Exponent Field – Why Biased?

- Use biased notation
  - Read exponent as unsigned, but with *bias* of $2^{w-1}-1$ = 127
  - Representable exponents roughly half positive and half negative
  - E = Exp + bias ↔ Exp = E – bias



- Why biased?
  - Sign-and-magnitude: encodings for Exp+Man are aligned with magnitude
  - Makes floating point arithmetic easier (somewhat compatible with two's complement hardware)

The Mantissa/Fraction Field (Review)



$$(-1)^{S} \times (1\,.\,M) \times 2^{(E-\text{bias})}$$

- Note the implicit leading 1 in front of the M bit vector
  - Gives us an extra bit of *precision*

- Examples:
  - Man of 1.10111_2 is encoded as M = 0b 101 1100 0000 0000 0000 0000
  - M = 110 1000 0000 0000 0000 0000 is decoded as a Man = 1.1101_2

Normalized Floating Point Conversions (Review)

- FP → Decimal
  1. Append the bits of M to implicit leading 1 to form the mantissa.
  2. Multiply the mantissa by $2^{E - bias}$.
  3. Multiply the sign $(-1)^{S}$.
  4. Multiply out the exponent by shifting the binary point.
  5. Convert from binary to decimal.

- Decimal → FP
  1. Convert decimal to binary.
  2. Convert binary to normalized scientific notation.
  3. Encode sign as S (0/1).
  4. Add the bias to exponent and encode E as unsigned.
  5. The first bits after the leading 1 that fit are encoded into M.

$2^{-1} = 0.5$
$2^{-2} = 0.25$
$2^{-3} = 0.125$
$2^{-4} = 0.0625$

Polling Questions (1/2)

- What is the value encoded by the following floating point number?

  0b 0 | 1000 0000 | 110 0000 0000 0000 0000 0000

  - bias = $2^{w-1}-1$
  - exponent = E – bias
  - mantissa = 1.M

- Convert the decimal number **-7.375 = -1.11011 x 2^2** into floating point representation.

Lecture Outline (3/5)

- Scientific Notation
- IEEE 754 Floating Point Encoding
- **Floating Point Special Cases**
- Floating Point Limitations and Dangers
- Floating Point in Real Life

Special Cases

- But wait… what happened to zero?
  - *Special case:* E and M all zeros = 0
  - Two zeros (sign and magnitude), but at least 0x00000000 = 0 like integers
- E = 0xFF, M = 0: $\pm\infty$
  - *e.g.*, division by 0
  - Still work in comparisons!
- E = 0xFF, M ≠ 0: Not a Number (NaN)
  - *e.g.*, square root of negative number, 0/0, $\infty-\infty$
  - NaN propagates through computations
  - Value of M can be useful in debugging

New Representation Limits (Review)

- New largest value (besides ∞)?
  - E = 0xFF taken; next largest is E = 0xFE
  - Largest will have M = 0b1…1 → $1.1\ldots1_2 \times 2^{254-127} = 2^{128} - 2^{104}$

- New value closest to 0:
  - E = 0x00 taken; next smallest is E = 0x01
  - Smallest will have M = 0 → $1.0\ldots0_2 \times 2^{1-127} = 2^{-126}$

- Can we go smaller?
  - Normalization and implicit 1 are to blame

Denorm Numbers

This is extra (non-testable) material

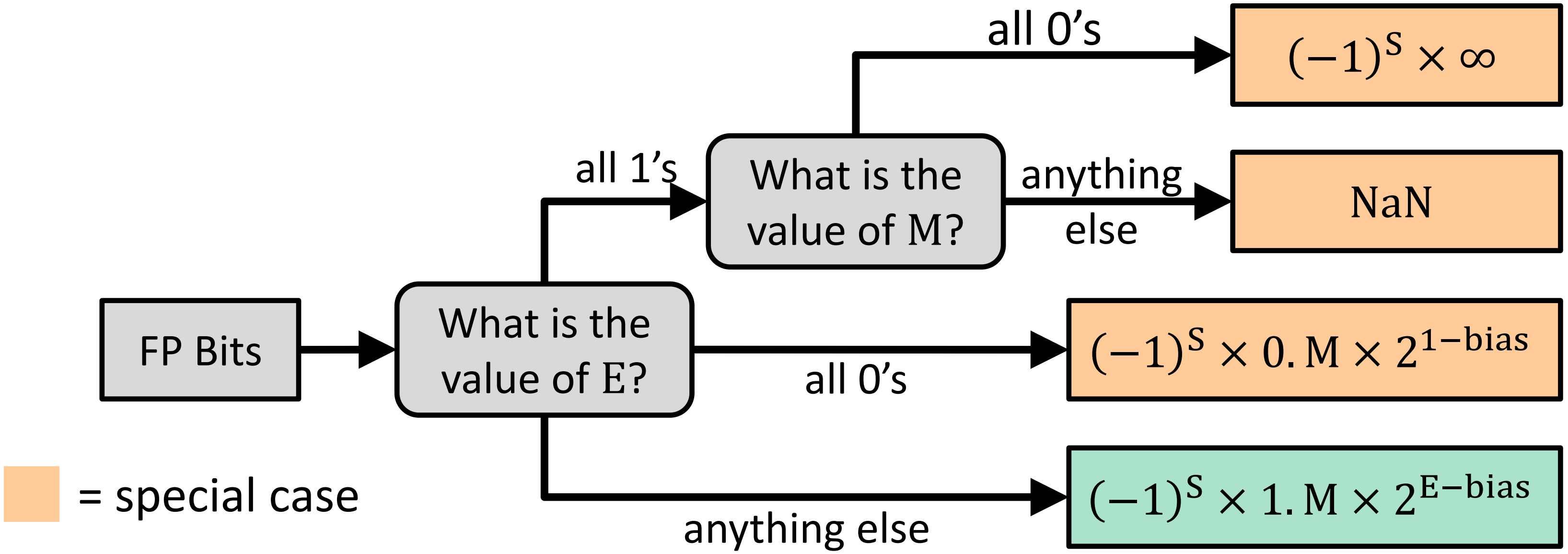
- *Special case:* E = 0, M ≠ 0 are denormalized numbers
  - No leading 1
  - Uses implicit exponent of –126 even though E = 0x00

- Denormalized numbers close the gap between zero and the smallest normalized number
  - Smallest norm: $\pm 1.0\ldots0_{two} \times 2^{-126} = \pm 2^{-126}$
  - Smallest denorm: $\pm 0.0\ldots01_{two} \times 2^{-126} = \pm 2^{-149}$
    - There is still a gap between zero and the smallest denormalized number

So much closer to 0

Floating Point Special Case Summary

E	M	Interpretation
0b0…0	0b0…0	± 0
0b0…0	non-zero	± denormalized num
everything else	anything	± normalized num
0b1…1	0b0…0	± ∞
0b1…1	non-zero	NaN



Lecture Outline (4/5)

- Scientific Notation
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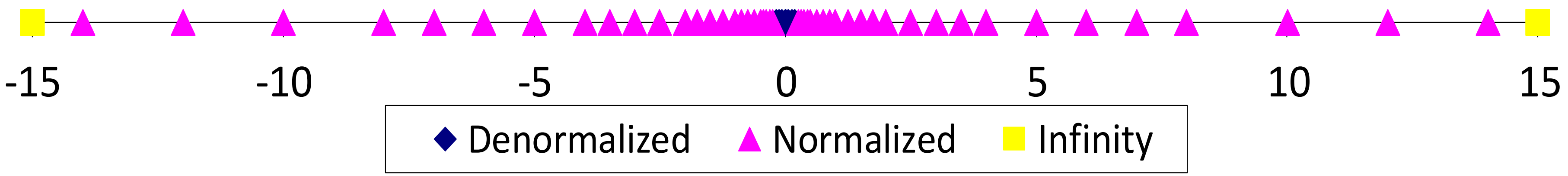
Distribution of Representable Values (Review)

- What ranges are NOT representable?
  - Between largest norm and infinity **Overflow** (Exp too large)
  - Between zero and smallest denorm **Underflow** (Exp too small)
  - Between norm numbers? **Rounding**
- Given a FP number, what's the next largest representable number?
  - What is this "step" when Exp = 0?
  - What is this "step" when Exp = 100?
- Distribution of values is denser closer to zero:



Precision and Accuracy

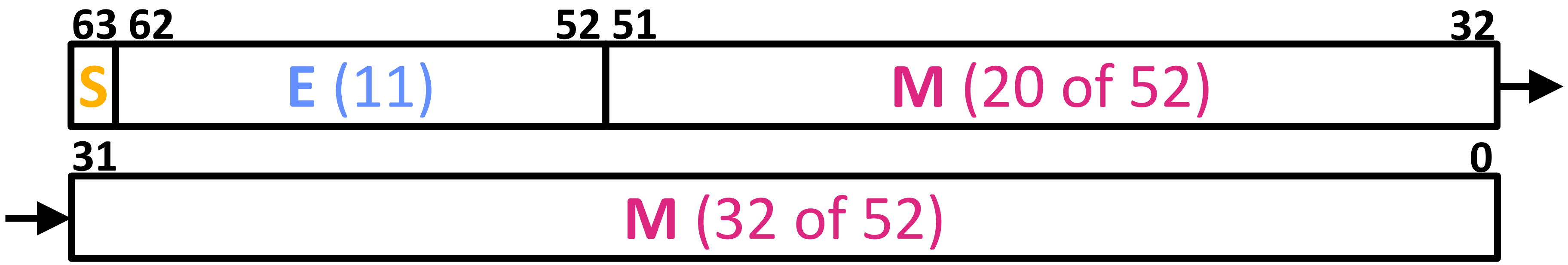
- Accuracy is a measure of the difference between the *actual value of a number* and its computer representation

- Precision is a count of the number of bits in a computer word used to represent a value
  - Capacity for accuracy

- *High precision permits high accuracy but doesn't guarantee it*
  - Example: `float pi = 3.14;` will be represented using all 24 bits of the mantissa (highly precise), but is only an approximation (not accurate)

Need Greater Precision?

- Double Precision (vs. Single Precision) in 64 bits



  - C variable declared as `double`
  - Exponent bias is now 2^{10}–1 = 1023
  - **Advantages:** greater precision (larger mantissa), greater range (larger exponent)
  - **Disadvantages:** more bits used, slower to manipulate

Floating Point Arithmetic (Review)

$$\text{Value} = (-1)^{S} \times \text{Mantissa} \times 2^{\text{Exponent}}$$

S	E	M

- Basic theoretical idea for floating point operations like + and ×:
  1) First, compute the exact result
  2) Then encode the result based on the specifics of your representation
     - If exponent is outside of range, then you will get *over/underflow*
     - If the exact result is not representable, then it will get *rounded* to fit the precision (width of M)

Properties of Floating Point Arithmetic (Review)

- Floats with value $\pm\infty$ and *NaN* can be used in operations
  - Result usually still $\pm\infty$ or NaN, but not always intuitive
- Floating point operations do not work like real math, due to *rounding*
  - Not associative: `(3.14+1e100)-1e100 != 3.14+(1e100-1e100)`
    `0` **`3.14`**
  - Not distributive: `100*(0.1+0.2) != 100*0.1+100*0.2`
    `30.000000000000003553` **`30`**
  - Not cumulative: repeatedly adding a small number to a large one may do nothing
- Equality (==) comparisons between floating point numbers are tricky, and often return unexpected results, so just avoid them!

Floating Point in C

- Two common data types: `float`, `double`
- Floating point literals indicated by decimal point (`double` by default)
  - Examples: `1.0` (double), `1.0f` (`float`)
- Related libraries:
  - `math.h` for `INFINITY` and `NAN` constants, `float.h` for additional constants
- Casting between `int`, `float`, and `double` **changes** the bit representation
  - Tries to preserve the value, but not always reversible
  - Integral → floating point: may get rounded if not enough precision
  - Floating point → integral: fractional part will get lost/truncated

Polling Questions (2/2)

- For the following code, what is the smallest value of n that will encounter a limit of representation?

```
float f = 1.0;  // 2^0
for (int i = 0; i < n; ++i)
    f *= 1024;  // 1024 = 2^10
printf("f = %f\n", f);
```

Lecture Outline (5/5)

- Scientific Notation
- IEEE 754 Floating Point Encoding
- Floating Point Special Cases
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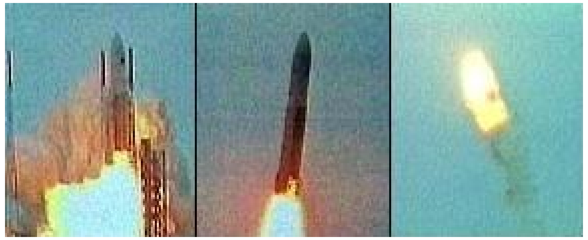
- **Floating Point in Real Life**

Floating Point Issues in Real Life

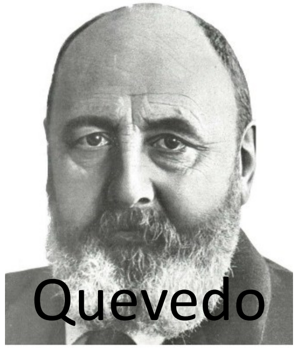
- **1991:** Patriot missile targeting error
  - Time in system stored in integer (tenths of a second since boot)
  - Converted to seconds by multiplying by 0.1 = $0.0\overline{0011}_2$ leading to erroneous time (error grows the longer system has been on)
- **1996:** V88 Ariane 501 rocket exploded 37 seconds after launch
  - Reused code from Ariane 4 inertial reference platform
  - Overflow when converting a 64-bit floating point number to a 16-bit integer (not protected by extra lines of code)

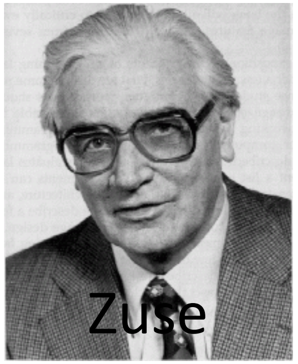


- **Other related bugs:**
  - 1982: Vancouver Stock Exchange 50% error in less than 2 years due to truncation
  - 1994: Intel Pentium FDIV (floating point division) hardware bug costs company $475 million in recall

More on Floating Point History

- Early days
  - First design with floating-point arithmetic in 1914 by Leonardo Torres y Quevedo
  - Implementations started in 1940 by Konrad Zuse, but with differing field lengths (usually not summing to 32 bits) and different subsets of the special cases
- IEEE 754 standard created in 1985
  - Primary architect was William Kahan, who won a Turing Award for this work
  - Standardized bit encoding, well-defined behavior for *all* arithmetic operations





Floating Point in the "Wild"

- 3 formats from IEEE 754 standard widely used in computer hardware and languages
  - In C, called `float`, `double`, `long double`
- Common applications:
  - 3D graphics: textures, rendering, rotation, translation
  - "Big Data": scientific computing at scale, machine learning
- Non-standard formats in domain-specific areas:
  - **Bfloat16:** training ML models; range more valuable than precision
  - **TensorFloat-32:** Nvidia-specific hardware for Tensor Core GPUs

Type	S bits	E bits	M bits	Total bits
Half-precision	1	5	10	16
Bfloat16	1	8	7	16
TensorFloat-32	1	8	10	19
Single-precision	1	8	23	32

Summary (1/2)

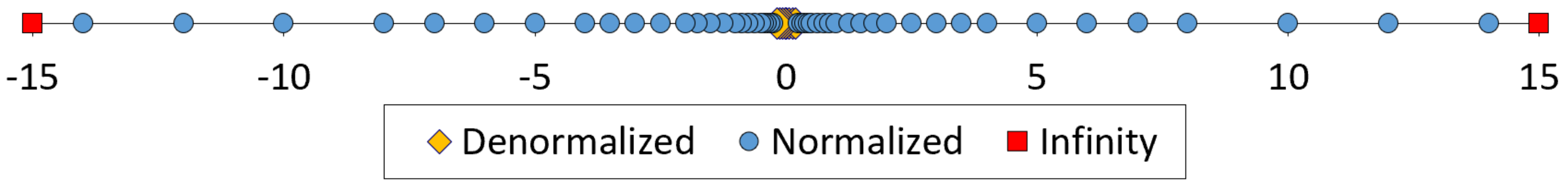
- Floating point approximates real numbers (large, small, & special):

  - Normalized case: ±1 × Mantissa × 2^{Exponent} = $(-1)^S \times 1.M \times 2^{(E-\text{bias})}$
  - Mantissa approximates fractional portion
    - Size of mantissa field determines our representable ***precision***
    - Exceeding mantissa length causes ***rounding***
  - Exponent in biased notation (bias = $2^{w-1} - 1$)
    - Size of exponent field determines our representable ***range***
    - Outside of representable exponents is ***overflow*** and ***underflow***
  - `double` (64 bits: [S (1)|E (11)|M (52)]) available if more precision needed

E	M	Meaning
0b0…0	anything	± denorm num (including 0)
anything else	anything	± norm num
0b1…1	0	± ∞
0b1…1	non-zero	NaN

Summary (2/2)

- Limitations of FP affect programmers all the time (!)
  - Overflow, underflow, rounding
    - Rounding is a HUGE issue due to limited mantissa bits and gaps that are scaled by the value of the exponent



  - Floating point arithmetic is NOT associative or distributive
    - ∞ and NaN are valid operands, but can produce unintuitive results
  - Do NOT use equality (==) with floating point numbers
  - Converting between integral and floating point data types *does* change the bits
    - *e.g.*, `int i = 2; // stored as 0x00000002,`
      `float f = i; // stored as 0x40000000`