Floating Point

CSE 351 Winter 2023

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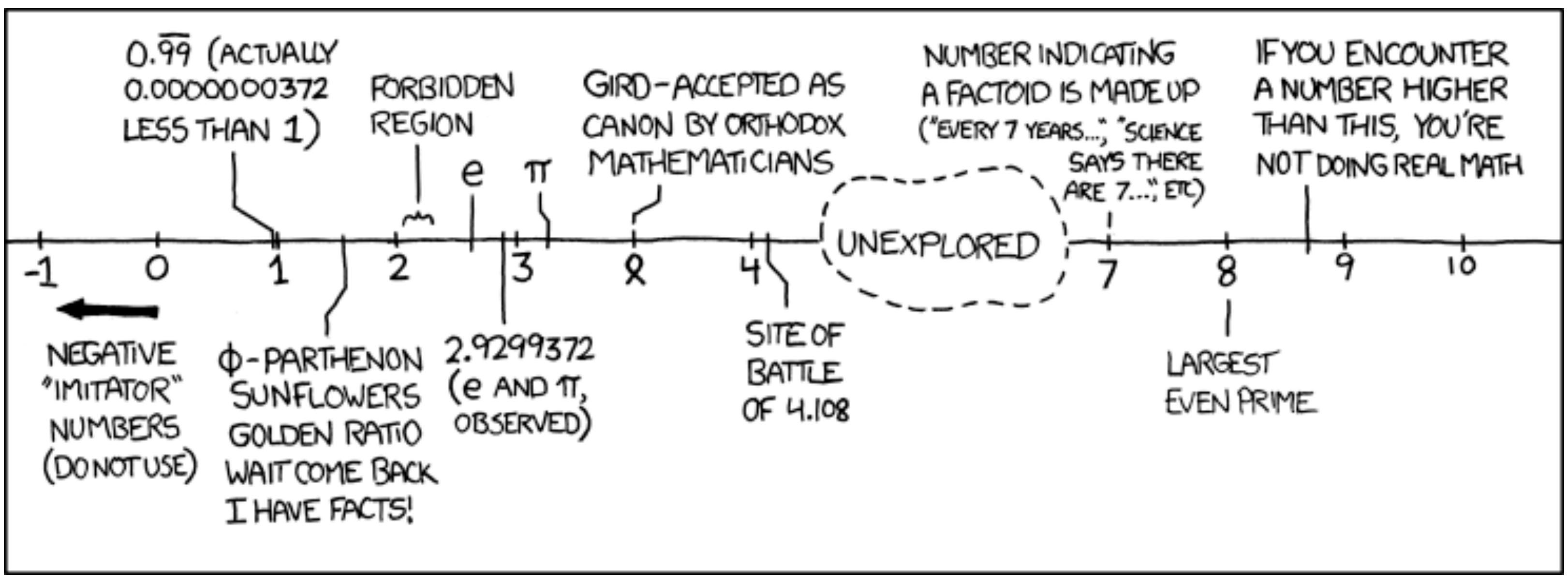


http://xkcd.com/899/

Relevant Course Information

- hw4 due tonight, hw5 due Friday (1/20)
- Lab 1a due tonight (1/18)
  - Submit `pointer.c` and `lab1Asynthesis.txt`
    - Make sure there are no lingering `printf` statements in your code!
  - Make sure you submit *something* to Gradescope before the deadline and that the file names are correct
  - Can use late day tokens to submit up until Friday 11:59 pm
  - If you are submitting with a partner, ensure that you add them to the submission
- Lab 1b due Friday (1/20)
  - Submit `aisle_manager.c`, `store_client.c`, and `lab1Bsynthesis.txt`

Exams

- Midterm and final exams will be taken on Gradescope
- Open for 72 (maybe 75) hours, no time limit
  - Designed to take 1-3 hours
  - Midterm due Friday, February 10
  - Final due Wednesday, March 15
- Open book, open notes, open (.*)
  - But not group work – taken individually
  - High-level discussion with classmates OK, but you must write answers on your own (like labs, but without a partner)
- Mixture of problem-solving, design, and personal reflection questions (short answer & open ended)

Lab 1b Aside: C Macros

- C macros basics: #define FOO 7
  - Basic syntax is of the form: `#define NAME expression`
  - Allows you to use "`NAME`" instead of "`expression`" in code
    - Does naïve copy and replace *before* compilation – everywhere the characters "NAME" appear in the code, the characters "expression" will now appear instead
    - NOT the same as a Java constant, but used in a similar way
  - Useful to help with readability/factoring in code

- You'll use C macros in Lab 1b for defining bit masks
  - See Lab 1b starter code and Lecture 4 slides (card operations) for examples

Reading Review

- Terminology:
  - normalized scientific binary notation
  - trailing zeros
  - sign, mantissa, exponent ↔ bit fields S, M, and E
  - `float`, `double`
  - biased notation (exponent), implicit leading one (mantissa)
  - rounding errors

- Questions from the Reading?

Review Questions

$2^3 \quad 2^1 \quad 2^0 \quad 2^{-2} \quad 2^{-3}$

$8 + 2 + 1 + 0.25 + .125$

$2^{-1} = 0.5$
$2^{-2} = 0.25$
$2^{-3} = 0.125$
$2^{-4} = 0.0625$

- Convert 11.375_{10} to normalized binary scientific notation

$1011.011 \quad 1.011011 \times 2^3$

- What is the value encoded by the following floating-point number?

0b 0 | 1000 0000 | 110 0000 0000 0000 0000 0000

  - bias = $2^{w-1}-1$ $\quad 2^7 - 1 = 127$
  - exponent = E – bias $\quad 128 - 127 = 1$
  - mantissa = 1.M

$1.11 \times 2^1 = 11.1$

$2^1 \, 2^0 \, 2^{-1}$

$\boxed{3.5}$

Number Representation Revisited

- What can we represent in one word?
  - Signed and Unsigned Integers
  - Characters (ASCII)
  - Addresses
- How do we encode the following:
  - Real numbers (e.g., 3.14159)
  - Very large numbers (e.g., 6.02×10^{23})
  - Very small numbers (e.g., 6.626×10^{-34})
  - Special numbers (e.g., ∞, NaN)

Floating Point

Floating Point Topics

- **Fractional binary numbers**
- **IEEE floating-point standard**
- Floating-point operations and rounding
- Floating-point in C

- There are many more details that we won't cover
  - It's a 58-page standard…

Representation of Fractions

- "Binary Point," like decimal point, signifies boundary between integer and fractional parts:

  Example 6-bit representation:

  xx.yyyy

  2^1 2^0 2^{-1} 2^{-2} 2^{-3} 2^{-4}

- Example: $10.1010_2 = 1{\times}2^1 + 1{\times}2^{-1} + 1{\times}2^{-3} = 2.625_{10}$

Binary Scientific Notation (Review)

exponent

$$\overbrace{1.01}^{\text{mantissa}}{}_2 \times 2^{-1}$$

binary point radix (base)

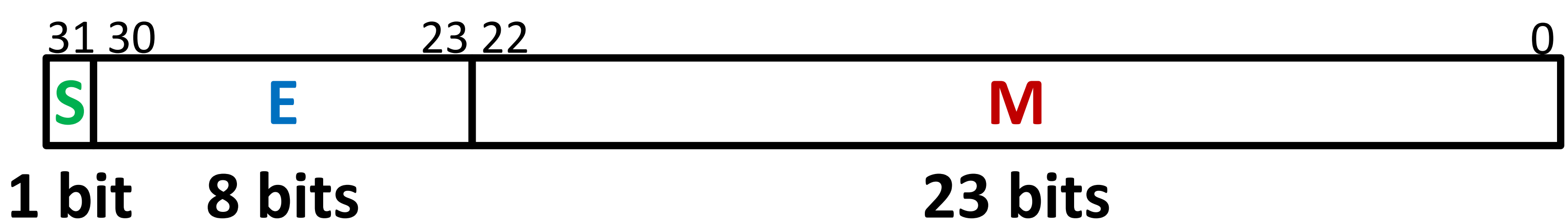
- *Normalized form*: exactly one digit (non-zero) to left of binary point
- Computer arithmetic that supports this called floating point due to the "floating" of the binary point
  - Declare such variable in C as `float` (or `double`)

IEEE Floating Point

- IEEE 754 (established in 1985)
  - Standard to make numerically-sensitive programs portable
  - Specifies two things: *representation scheme* and result of *floating point operations*
  - Supported by all major CPUs
- Driven by numerical concerns
  - **Scientists**/numerical analysts want them to be as **real** as possible
  - **Engineers** want them to be **easy to implement** and **fast**
  - Scientists mostly won out:
    - Nice standards for rounding, overflow, underflow, but...
    - Hard to make fast in hardware
    - **Float operations can be an order of magnitude slower than integer ops**

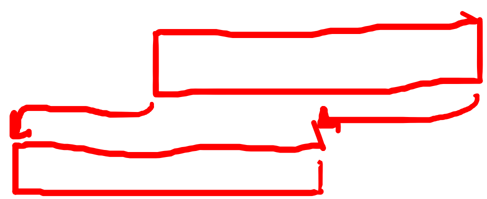
Floating Point Encoding (Review)

- Use normalized, base 2 scientific notation:
  - Value: $\pm 1 \times \text{Mantissa} \times 2^{\text{Exponent}}$
  - Bit Fields: $(-1)^{S} \times 1.M \times 2^{(E-\text{bias})}$
- Representation Scheme:
  - Sign bit (0 is positive, 1 is negative)
  - Mantissa (a.k.a. significand) is the fractional part of the number in normalized form and encoded in bit vector **M**
  - Exponent weights the value by a (possibly negative) power of 2 and encoded in the bit vector **E**



The Exponent Field (Review)

- Use biased notation
  - Read exponent as unsigned, but with *bias* of $2^{w-1}-1$ = 127
  - Representable exponents roughly ½ positive and ½ negative
  - Exp = E – bias ↔ E = Exp + bias
    - Exponent 0 (Exp = 0) is represented as E = 0b 0111 1111



- Why biased?
  - Now it's a sign-and-magnitude representation!
  - Makes floating point arithmetic easier (somewhat compatible with two's complement hardware)

The Mantissa (Fraction) Field (Review)

31	30 … 23	22 … 0
S	E	M
1 bit	8 bits	23 bits

$$(-1)^{S} \times (1\,.\,M) \times 2^{(E-\text{bias})}$$

- Note the implicit leading 1 in front of the M bit vector
  - Example: 0b 0011 1111 1100 0000 0000 0000 0000 0000 is read as $1.1_2 = 1.5_{10}$, *not* $0.1_2 = 0.5_{10}$
  - Gives us an extra bit of *precision*

$2^{Exp+1} - 2^{Exp-23}$

- Mantissa "limits"
  - Low values near M = 0b0…0 are close to 2^{Exp}

    $1.000\ldots \times 2^{Exp}$
  - High values near M = 0b1…1 are close to 2^{Exp+1}

    $1.111\ldots \times 2^{Exp}$

Normalized Floating Point Conversions

- FP → Decimal
  1. Append the bits of M to implicit leading 1 to form the mantissa.
  2. Multiply the mantissa by $2^{E-\text{bias}}$.
  3. Multiply the sign $(-1)^S$.
  4. Multiply out the exponent by shifting the binary point.
  5. Convert from binary to decimal.

- Decimal → FP
  1. Convert decimal to binary.
  2. Convert binary to normalized scientific notation.
  3. Encode sign as S (0/1).
  4. Add the bias to exponent and encode E as unsigned.
  5. The first bits after the leading 1 that fit are encoded into M.

Practice Question

$2^{-1} = 0.5$
$2^{-2} = 0.25$
$2^{-3} = 0.125$
$2^{-4} = 0.0625$

- Convert the decimal number **-7.375** into floating point representation

binary point

sum of powers of 2: $2^2 + 2^1 + 2^0 + 2^{-2} + 2^{-3}$

1 1 1 . 0 1 1

$\Rightarrow -111.011_2$

$= 1.11011 \times 2^2$

S = 1 (negative)

M = 11011 (implicit 1)

(add bias)

E = 2 + 127 = 129 = 10000001

Overall: 11000000111011 0...0

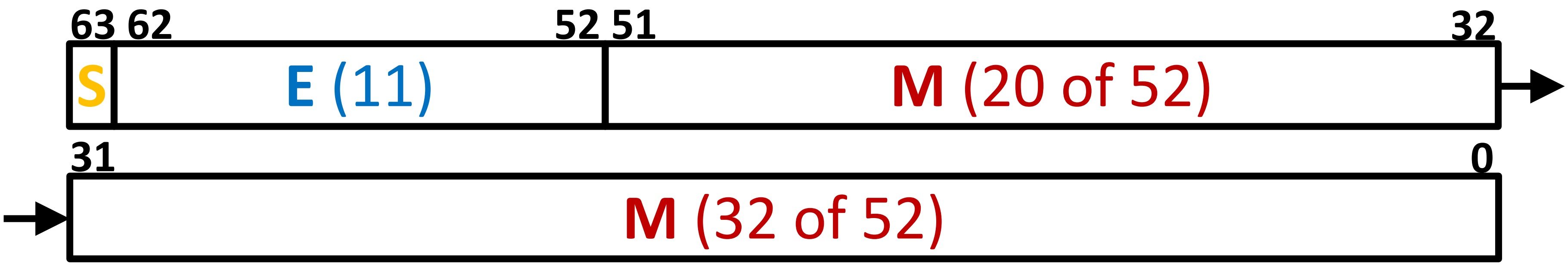
Precision and Accuracy

- Precision is a count of the number of bits in a computer word used to represent a value
  - Capacity for accuracy
- Accuracy is a measure of the difference between the *actual value of a number* and its computer representation

  - *High precision permits high accuracy but doesn't guarantee it. It is possible to have high precision but low accuracy.*
  - **Example:** `float pi = 3.14;`
    - `pi` will be represented using all 24 bits of the mantissa (highly precise), but is only an approximation (not accurate)

Need Greater Precision?

- Double Precision (vs. Single Precision) in 64 bits



  - C variable declared as `double`
  - Exponent bias is now $2^{10}-1 = 1023$
  - **Advantages:** greater precision (larger mantissa), greater range (larger exponent)
  - **Disadvantages:** more bits used, slower to manipulate

Floating Point Topics

- Fractional binary numbers
- IEEE floating-point standard
- **Floating-point operations and rounding**
- **Floating-point in C**

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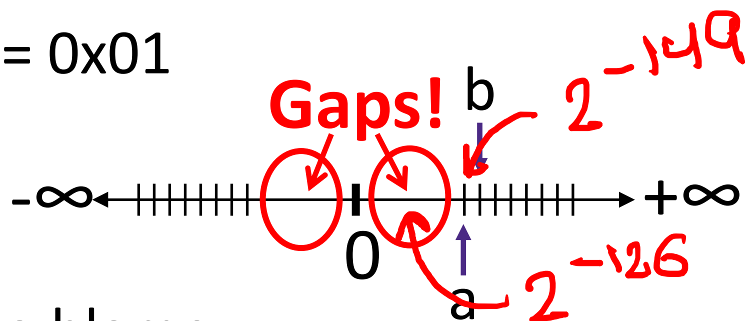
Special Cases

- But wait… what happened to zero?
  - *Special case:* E and M all zeros = 0
  - Two zeros! But at least 0x00000000 = 0 like integers
- E = 0xFF, M = 0: ± ∞
  - *e.g.*, division by 0
  - Still work in comparisons!
- E = 0xFF, M ≠ 0: Not a Number (NaN)
  - *e.g.*, square root of negative number, 0/0, ∞–∞
  - NaN propagates through computations
  - Value of M can be useful in debugging

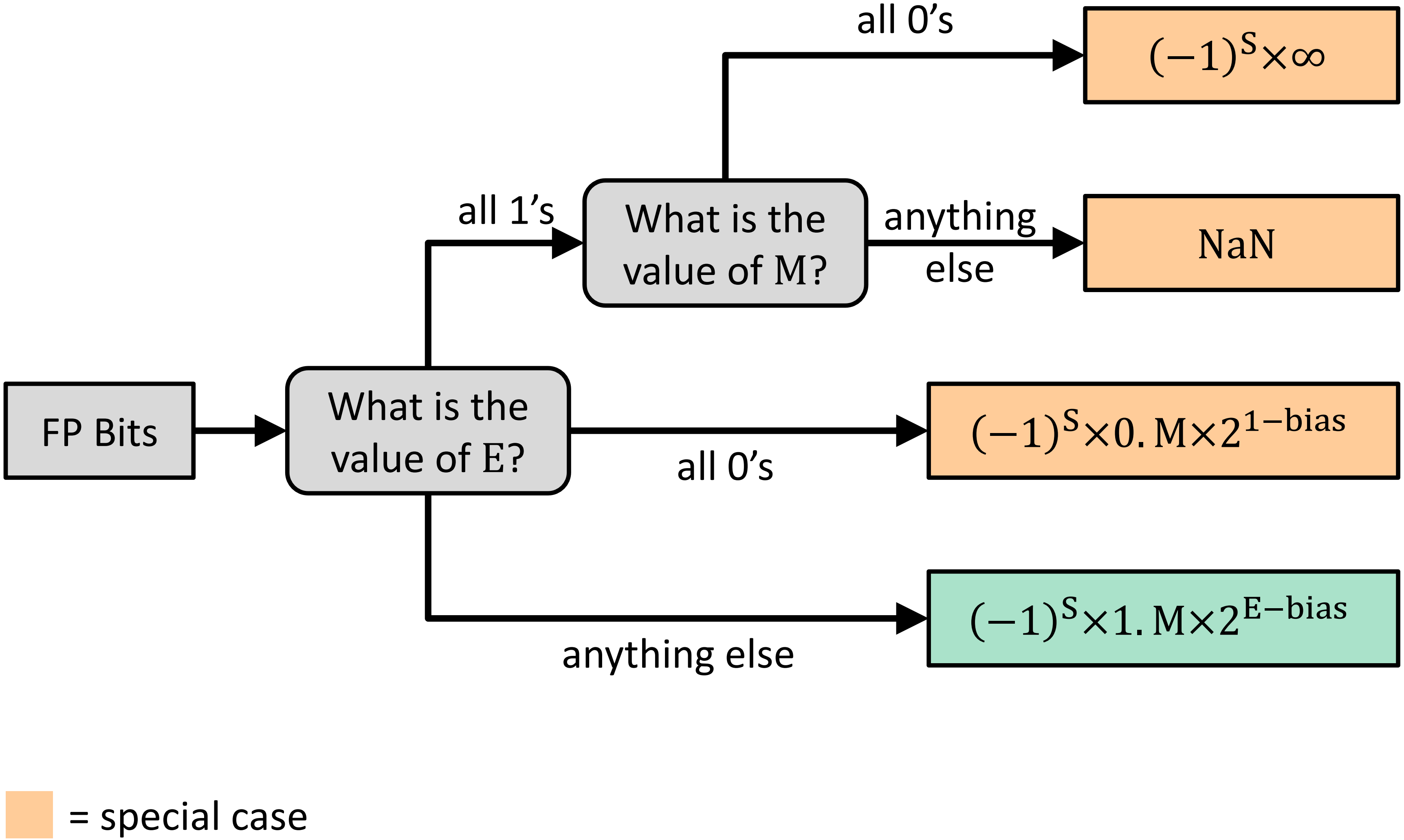
New Representation Limits

- New largest value (besides ∞)?
  - E = 0xFF has now been taken!
  - E = 0xFE has largest: $1.1\ldots1_2 \times 2^{127} = 2^{128} - 2^{104}$

- New numbers closest to 0:
  - E = 0x00 taken; next smallest is E = 0x01
  - $a = 1.0\ldots00_2 \times 2^{-126} = 2^{-126}$
  - $b = 1.0\ldots01_2 \times 2^{-126} = 2^{-126} + 2^{-149}$
  - Normalization and implicit 1 are to blame
  - *Special case:* E = 0, M ≠ 0 are denormalized numbers
    - Mantissa has implicit 0 instead of implicit 1
    - Store much smaller numbers

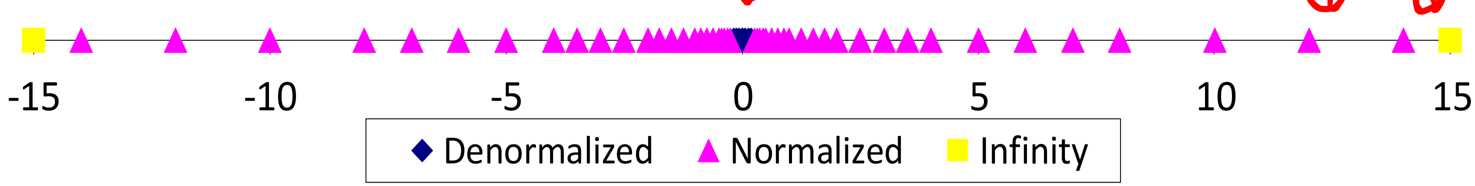


Floating Point Decoding Flow Chart



Distribution of Values (Review)

- What ranges are NOT representable?
  - Between largest norm and infinity **Overflow** (Exp too large)
  - Between zero and smallest denorm **Underflow** (Exp too small)
  - Between norm numbers? **Rounding**
- Given a FP number, what's the next largest representable number?
  - What is this "step" when Exp = 0? 2^{-23}
  - What is this "step" when Exp = 100? 2^{77}
- Distribution of values is denser toward zero



Floating Point Operations: Basic Idea

$$\text{Value} = (-1)^{S} \times \text{Mantissa} \times 2^{\text{Exponent}}$$

S	E	M

- $x +_f y = \text{Round}(x + y)$
- $x *_f y = \text{Round}(x * y)$

- Basic idea for floating point operations:
  - First, compute the exact result
  - Then *round* the result to make it fit into the specified precision (width of M)
    - Possibly over/underflow if exponent outside of range

Mathematical Properties of FP Operations

- Overflow yields $\pm\infty$ and underflow yields 0
- Floats with value $\pm\infty$ and NaN can be used in operations
  - Result usually still $\pm\infty$ or NaN, but not always intuitive
- Floating point operations do not work like real math, due to rounding

  rounded off
  - Not associative: `(3.14+1e100)-1e100 != 3.14+(1e100-1e100)`

    `0` `3.14`
  - Not distributive: `100*(0.1+0.2) != 100*0.1+100*0.2`

    `30.000000000000003553` `30`

    no exact binary fraction
  - Not cumulative
    - Repeatedly adding a very small number to a large one may do nothing

Floating Point in C

!!!

- Two common levels of precision:

`float`	`1.0f`	single precision (32-bit)
`double`	`1.0`	double precision (64-bit)

- `#include <math.h>` to get `INFINITY` and `NAN` constants
- `#include <float.h>` for additional constants
- **Equality (==) comparisons between floating point numbers are tricky, and often return unexpected results, so just avoid them!**

some small threshold

instead: $|a-b| < \varepsilon$

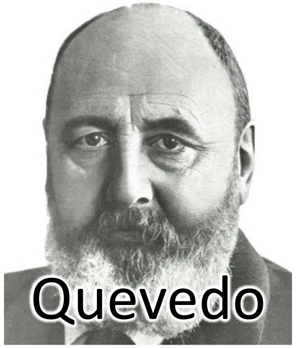
Floating Point Conversions in C

!!!

- Casting between `int`, `float`, and `double` **changes the bit representation**
  - `int` → `float`
    - May be rounded (not enough bits in mantissa: 23)
    - Overflow impossible
  - `int` or `float` → `double`
    - Exact conversion (all 32-bit `int`s are representable)
  - `long` → `double`
    - Depends on word size (32-bit is exact, 64-bit may be rounded)
  - `double` or `float` → `int`
    - Truncates fractional part (rounded toward zero)
    - “Not defined” when out of range or NaN: generally sets to TMin (even if the value is a very big positive)

More on Floating Point History

- Early days
  - First design with floating-point arithmetic in 1914 by Leonardo Torres y Quevedo
  - Implementations started in 1940 by Konrad Zuse, but with differing field lengths (usually not summing to 32 bits) and different subsets of the special cases
- IEEE 754 standard created in 1985
  - Primary architect was William Kahan, who won a Turing Award for this work
  - Standardized bit encoding, well-defined behavior for *all* arithmetic operations



Number Representation Really Matters

- **1991:** Patriot missile targeting error
  - clock skew due to conversion from integer to floating point
- **1996:** Ariane 5 rocket exploded ($1 billion)
  - overflow converting 64-bit floating point to 16-bit integer
- **2000:** Y2K problem
  - limited (decimal) representation: overflow, wrap-around
- **2038:** Unix epoch rollover
  - Unix epoch = seconds since 12am, January 1, 1970
  - signed 32-bit integer representation rolls over to TMin in 2038
- **Other related bugs:**
  - 1982: Vancouver Stock Exchange 10% error in less than 2 years
  - 1994: Intel Pentium FDIV (floating point division) HW bug ($475 million)
  - 1997: USS Yorktown "smart" warship stranded: divide by zero
  - 1998: Mars Climate Orbiter crashed: unit mismatch ($193 million)

Summary

- Floating point approximates real numbers:

  - Handles large numbers, small numbers, special numbers
  - Exponent in biased notation (bias = $2^{w-1} - 1$)
    - Size of exponent field determines our representable *range*
    - Outside of representable exponents is *overflow* and *underflow*
  - Mantissa approximates fractional portion of binary point
    - Size of mantissa field determines our representable *precision*
    - Implicit leading 1 (normalized) except in special cases
    - Exceeding length causes *rounding*

Summary

- Floats also suffer from the fixed number of bits available to represent them
  - Can get overflow/underflow
  - "Gaps" produced in representable numbers means we can lose precision, unlike `int`s
    - Some "simple fractions" have no exact representation (*e.g.*, 0.2)
    - "Every operation gets a slightly wrong result"
- Floating point arithmetic not associative or distributive
  - Mathematically equivalent ways of writing an expression may compute different results
- Never test floating point values for equality!
- Careful when converting between `int`s and `float`s!

Summary

E	M	Meaning
0x00	0	± 0
0x00	non-zero	± denorm num
0x01 – 0xFE	anything	± norm num
0xFF	0	± ∞
0xFF	non-zero	NaN

- Floating point encoding has many limitations
  - Overflow, underflow, rounding
  - Rounding is a HUGE issue due to limited mantissa bits and gaps that are scaled by the value of the exponent
  - Floating point arithmetic is NOT associative or distributive
- Converting between integral and floating point data types *does* change the bits

BONUS SLIDES

Some additional information about floating point numbers. We won't test you on this, but you may find it interesting ☺

Floating Point Rounding

This is extra (non-testable) material

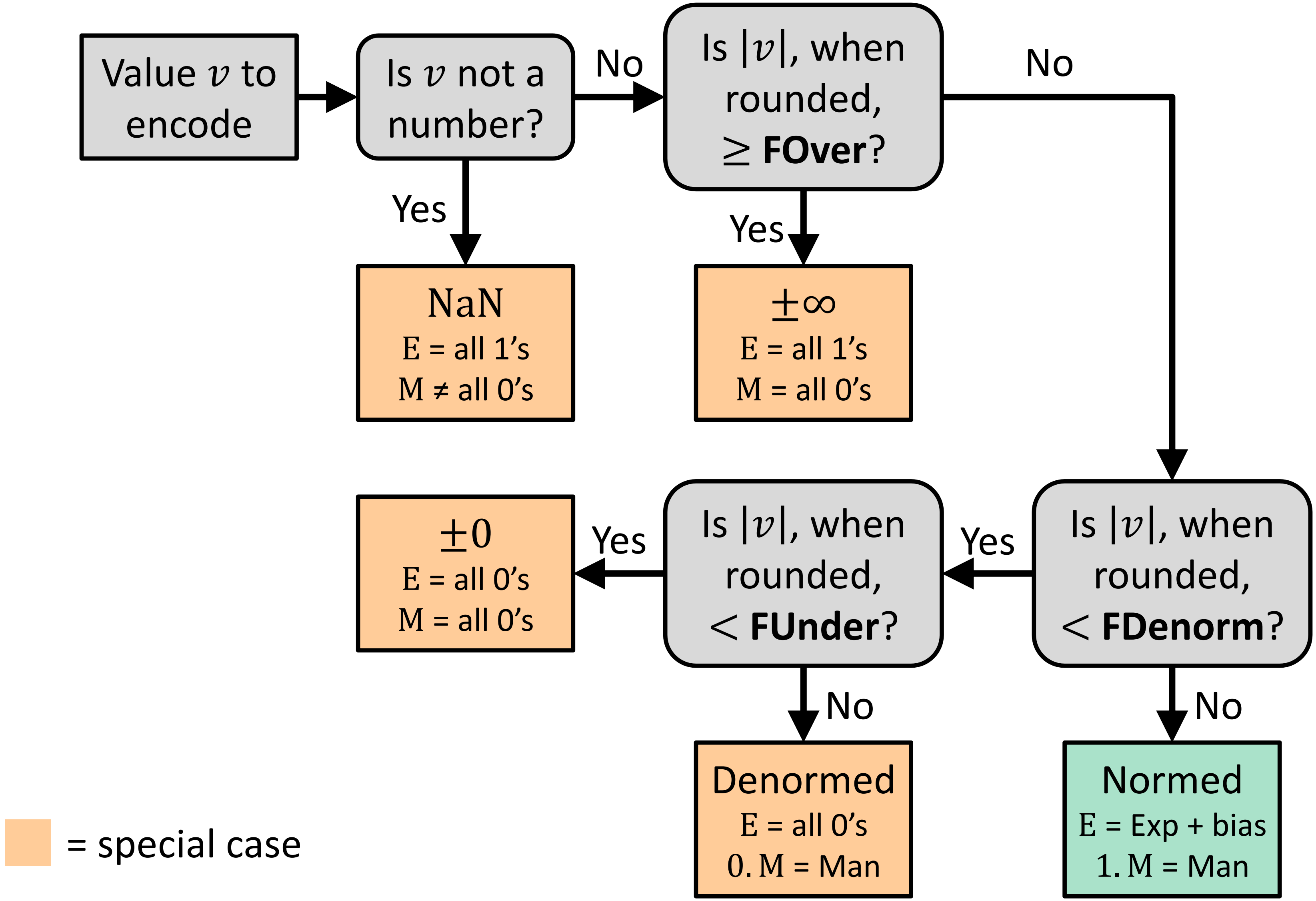
- The IEEE 754 standard actually specifies different rounding modes:
  - Round to nearest, ties to nearest even digit
  - Round toward $+\infty$ (round up)
  - Round toward $-\infty$ (round down)
  - Round toward 0 (truncation)

S	E	M
1	4	3

- In our tiny example:
  - Man = 1.001 01 rounded to M = 0b001
  - Man = 1.001 11 rounded to M = 0b010
  - Man = 1.001 10 rounded to M = 0b010
  - Man = 1.000 10 rounded to M = 0b000

Floating Point Encoding Flow Chart

This is extra (non-testable) material



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Limits of Interest

- The following thresholds will help give you a sense of when certain outcomes come into play, but don't worry about the specifics:
  - **FOver** = $2^{\text{bias}+1}$ = 2^{8}
    - This is just larger than the largest representable normalized number
  - **FDenorm** = $2^{1-\text{bias}}$ = 2^{-6}
    - This is the smallest representable normalized number
  - **FUnder** = $2^{1-\text{bias}-m}$ = 2^{-9}
    - m is the width of the mantissa field
    - This is the smallest representable denormalized number

Denorm Numbers

This is extra (non-testable) material

- Denormalized numbers
  - No leading 1
  - Uses implicit exponent of –126 even though E = 0x00

- Denormalized numbers close the gap between zero and the smallest normalized number
  - Smallest norm: $\pm\ 1.0\ldots0_{two}\times2^{-126} = \pm\ 2^{-126}$
  - Smallest denorm: $\pm\ 0.0\ldots01_{two}\times2^{-126} = \pm\ 2^{-149}$
    - There is still a gap between zero and the smallest denormalized number

So much closer to 0

Floating Point in the "Wild"

This is extra (non-testable) material

- 3 formats from IEEE 754 standard widely used in computer hardware and languages
  - In C, called `float`, `double`, `long double`
- Common applications:
  - 3D graphics: textures, rendering, rotation, translation
  - "Big Data": scientific computing at scale, machine learning
- Non-standard formats in domain-specific areas:
  - **Bfloat16:** training ML models; range more valuable than precision
  - **TensorFloat-32:** Nvidia-specific hardware for Tensor Core GPUs

Type	S bits	E bits	M bits	Total bits
Half-precision	1	5	10	16
Bfloat16	1	8	7	16
TensorFloat-32	1	8	10	19
Single-precision	1	8	23	32