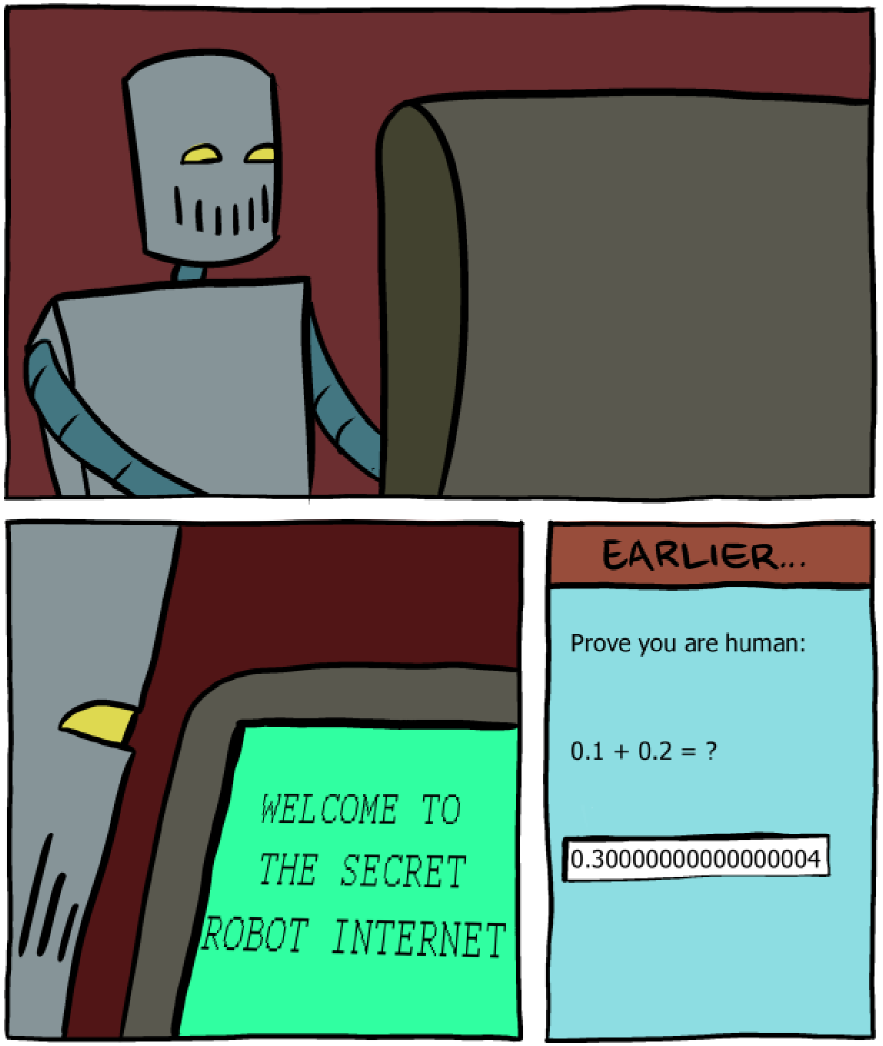
Floating Point II

CSE 351 Spring 2022

Instructor:
Ruth Anderson

Teaching Assistants:
Melissa Birchfield
Jacob Christy
Alena Dickmann
Kyrie Dowling
Ellis Haker
Maggie Jiang
Diya Joy
Anirudh Kumar
Jim Limprasert
Armin Magness
Hamsa Shankar
Dara Stotland
Jeffery Tian
Assaf Vayner
Tom Wu
Angela Xu
Effie Zheng



http://www.smbc-comics.com/?id=2999

Relevant Course Information

- hw5 due Monday (4/11) @ 11:59 pm
- Lab 1a due TONIGHT (4/11) @ 11:59 pm
  - Submit `pointer.c` and `lab1Asynthesis.txt`
  - Make sure you check the Gradescope autograder output!
  - Can use late day tokens to submit up until Wed 11:59 pm
- Lab 1b, due 4/18
  - Submit `aisle_manager.c`, `store_client.c`, and `lab1Bsynthesis.txt`

Getting Help with 351

- Lecture recordings, readings, inked slides, textbook readings
- Form a study group!
  - Good for everything but labs, which should be done in pairs
  - Communicate regularly, use the class terminology, ask and answer each others' questions, show up to OH together
- Attend office hours
  - Use the OH queue, but can also chat with other students there – help each other learn!
- Post on Ed Discussion
- Request a 1-on-1 meeting
  - Available on a limited basis for special circumstances

Reading Review

- Terminology:
  - Special cases
    - Denormalized numbers
    - $\pm\infty$
    - Not-a-Number (NaN)
  - Limits of representation
    - Overflow
    - Underflow
    - Rounding

Review Questions

- What is the value of the following floats?
  - 0x00000000
  - 0xFF800000

- For the following code, what is the smallest value of n that will encounter a limit of representation?

```
float f = 1.0;  // 2^0
for (int i = 0; i < n; ++i)
    f *= 1024;  // 1024 = 2^10
printf("f = %f\n", f);
```

Floating Point Encoding Summary (Review)

E	M	Meaning
0x00	0	± 0
0x00	non-zero	± denorm num
0x01 – 0xFE	anything	± norm num
0xFF	0	± ∞
0xFF	non-zero	NaN

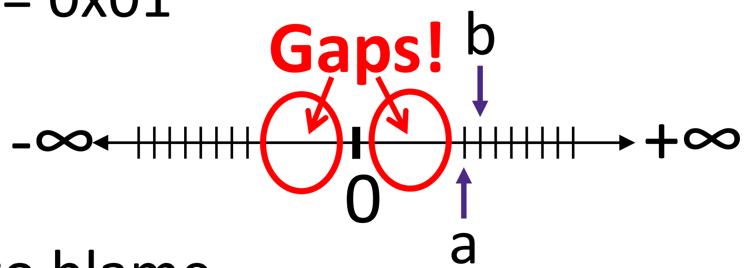
Special Cases

- But wait… what happened to zero?
  - *Special case:* E and M all zeros = 0
  - Two zeros! But at least 0x00000000 = 0 like integers
- E = 0xFF, M = 0: ± ∞
  - *e.g.*, division by 0
  - Still work in comparisons!
- E = 0xFF, M ≠ 0: Not a Number (NaN)
  - *e.g.*, square root of negative number, 0/0, ∞–∞
  - NaN propagates through computations
  - Value of M can be useful in debugging

New Representation Limits

- New largest value (besides ∞)?
  - E = 0xFF has now been taken!
  - E = 0xFE has largest: $1.1\ldots1_2 \times 2^{127} = 2^{128} - 2^{104}$

- New numbers closest to 0:
  - E = 0x00 taken; next smallest is E = 0x01
  - $a = 1.0\ldots00_2 \times 2^{-126} = 2^{-126}$
  - $b = 1.0\ldots01_2 \times 2^{-126} = 2^{-126} + 2^{-149}$
  - Normalization and implicit 1 are to blame
  - *Special case:* E = 0, M ≠ 0 are denormalized numbers



Denorm Numbers

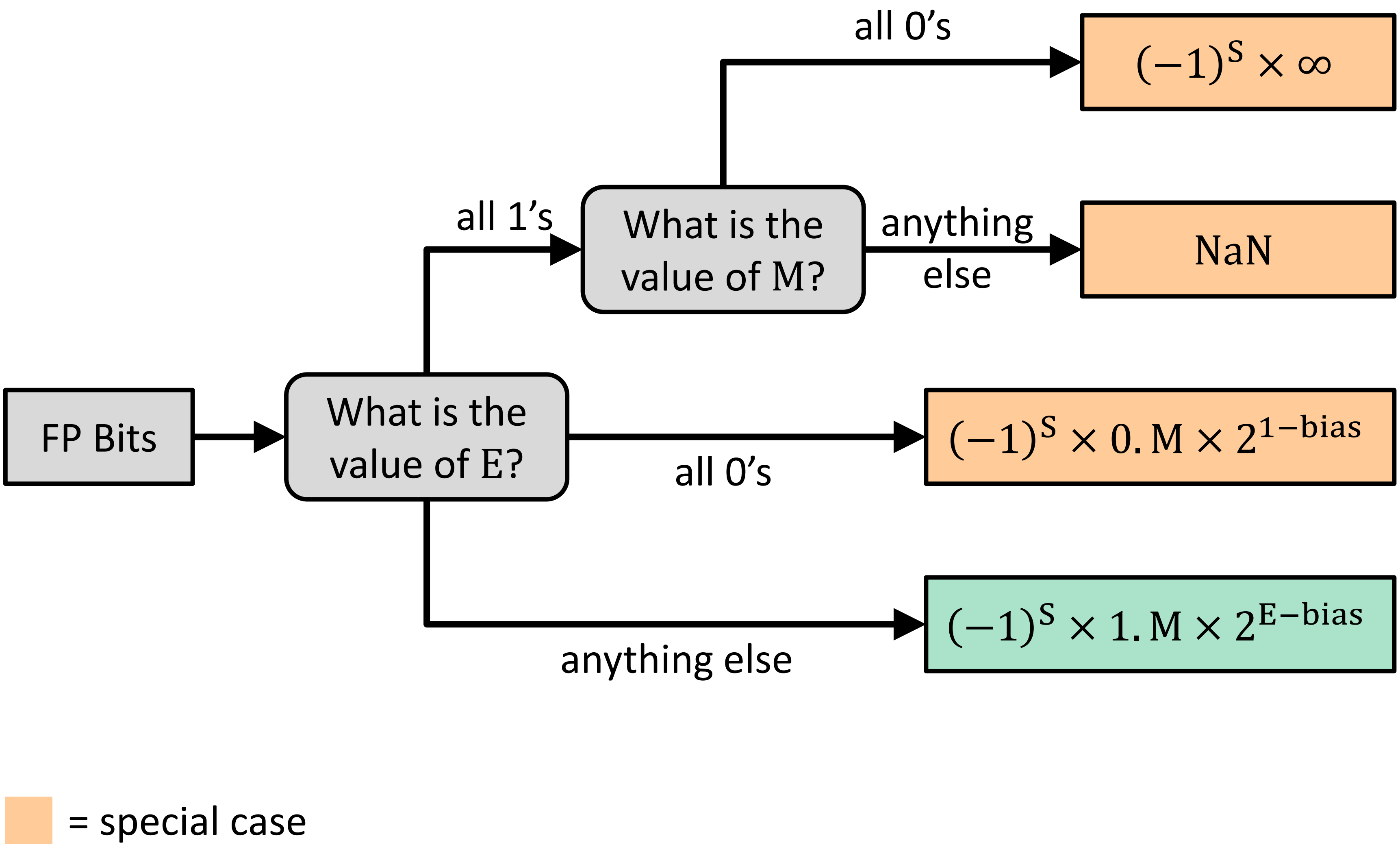
This is extra (non-testable) material

- Denormalized numbers
  - No leading 1
  - Uses implicit exponent of –126 even though E = 0x00

- Denormalized numbers close the gap between zero and the smallest normalized number
  - Smallest norm: $\pm\ 1.0\ldots00_{\text{two}}\times 2^{-126} = \pm\ 2^{-126}$
  - Smallest denorm: $\pm\ 0.0\ldots01_{\text{two}}\times 2^{-126} = \pm\ 2^{-149}$
    - There is still a gap between zero and the smallest denormalized number

So much closer to 0

Floating Point Interpretation Flow Chart



Floating point topics

- Fractional binary numbers
- IEEE floating-point standard
- **Floating-point operations and rounding**
- **Floating-point in C**

- There are many more details that we won’t cover
  - It’s a 58-page standard…

Tiny Floating Point Representation

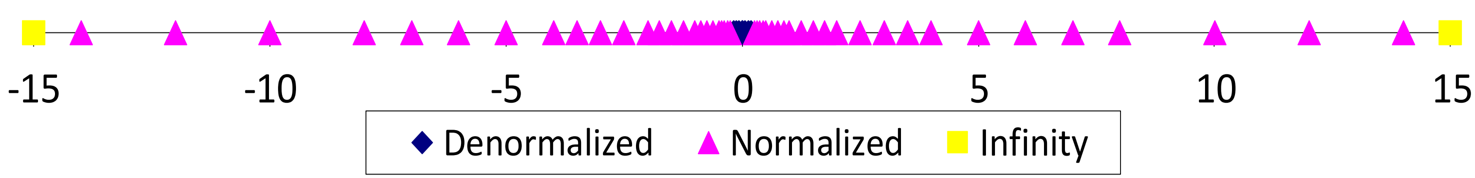
- We will use the following **8-bit** floating point representation to illustrate some key points:

S	E	M
1	4	3

- Assume that it has the same properties as IEEE floating point:
  - bias =
  - encoding of -0 =
  - encoding of $+\infty$ =
  - encoding of the largest (+) normalized # =
  - encoding of the smallest (+) normalized # =

Distribution of Values (Review)

- What ranges are NOT representable?
  - Between largest norm and infinity **Overflow** (Exp too large)
  - Between zero and smallest denorm **Underflow** (Exp too small)
  - Between norm numbers? **Rounding**
- Given a FP number, what's the bit pattern of the next largest representable number?
  - What is this "step" when Exp = 0?
  - What is this "step" when Exp = 100?
- Distribution of values is denser toward zero



Floating Point Rounding

This is extra (non-testable) material

- The IEEE 754 standard actually specifies different rounding modes:
  - Round to nearest, ties to nearest even digit
  - Round toward $+\infty$ (round up)
  - Round toward $-\infty$ (round down)
  - Round toward 0 (truncation)

- In our tiny example:
  - Man = 1.001 01 rounded to M = 0b001
  - Man = 1.001 11 rounded to M = 0b010
  - Man = 1.001 10 rounded to M = 0b010
  - Man = 1.000 10 rounded to M = 0b000

S	E	M
1	4	3

Floating Point Operations: Basic Idea

$$\text{Value} = (-1)^{S} \times \text{Mantissa} \times 2^{\text{Exponent}}$$

S	E	M

- $x +_f y = \text{Round}(x + y)$
- $x *_f y = \text{Round}(x * y)$

- Basic idea for floating point operations:
  - First, compute the exact result
  - Then *round* the result to make it fit into the specified precision (width of M)
    - Possibly over/underflow if exponent outside of range

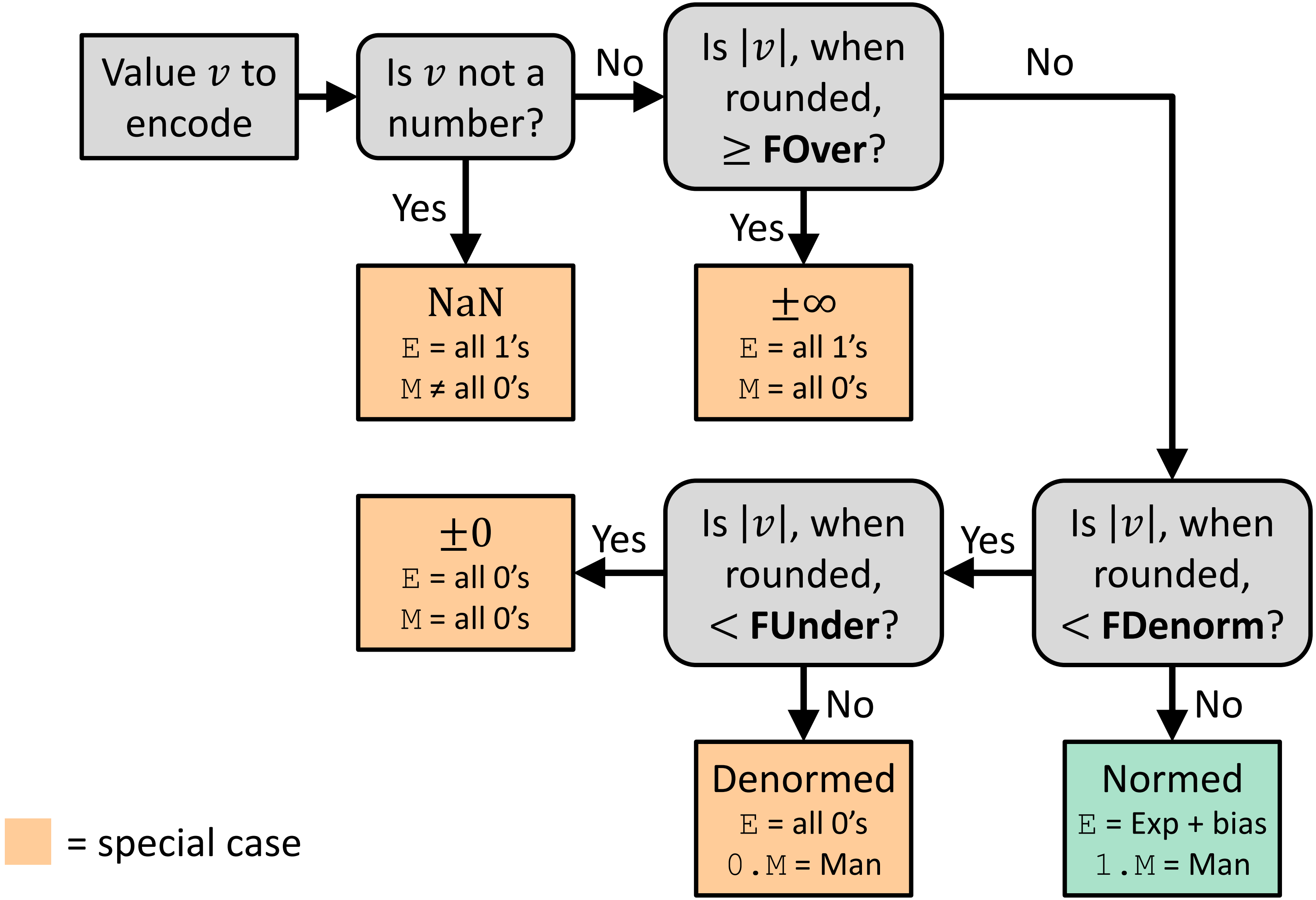
Mathematical Properties of FP Operations

- Overflow yields $\pm\infty$ and underflow yields 0
- Floats with value $\pm\infty$ and NaN can be used in operations
  - Result usually still $\pm\infty$ or NaN, but not always intuitive
- Floating point operations do not work like real math, due to rounding
  - Not associative: `(3.14+1e100)-1e100 != 3.14+(1e100-1e100)`

    `0` **`3.14`**
  - Not distributive: `100*(0.1+0.2) != 100*0.1+100*0.2`

    `30.000000000000003553` **`30`**
  - Not cumulative
    - Repeatedly adding a very small number to a large one may do nothing

Floating Point Encoding Flow Chart



Limits of Interest

This is extra (non-testable) material

- The following thresholds will help give you a sense of when certain outcomes come into play, but don't worry about the specifics:
  - **FOver** = $2^{\text{bias}+1}$ = 2^{8}
    - This is just larger than the largest representable normalized number
  - **FDenorm** = $2^{1-\text{bias}}$ = 2^{-6}
    - This is the smallest representable normalized number
  - **FUnder** = $2^{1-\text{bias}-m}$ = 2^{-9}
    - m is the width of the mantissa field
    - This is the smallest representable denormalized number

Floating Point in C

!!!

- Two common levels of precision:

`float`	`1.0f`	single precision (32-bit)
`double`	`1.0`	double precision (64-bit)

- `#include <math.h>` to get `INFINITY` and `NAN` constants

- Equality (==) comparisons between floating point numbers are tricky, and often return unexpected results, so just avoid them!

Floating Point Conversions in C

!!!

- Casting between `int`, `float`, and `double` **changes** the bit representation
  - `int` → `float`
    - May be rounded (not enough bits in mantissa: 23)
    - Overflow impossible
  - `int` or `float` → `double`
    - Exact conversion (all 32-bit `int`s representable)
  - `long` → `double`
    - Depends on word size (32-bit is exact, 64-bit may be rounded)
  - `double` or `float` → `int`
    - Truncates fractional part (rounded toward zero)
    - "Not defined" when out of range or NaN: generally sets to `Tmin` (even if the value is a very big positive)

Exploration Question

- We execute the following code in C. How many bytes are the same (value and position) between `i` and `f`?

```
int i = 384;  // 2^8 + 2^7
float f = (float) i;
```

A. 0 bytes
B. 1 byte
C. 2 bytes
D. 3 bytes
E. We're lost…

Floating Point Summary

- Floats also suffer from the fixed number of bits available to represent them
  - Can get overflow/underflow
  - "Gaps" produced in representable numbers means we can lose precision, unlike `ints`
    - Some "simple fractions" have no exact representation (*e.g.* 0.2)
    - "Every operation gets a slightly wrong result"
- Floating point arithmetic not associative or distributive
  - Mathematically equivalent ways of writing an expression may compute different results
- Never test floating point values for equality!
- Careful when converting between `ints` and `floats`!

Number Representation Really Matters

- **1991:** Patriot missile targeting error
  - clock skew due to conversion from integer to floating point
- **1996:** Ariane 5 rocket exploded ($1 billion)
  - overflow converting 64-bit floating point to 16-bit integer
- **2000:** Y2K problem
  - limited (decimal) representation: overflow, wrap-around
- **2038:** Unix epoch rollover
  - Unix epoch = seconds since 12am, January 1, 1970
  - signed 32-bit integer representation rolls over to TMin in 2038
- **Other related bugs:**
  - 1982: Vancouver Stock Exchange 10% error in less than 2 years
  - 1994: Intel Pentium FDIV (floating point division) HW bug ($475 million)
  - 1997: USS Yorktown "smart" warship stranded: divide by zero
  - 1998: Mars Climate Orbiter crashed: unit mismatch ($193 million)

Summary

E	M	Meaning
0x00	0	± 0
0x00	non-zero	± denorm num
0x01 – 0xFE	anything	± norm num
0xFF	0	± ∞
0xFF	non-zero	NaN

- Floating point encoding has many limitations
  - Overflow, underflow, rounding
  - Rounding is a HUGE issue due to limited mantissa bits and gaps that are scaled by the value of the exponent
  - Floating point arithmetic is NOT associative or distributive
- Converting between integral and floating point data types *does* change the bits

BONUS SLIDES

An example that applies the IEEE Floating Point concepts to a smaller (8-bit) representation scheme.

These slides expand on material covered today, so while you don't need to read these, the information is "fair game."

Tiny Floating Point Example

S	E	M
1	4	3

- 8-bit Floating Point Representation
  - The sign bit is in the most significant bit (MSB)
  - The next four bits are the exponent, with a bias of $2^{4-1}-1 = 7$
  - The last three bits are the mantissa
- Same general form as IEEE Format
  - Normalized binary scientific point notation
  - Similar special cases for 0, denormalized numbers, NaN, ∞

Dynamic Range (Positive Only)

	S E M	Exp	Value	
Denormalized numbers	0 0000 000	-6	0	
	0 0000 001	-6	1/8*1/64 = 1/512	closest to zero
	0 0000 010	-6	2/8*1/64 = 2/512	
	…			
	0 0000 110	-6	6/8*1/64 = 6/512	
	0 0000 111	-6	7/8*1/64 = 7/512	largest denorm
Normalized numbers	0 0001 000	-6	8/8*1/64 = 8/512	smallest norm
	0 0001 001	-6	9/8*1/64 = 9/512	
	…			
	0 0110 110	-1	14/8*1/2 = 14/16	
	0 0110 111	-1	15/8*1/2 = 15/16	closest to 1 below
	0 0111 000	0	8/8*1 = 1	
	0 0111 001	0	9/8*1 = 9/8	closest to 1 above
	0 0111 010	0	10/8*1 = 10/8	
	…			
	0 1110 110	7	14/8*128 = 224	
	0 1110 111	7	15/8*128 = 240	largest norm
	0 1111 000	n/a	inf	

Special Properties of Encoding

- Floating point zero (0^+) exactly the same bits as integer zero
  - All bits = 0
- Can (Almost) Use Unsigned Integer Comparison
  - Must first compare sign bits
  - Must consider $0^- = 0^+ = 0$
  - NaNs problematic
    - Will be greater than any other values
    - What should comparison yield?
  - Otherwise OK
    - Denorm vs. normalized
    - Normalized vs. infinity