Floating Point II

CSE 351 Spring 2022

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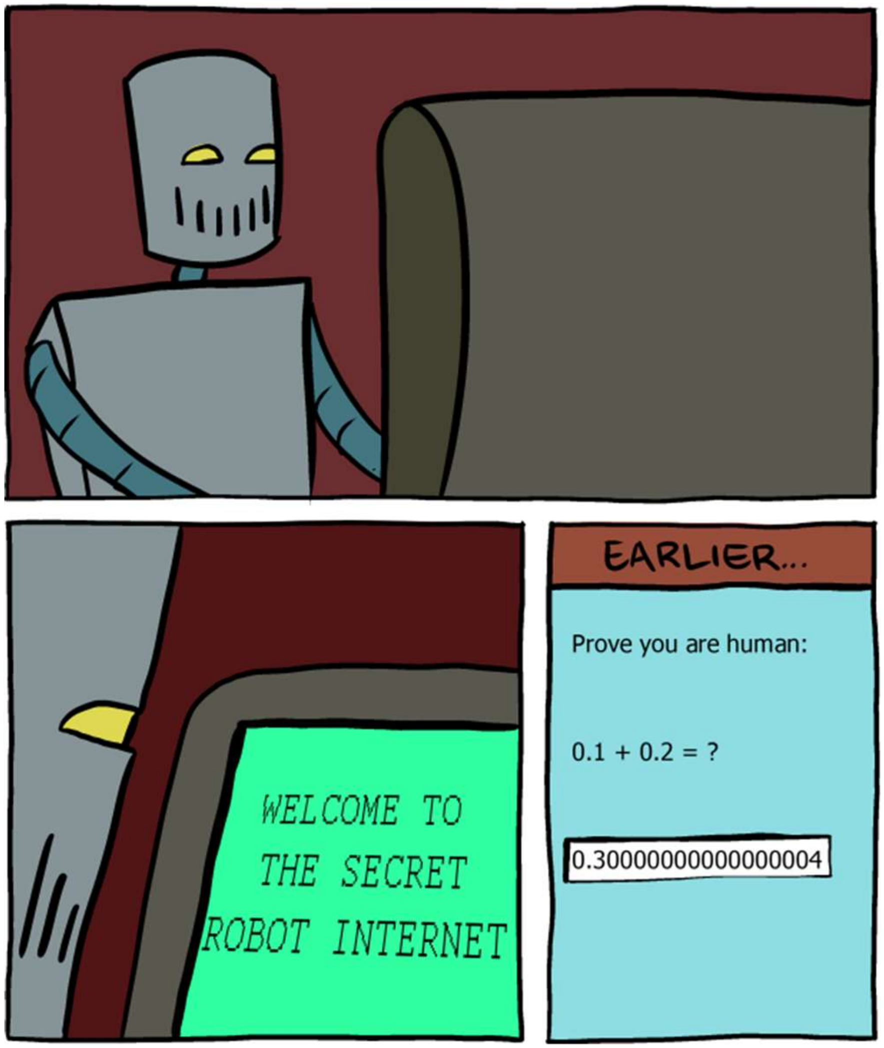


http://www.smbc-comics.com/?id=2999

Relevant Course Information

- hw5 due Monday (4/11) @ 11:59 pm
- Lab 1a due TONIGHT (4/11) @ 11:59 pm
  - Submit `pointer.c` and `lab1Asynthesis.txt`
  - Make sure you check the Gradescope autograder output!
  - Can use late day tokens to submit up until Wed 11:59 pm
- Lab 1b, due 4/18
  - Submit `aisle_manager.c`, `store_client.c`, and `lab1Bsynthesis.txt`

Getting Help with 351

- Lecture recordings, readings, inked slides, textbook readings
- Form a study group!
  - Good for everything but labs, which should be done in pairs
  - Communicate regularly, use the class terminology, ask and answer each others' questions, show up to OH together
- Attend office hours
  - Use the OH queue, but can also chat with other students there – help each other learn!
- Post on Ed Discussion
- Request a 1-on-1 meeting
  - Available on a limited basis for special circumstances

Reading Review

- Terminology:
  - Special cases
    - Denormalized numbers
    - ±∞
    - Not-a-Number (NaN)
  - Limits of representation
    - Overflow
    - Underflow
    - Rounding

Review Questions

- What is the value of the following floats?
  - 0x00000000 S=0 E=0 M=0 $\boxed{+0}$
  - 0xFF800000 S=1 E=all 1s M=0 $\boxed{-\infty}$

$\underbrace{1111\ 1111}\ 1000$

- For the following code, what is the smallest value of n that will encounter a limit of representation?

```
float f = 1.0;  // 2^0
for (int i = 0; i < n; ++i)
    f *= 1024;  // 1024 = 2^10
printf("f = %f\n", f);
```

$E_{max} = 0xFE$, $Exp_{max} = 254 - 127 = 127$

for $\boxed{n=13}$, we hit 2^{130}, which causes $\boxed{\text{overflow}}$

n	f
1	2^{10}
2	2^{20}
3	2^{30}
⋮	
12	2^{120}
13	2^{130} ←

Floating Point Encoding Summary (Review)

E	M	Meaning
0x00	0	± 0
0x00	non-zero	± denorm num
0x01 – 0xFE	anything	± norm num
0xFF	0	± ∞
0xFF	non-zero	NaN

smallest E (all 0's)

everything else

largest E (all 1's)

0000 0001 1111 1110

1 . . . 254 ← E

1-127 254-127

-126 . . 127 ← Exp

Special Cases

- But wait… what happened to zero?
    - *Special case:* E and M all zeros = 0
    - Two zeros! But at least 0x00000000 = 0 like integers

      1111 1111 0x8000 0000 = -0
- E = 0xFF, M = 0: ± ∞

  all ones
    - *e.g.*, division by 0
    - Still work in comparisons!
- E = 0xFF, M ≠ 0: Not a Number (NaN)
    - *e.g.*, square root of negative number, 0/0, ∞–∞
    - NaN propagates through computations
    - Value of M can be useful in debugging (tells you cause of NaN)

New Representation Limits

- New largest value (besides ∞)?
  - E = 0xFF has now been taken!
  - E = 0xFE has largest: $1.\overbrace{1\ldots1}^{\text{23 ones}}{}_2\times 2^{127} = 2^{128} - 2^{104}$

    ↳ 254 - bias

- New numbers closest to 0:

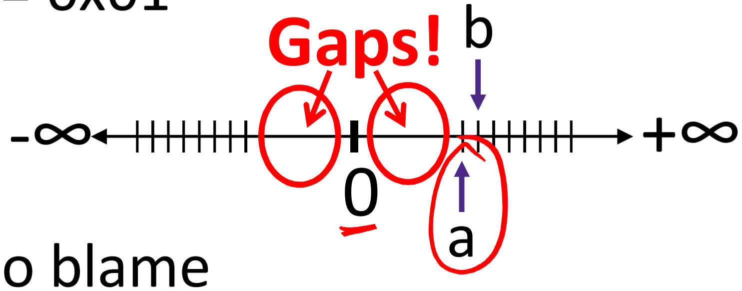
  Exp = -126

  - E = 0x00 taken; next smallest is E = 0x01
  - a = $1.0\ldots00_2\times 2^{\boxed{-126}} = 2^{-126}$

    23

  - b = $1.0\ldots01_2\times 2^{-126} = 2^{-126} + 2^{-149}$
  - Normalization and implicit 1 are to blame
  - *Special case:* E = 0, M ≠ 0 are denormalized numbers (0.M)

    normalized: 1.M



This is extra (non-testable) material

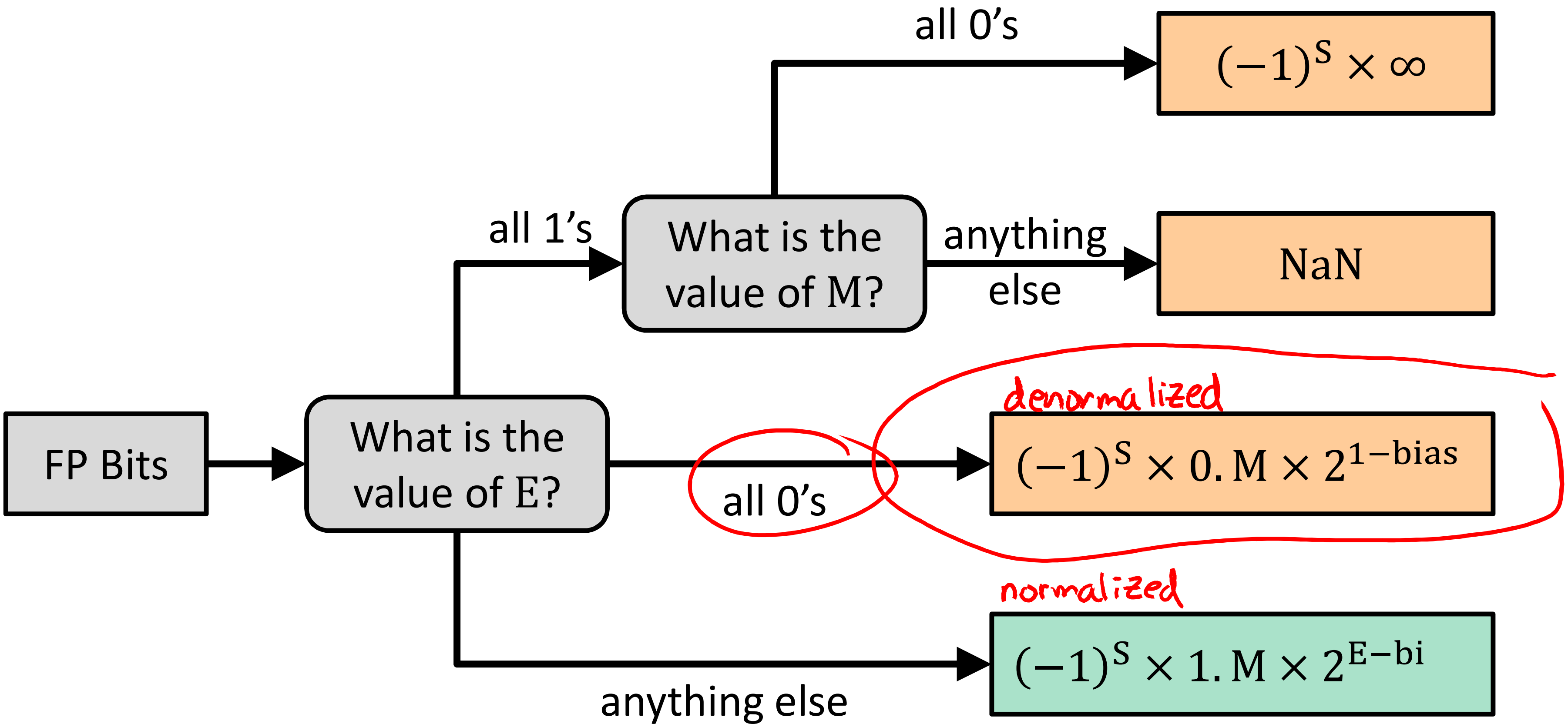
Denorm Numbers

- Denormalized numbers
  - No leading 1
  - Uses implicit exponent of –126 even though E = 0x00

- Denormalized numbers close the gap between zero and the smallest normalized number
  - Smallest norm: $\pm 1.0\ldots00_{\text{two}} \times 2^{-126} = \pm 2^{-126}$
  - Smallest denorm: $\pm 0.0\ldots01_{\text{two}} \times 2^{-126} = \pm 2^{-149}$
    - There is still a gap between zero and the smallest denormalized number

So much closer to 0

Floating Point Interpretation Flow Chart



FP Bits

What is the value of E?

all 1's → What is the value of M?

all 0's → $(-1)^S \times \infty$

anything else → NaN

all 0's → denormalized $(-1)^S \times 0.M \times 2^{1-\mathrm{bias}}$

anything else → normalized $(-1)^S \times 1.M \times 2^{E-\mathrm{bi}}$

🟧 = special case

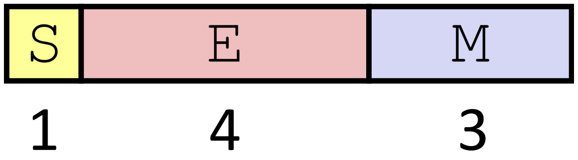
Floating point topics

- Fractional binary numbers
- IEEE floating-point standard
- **Floating-point operations and rounding**
- **Floating-point in C**

- There are many more details that we won’t cover
  - It’s a 58-page standard…

Tiny Floating Point Representation

- We will use the following **8-bit** floating point representation to illustrate some key points:



- Assume that it has the same properties as IEEE floating point:
  - bias = $2^{w-1}-1 = 2^{4-1}-1 = 2^{3}-1 = 7$
  - encoding of -0 = 0b 1 0000 000 → 0x80
  - encoding of $+\infty$ = 0b 0 1111 000 → 0x78
  - encoding of the largest (+) normalized # = 0b 0 1110 111 → $\underline{1}.\underline{111}_2 \times 2^{14-7}$
  - encoding of the smallest (+) normalized # = 0b 0 0001 000 → $\underline{1}.0_2 \times 2^{1-7}$

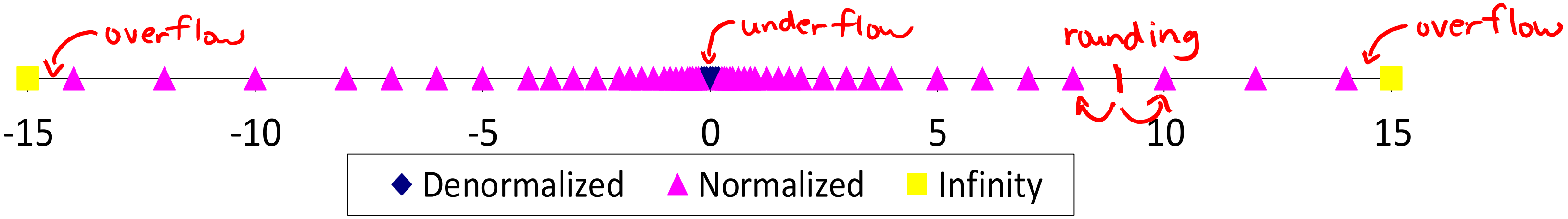
Distribution of Values (Review)

- What ranges are NOT representable?
  - Between largest norm and infinity **Overflow** (Exp too large)
  - Between zero and smallest denorm **Underflow** (Exp too small)
  - Between norm numbers? **Rounding**
- Given a FP number, what's the bit pattern of the next largest representable number?

  if M = 0b 0…00, then $2^{Exp} \times 1.0$

  if M = 0b 0…01, then $2^{Exp} \times (1+2^{-23})$

  diff = 2^{Exp-23}
  - What is this "step" when Exp = 0? 2^{-23}
  - What is this "step" when Exp = 100? 2^{77}
- Distribution of values is denser toward zero



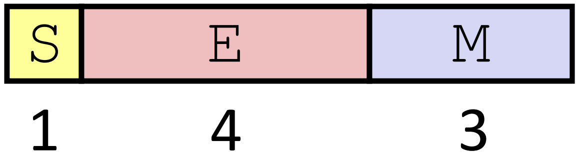
Floating Point Rounding

This is extra (non-testable) material

- The IEEE 754 standard actually specifies different rounding modes:
  - Round to nearest, ties to nearest even digit
  - Round toward $+\infty$ (round up)
  - Round toward $-\infty$ (round down)
  - Round toward 0 (truncation)

→ 0.5_{10}

→ 0.1_2



- In our tiny example:
  - Man = 1.001/01 rounded to M = 0b001 (down) — < half
  - Man = 1.001/11 rounded to M = 0b010 (up) — > half
  - Man = 1.001/10 rounded to M = 0b010 (up) — == half
  - Man = 1.000/10 rounded to M = 0b000 (down)

even digit

Floating Point Operations: Basic Idea

$$\text{Value} = (-1)^{S} \times \text{Mantissa} \times 2^{\text{Exponent}}$$

- $x +_f y = \text{Round}(x + y)$
- $x *_f y = \text{Round}(x * y)$

- Basic idea for floating point operations:
  - First, compute the exact result
  - Then *round* the result to make it fit into the specified precision (width of M)
    - Possibly over/underflow if exponent outside of range

Mathematical Properties of FP Operations

- Overflow yields ±∞ and underflow yields 0
- Floats with value ±∞ and NaN can be used in operations
  - Result usually still ±∞ or NaN, but not always intuitive
- Floating point operations do not work like real math, due to rounding

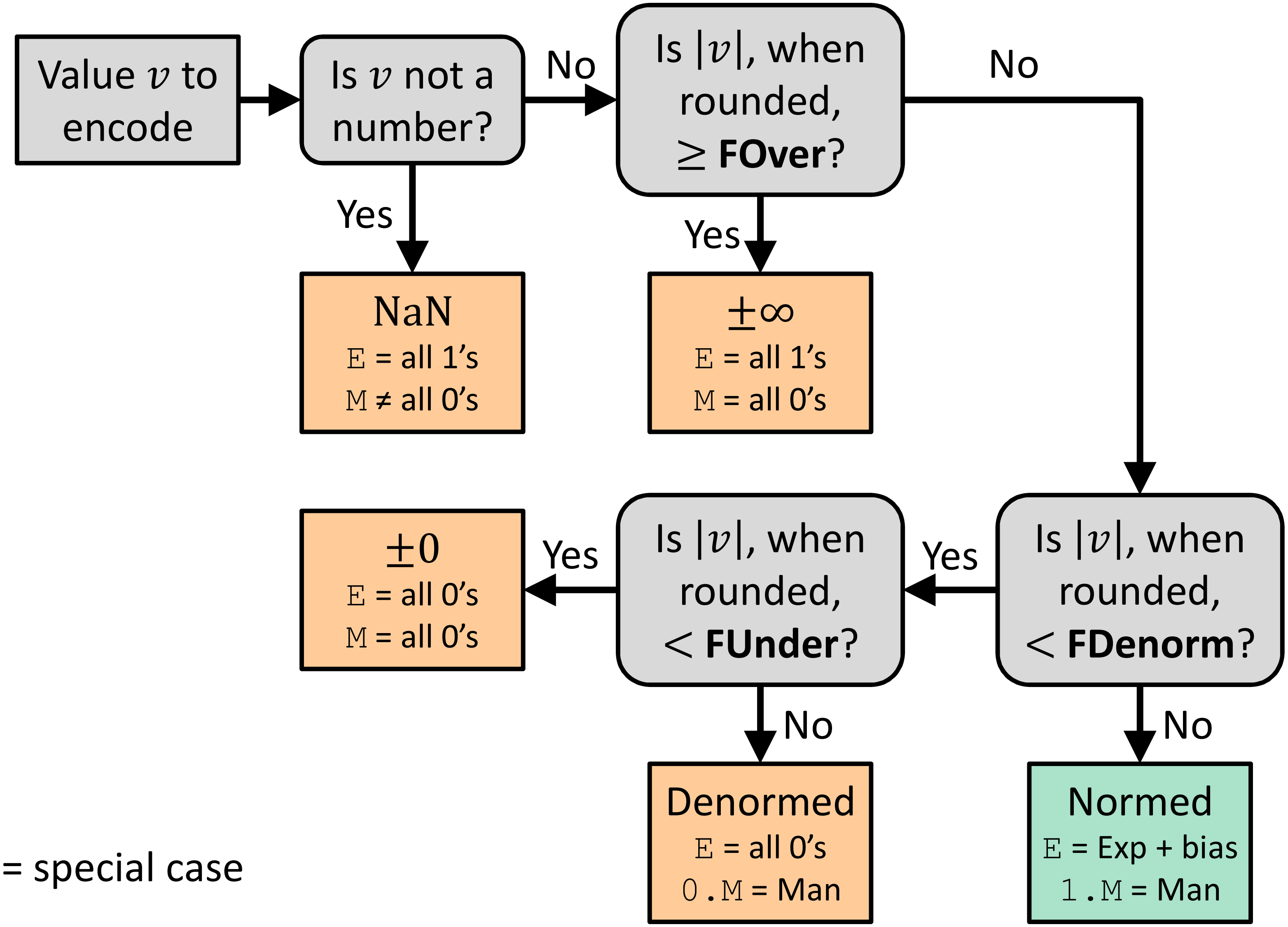
  3.14 10^{100} 0

  - Not associative: `(3.14+1e100)-1e100 != 3.14+(1e100-1e100)`

    `0` **`3.14`**
  - Not distributive: `100*(0.1+0.2) != 100*0.1+100*0.2`

    `30.000000000000003553` **`30`**
  - Not cumulative
    - Repeatedly adding a very small number to a large one may do nothing

Floating Point Encoding Flow Chart



Value v to encode

Is v not a number?

Yes

NaN
`E` = all 1's
`M` ≠ all 0's

No

Is $|v|$, when rounded, $\geq$ **FOver**?

Yes

$\pm\infty$
`E` = all 1's
`M` = all 0's

No

Is $|v|$, when rounded, $<$ **FDenorm**?

No

Normed
`E` = Exp + bias
`1.M` = Man

Yes

Is $|v|$, when rounded, $<$ **FUnder**?

Yes

± 0
`E` = all 0's
`M` = all 0's

No

Denormed
`E` = all 0's
`0.M` = Man

🟧 = special case

Limits of Interest

This is extra (non-testable) material

- The following thresholds will help give you a sense of when certain outcomes come into play, but don't worry about the specifics:
  - **FOver** = $2^{\text{bias}+1}$ = 2^{8}
    - This is just larger than the largest representable normalized number
  - **FDenorm** = $2^{1-\text{bias}}$ = 2^{-6}
    - This is the smallest representable normalized number
  - **FUnder** = $2^{1-\text{bias}-m}$ = 2^{-9}
    - m is the width of the mantissa field
    - This is the smallest representable denormalized number

Floating Point in C

!!!

- Two common levels of precision:

`float`	`1.0f`	single precision (32-bit)
`double`	`1.0`	double precision (64-bit)

- `#include <math.h>` to get `INFINITY` and `NAN` constants <float.h> for additional constants

- Equality (==) comparisons between floating point numbers are tricky, and often return unexpected results, so just avoid them!

instead use abs(f1 - f2) < 2^{-20}

↑ some arbitrary threshold

Floating Point Conversions in C

!!!

- Casting between `int`, `float`, and `double` **changes** the bit representation
  - `int` → `float`
    - May be rounded (not enough bits in mantissa: 23)
    - Overflow impossible
  - `int` or `float` → `double`
    - Exact conversion (all 32-bit `int`s representable)
  - `long` → `double`
    - Depends on word size (32-bit is exact, 64-bit may be rounded)
  - `double` or `float` → `int`
    - Truncates fractional part (rounded toward zero)
    - "Not defined" when out of range or NaN: generally sets to `Tmin` (even if the value is a very big positive)

Exploration Question

- We execute the following code in C. How many bytes are the same (value and position) between `i` and `f`?

```
int i = 384;  // 2^8 + 2^7
float f = (float) i;
```

8 7 6 5 4 3 2 1 0

= 0b 1 1000 0000

$= 1.1_2 \times 2^8$

S = 0

E = 8+127 = 135

= 0b1000 0111

M = 0b 10...0

0b 0 1000 0111 100...0

[✓ **A.** 0 bytes]

B. 1 byte

C. 2 bytes

D. 3 bytes

E. We're lost…

i stored as 0x 00 00 01 80

f stored as 0x 43 C0 00 00

Floating Point Summary

- Floats also suffer from the fixed number of bits available to represent them
  - Can get overflow/underflow
  - “Gaps” produced in representable numbers means we can lose precision, unlike `ints`
    - Some “simple fractions” have no exact representation (*e.g.* 0.2)
    - “Every operation gets a slightly wrong result”
- Floating point arithmetic not associative or distributive
  - Mathematically equivalent ways of writing an expression may compute different results
- Never test floating point values for equality!
- Careful when converting between `ints` and `floats`!

Number Representation Really Matters

- **1991:** Patriot missile targeting error
  - clock skew due to conversion from integer to floating point
- **1996:** Ariane 5 rocket exploded ($1 billion)
  - overflow converting 64-bit floating point to 16-bit integer
- **2000:** Y2K problem
  - limited (decimal) representation: overflow, wrap-around
- **2038:** Unix epoch rollover
  - Unix epoch = seconds since 12am, January 1, 1970
  - signed 32-bit integer representation rolls over to TMin in 2038
- **Other related bugs:**
  - 1982: Vancouver Stock Exchange 10% error in less than 2 years
  - 1994: Intel Pentium FDIV (floating point division) HW bug ($475 million)
  - 1997: USS Yorktown “smart” warship stranded: divide by zero
  - 1998: Mars Climate Orbiter crashed: unit mismatch ($193 million)

Summary

`E`	`M`	Meaning
0x00	0	± 0
0x00	non-zero	± denorm num
0x01 – 0xFE	anything	± norm num
0xFF	0	± ∞
0xFF	non-zero	NaN

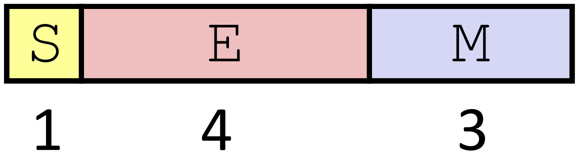
- Floating point encoding has many limitations
  - Overflow, underflow, rounding
  - Rounding is a HUGE issue due to limited mantissa bits and gaps that are scaled by the value of the exponent
  - Floating point arithmetic is NOT associative or distributive
- Converting between integral and floating point data types *does* change the bits

BONUS SLIDES

An example that applies the IEEE Floating Point concepts to a smaller (8-bit) representation scheme.

These slides expand on material covered today, so while you don't need to read these, the information is "fair game."

Tiny Floating Point Example



- 8-bit Floating Point Representation
  - The sign bit is in the most significant bit (MSB)
  - The next four bits are the exponent, with a bias of $2^{4-1}-1 = 7$
  - The last three bits are the mantissa

- Same general form as IEEE Format
  - Normalized binary scientific point notation
  - Similar special cases for 0, denormalized numbers, NaN, ∞

Dynamic Range (Positive Only)

	S	E	M	Exp	Value	
Denormalized numbers	0	0000	000	-6	0	
	0	0000	001	-6	1/8*1/64 = 1/512	closest to zero
	0	0000	010	-6	2/8*1/64 = 2/512	
	…					
	0	0000	110	-6	6/8*1/64 = 6/512	
	0	0000	111	-6	7/8*1/64 = 7/512	largest denorm
Normalized numbers	0	0001	000	-6	8/8*1/64 = 8/512	smallest norm
	0	0001	001	-6	9/8*1/64 = 9/512	
	…					
	0	0110	110	-1	14/8*1/2 = 14/16	
	0	0110	111	-1	15/8*1/2 = 15/16	closest to 1 below
	0	0111	000	0	8/8*1 = 1	
	0	0111	001	0	9/8*1 = 9/8	closest to 1 above
	0	0111	010	0	10/8*1 = 10/8	
	…					
	0	1110	110	7	14/8*128 = 224	
	0	1110	111	7	15/8*128 = 240	largest norm
	0	1111	000	n/a	inf	

Special Properties of Encoding

- Floating point zero (0^+) exactly the same bits as integer zero
  - All bits = 0

- Can (Almost) Use Unsigned Integer Comparison
  - Must first compare sign bits
  - Must consider $0^- = 0^+ = 0$
  - NaNs problematic
    - Will be greater than any other values
    - What should comparison yield?
  - Otherwise OK
    - Denorm vs. normalized
    - Normalized vs. infinity