Floating Point II
CSE 351 Autumn 2022

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http://www.smbc-comics.com/?id=2999
Relevant Course Information

❖ hw6 due Friday, hw7 due Monday

❖ Lab 1a: last chance to submit is tonight @ 11:59 pm
  ▪ One submission per partnership
  ▪ Make sure you check the Gradescope autograder output!
  ▪ Grades hopefully released by end of Sunday (10/16)

❖ Lab 1b due Monday (10/17)
  ▪ Submit aisle_manager.c, store_client.c, and lab1Bsynthesis.txt

❖ Section tomorrow on Integers and Floating Point
Getting Help with 351

- Lecture recordings, readings, inked slides, section presentation recordings, worksheet solutions
- Form a study group!
  - Good for everything but labs, which should be done in pairs
  - Communicate regularly, use the class terminology, ask and answer each others’ questions, show up to OH together
- Attend office hours
  - Can also chat with other students—help each other learn!
- Post on Ed Discussion
- Request a 1-on-1 meeting
  - Available on a limited basis for special circumstances
Reading Review

❖ Terminology:
  ▪ Special cases
    • Denormalized numbers
    • ±∞
    • Not-a-Number (NaN)
  ▪ Limits of representation
    • Overflow
    • Underflow
    • Rounding

❖ Questions from the Reading?
Review Questions

❖ What is the value of the following floats?

- \(0x0000000000\) \(\Rightarrow S=0, E=0, M=0\) \(\Rightarrow +0\)
- \(0xFF800000\) \(\Rightarrow S=1, E=\text{all 1's}, M=0\) \(\Rightarrow -\infty\)

❖ For the following code, what is the smallest value of \(n\) that will encounter a limit of representation?

```c
float f = 1.0; // 2^0
for (int i = 0; i < n; ++i)
    f *= 1024; // 1024 = 2^10
printf("f = %f\n", f);
```

\[\begin{array}{c|c}
 n & f \\
 \hline
 1 & 2^0 \\
 2 & 2^{12} \\
 3 & 2^{20} \\
 4 & 2^{30} \\
\end{array}\]

\(E_{\text{max}} = 0x\text{FE},\) \(E_{\text{max}} = 254 - 127 = 127\)

For \(n = 13\), we hit \(2^{130}\), which causes overflow.

Always \(M = 0\).
## Floating Point Encoding Summary (Review)

<table>
<thead>
<tr>
<th>E</th>
<th>M</th>
<th>Interpretation</th>
</tr>
</thead>
<tbody>
<tr>
<td>0x00</td>
<td>0</td>
<td>± 0</td>
</tr>
<tr>
<td>0x00</td>
<td>non-zero</td>
<td>± denorm num</td>
</tr>
<tr>
<td>0x01 – 0xFE</td>
<td>anything</td>
<td>± norm num</td>
</tr>
<tr>
<td>0xFF</td>
<td>0</td>
<td>± ∞</td>
</tr>
<tr>
<td>0xFF</td>
<td>non-zero</td>
<td>NaN</td>
</tr>
</tbody>
</table>
Special Cases

❖ But wait... what happened to zero?
  ▪ *Special case:* \( E \) and \( M \) all zeros = 0  
  ▪ Two zeros! But at least 0x00000000 = 0 like integers  

❖ \( E = 0xFF, M = 0 \): ± ∞  
  ▪ e.g., division by 0  
  ▪ Still work in comparisons!

❖ \( E = 0xFF, M ≠ 0 \): Not a Number (\( NaN \))  
  ▪ e.g., square root of negative number, 0/0, ∞–∞  
  ▪ NaN propagates through computations  
  ▪ Value of \( M \) can be useful in debugging (tells you cause of \( NaN \))
New Representation Limits (Review)

❖ New largest value (besides $\infty$)?
  - $E = 0xFF$ has now been taken!
  - $E = 0xFE$ has largest: $1.1\ldots1 \times 2^{127} = 2^{128} - 2^{104}$

❖ New numbers closest to 0:
  - $E = 0x00$ taken; next smallest is $E = 0x01$
  - $a = 1.0\ldots00 \times 2^{-126} = 2^{-126}$
  - $b = 1.0\ldots01 \times 2^{-126} = 2^{-126} + 2^{-149}$
  - Normalization and implicit 1 are to blame
  - Special case: $E = 0$, $M \neq 0$ are denormalized numbers $(0.M)$
Denorm Numbers (Review)

- Denormalized numbers
  - No leading 1
  - Uses implicit exponent of −126 even though $E = 0x00$

- Denormalized numbers close the gap between zero and the smallest normalized number
  - Smallest norm: $\pm 1.0...0_{\text{two}} \times 2^{-126} = \pm 2^{-126}$
  - Smallest denorm: $\pm 0.0...01_{\text{two}} \times 2^{-126} = \pm 2^{-149}$
    - There is still a gap between zero and the smallest denormalized number

This is extra (non-testable) material
Floating Point Decoding Flow Chart

FP Bits → What is the value of E?

- If all 0's:
  - If all 1's:
    - What is the value of M?
      - If all 0's:
        - denormalized: \((-1)^S \times 0. M \times 2^{1-bias}\)
      - anything else:
        - normalized: \((-1)^S \times 1. M \times 2^{E-bias}\)
    - anything else:
      - NaN
  - anything else:
    - \((-1)^S \times \infty\)

= special case
Floating Point Topics

- Fractional binary numbers
- IEEE floating-point standard
- Floating-point operations and limitations
- Floating-point in C

- There are many more details that we won’t cover
  - It’s a 58-page standard...
Tiny Floating Point Representation

❖ We will use the following 8-bit floating point representation to illustrate some key points:

- Assume that it has the same properties as IEEE floating point:
  - bias = $2^{w-1} - 1 = 2^3 - 1 = 7$
  - encoding of $-0 = 0b 1 0000 000$
  - encoding of $+\infty = 0b 0 1111 000$
  - encoding of the largest (+) normalized # = $0b 0 1110 111$
  - encoding of the smallest (+) normalized # = $0b 0 0001 000$

\[ \begin{array}{ccc}
S & E & M \\
1 & 4 & 3 \\
\end{array} \]

\[ 1.1112 \times 2^{4-7} \]
Distribution of Values (Review)

- What ranges are NOT representable?
  - Between largest norm and infinity → **Overflow** (Exp too large)
  - Between zero and smallest denorm → **Underflow** (Exp too small)
  - Between norm numbers?

- Given a FP number, what’s the next largest representable number?
  - What is this “step” when Exp = 0?
    - $2^{-23}$
  - What is this “step” when Exp = 100?
    - $2^{77}$

- Distribution of values is denser toward zero

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![Diagram showing distribution of values with overflow, underflow, rounding, and denormalized, normalized, infinity labels.]
Floating Point Operations: Basic Idea

Value = \((-1)^s \times \text{Mantissa} \times 2^{\text{Exponent}}\)

\[
\begin{array}{c|c|c}
S & E & M \\
\end{array}
\]

- \(x +_f y = \text{Round}(x + y)\)
- \(x \times_f y = \text{Round}(x \times y)\)

Basic idea for floating point operations:
- First, **compute the exact result**
- Then **round** the result to make it fit into the specified precision (width of M)
  - Possibly over/underflow if exponent outside of range
Mathematical Properties of FP Operations

- **Overflow** yields $\pm\infty$ and **underflow** yields 0
- **Floats with value** $\pm\infty$ and NaN can be used in operations
  - Result usually still $\pm\infty$ or NaN, but not always intuitive
- **Floating point operations do not work like real math, due to rounding**
  - Not associative: $(3.14 + 1e100) - 1e100 \neq 3.14 + (1e100 - 1e100)$
  - Not distributive: $100 \times (0.1 + 0.2) \neq 100 \times 0.1 + 100 \times 0.2$
  - Not cumulative
    - Repeatedly adding a very small number to a large one may do nothing
Floating Point Rounding

❖ The IEEE 754 standard actually specifies different rounding modes:

❖ Round to nearest, ties to nearest even digit
  ▪ Round toward $+\infty$ (round up)
  ▪ Round toward $-\infty$ (round down)
  ▪ Round toward 0 (truncation)

❖ In our tiny example:
  ▪ $\text{Man} = 1.001\ 01$ rounded to $M = 0b001$ (down)
  ▪ $\text{Man} = 1.001\ 11$ rounded to $M = 0b010$ (up)
  ▪ $\text{Man} = 1.001\ 10$ rounded to $M = 0b010$ (up)
  ▪ $\text{Man} = 1.000\ 10$ rounded to $M = 0b000$ (down)
Floating Point in C

- Two common levels of precision:
  - `float 1.0f` single precision (32-bit)
  - `double 1.0` double precision (64-bit)

- `#include <math.h>` to get INFINITY and NAN constants
- `#include <float.h>` for additional constants

- Equality (==) comparisons between floating point numbers are tricky, and often return unexpected results, so just avoid them!
  
  Instead use `abs(f1 - f2) < 2^{-20}`

  ~some arbitrary threshold
Floating Point Conversions in C

- Casting between `int`, `float`, and `double` changes the bit representation (tries to preserve the value)
  - `int` → `float`
    - May be rounded (not enough bits in mantissa: 23)
    - Overflow impossible
  - `int` or `float` → `double`
    - Exact conversion (all 32-bit `ints` are representable)
  - `long` → `double`
    - Depends on word size (32-bit is exact, 64-bit may be rounded)
  - `double` or `float` → `int`
    - Truncates fractional part (rounded toward zero)
    - “Not defined” when out of range or NaN: generally sets to TMin (even if the value is a very big positive)
Casting Example

❖ We execute the following code in C. How are i and f represented in hex?

```c
int i = 384;  // 2^8 + 2^7
float f = (float) i;
```

i stored as 0x00 00 01 80
f stored as 0x43 CO 00 00
Discussion Questions

❖ How do you feel about floating point?
  ▪ Do you feel like the limitations are acceptable?

❖ Does this affect the way you’ll think about non-integer arithmetic in the future?

❖ Are there any changes or different encoding schemes that you think would be an improvement?
More on Floating Point History

❖ Early days
  ▪ First design with floating-point arithmetic in 1914 by Leonardo Torres y Quevedo
  ▪ Implementations started in 1940 by Konrad Zuse, but with differing field lengths (usually not summing to 32 bits) and different subsets of the special cases

❖ IEEE 754 standard created in 1985
  ▪ Primary architect was William Kahan, who won a Turing Award for this work
  ▪ Standardized bit encoding, well-defined behavior for all arithmetic operations
Floating Point in the “Wild”

- 3 formats from IEEE 754 standard widely used in computer hardware and languages
  - In C, called float, double, long double
- Common applications:
  - 3D graphics: textures, rendering, rotation, translation
  - “Big Data”: scientific computing at scale, machine learning
- Non-standard formats in domain-specific areas:
  - **Bfloat16**: training ML models; range more valuable than precision
  - **TensorFloat-32**: Nvidia-specific hardware for Tensor Core GPUs

<table>
<thead>
<tr>
<th>Type</th>
<th>S bits</th>
<th>E bits</th>
<th>M bits</th>
<th>Total bits</th>
</tr>
</thead>
<tbody>
<tr>
<td>Half-precision</td>
<td>1</td>
<td>5</td>
<td>10</td>
<td>16</td>
</tr>
<tr>
<td>Bfloat16</td>
<td>1</td>
<td>8</td>
<td>7</td>
<td>16</td>
</tr>
<tr>
<td>TensorFloat-32</td>
<td>1</td>
<td>8</td>
<td>10</td>
<td>19</td>
</tr>
<tr>
<td>Single-precision</td>
<td>1</td>
<td>8</td>
<td>23</td>
<td>32</td>
</tr>
</tbody>
</table>
Floating Point Summary

❖ Floats also suffer from the fixed number of bits available to represent them
  ▪ Can get overflow/underflow
  ▪ “Gaps” produced in representable numbers means we can lose precision, unlike ints
    • Some “simple fractions” have no exact representation (e.g., 0.2)
    • “Every operation gets a slightly wrong result”

❖ Floating point arithmetic not associative or distributive
  ▪ Mathematically equivalent ways of writing an expression may compute different results

❖ Never test floating point values for equality!
❖ Careful when converting between ints and floats!
Number Representation Really Matters

- **1991**: Patriot missile targeting error
  - clock skew due to conversion from integer to floating point
- **1996**: Ariane 5 rocket exploded ($1 billion)
  - overflow converting 64-bit floating point to 16-bit integer
- **2000**: Y2K problem
  - limited (decimal) representation: overflow, wrap-around
- **2038**: Unix epoch rollover
  - Unix epoch = seconds since 12am, January 1, 1970
  - signed 32-bit integer representation rolls over to Tmin in 2038
- **Other related bugs:**
  - 1982: Vancouver Stock Exchange 10% error in less than 2 years
  - 1994: Intel Pentium FDIV (floating point division) HW bug ($475 million)
  - 1997: USS Yorktown “smart” warship stranded: divide by zero
  - 1998: Mars Climate Orbiter crashed: unit mismatch ($193 million)
Summary

- Floating point encoding has many limitations
  - Overflow, underflow, rounding
  - Rounding is a HUGE issue due to limited mantissa bits and gaps that are scaled by the value of the exponent
  - Floating point arithmetic is NOT associative or distributive

- Converting between integral and floating point data types *does* change the bits