Floating Point I

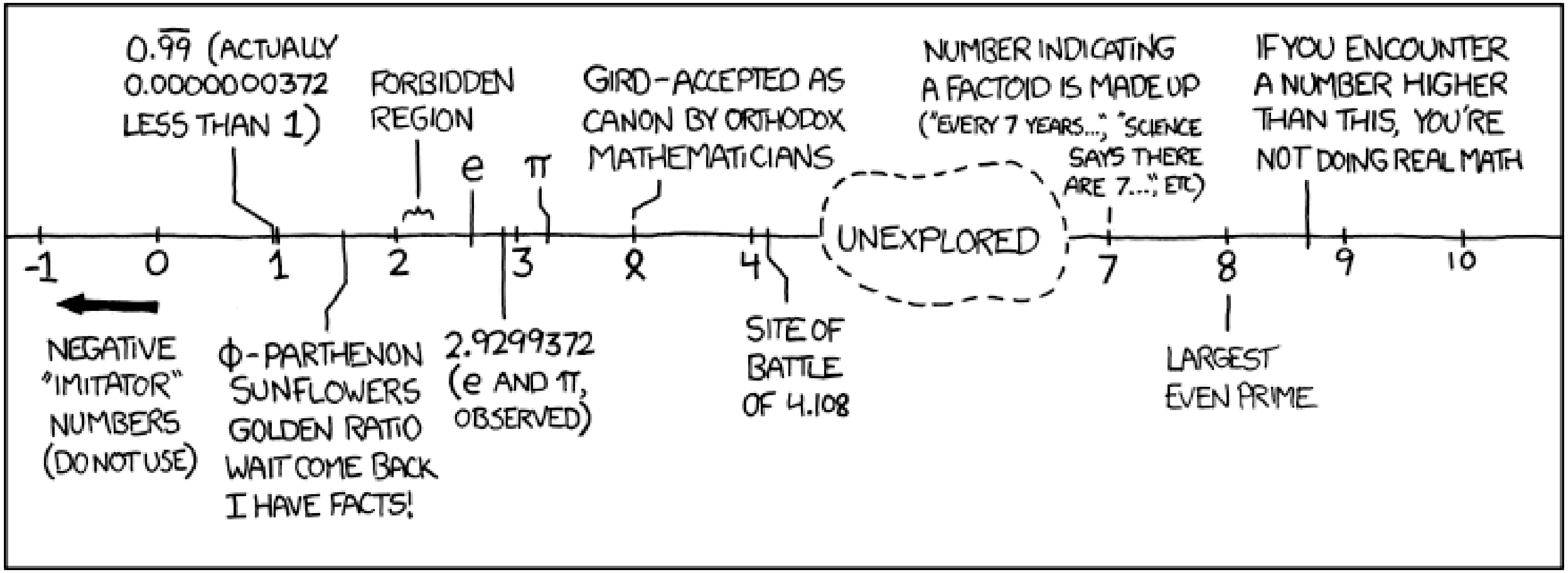
CSE 351 Autumn 2022

Instructor:

Justin Hsia

Teaching Assistants:

Angela Xu, Arjun Narendra, Armin Magness, Assaf Vayner, Carrie Hu, Clare Edmonds, David Dai, Dominick Ta, Effie Zheng, James Froelich, Jenny Peng, Kristina Lansang, Paul Stevans, Renee Ruan, Vincent Xiao



http://xkcd.com/899/

Relevant Course Information

- hw5 due Wednesday, hw6 due Friday
- Don’t change your poll answers after-the-fact!
  - Graded on completion; misrepresents your understanding
- Lab 1a due tonight at 11:59 pm
  - Submit `pointer.c` and `lab1Asynthesis.txt`
    - Make sure there are no lingering `printf` statements in your code!
  - Make sure you submit *something* to Gradescope before the deadline and that the file names are correct
  - Can use late days to submit up until Wed 11:59 pm
- Lab 1b due next Monday (10/17)
  - Submit `aisle_manager.c`, `store_client.c`, and `lab1Bsynthesis.txt`

Lab 1b Aside: C Macros

- C macros basics:
  - Basic syntax is of the form: `#define NAME expression`
  - Allows you to use "`NAME`" instead of "`expression`" in code
    - Does naïve copy and replace *before* compilation – everywhere the characters "NAME" appear in the code, the characters "expression" will now appear instead
    - NOT the same as a Java constant
  - Useful to help with readability/factoring in code

- You'll use C macros in Lab 1b for defining bit masks
  - See Lab 1b starter code and Lecture 4 slides (card operations) for examples

Reading Review

- Terminology:
  - normalized scientific binary notation
  - trailing zeros
  - sign, mantissa, exponent ↔ bit fields S, M, and E
  - `float`, `double`
  - biased notation (exponent), implicit leading one (mantissa)
  - rounding errors

- Questions from the Reading?

$2^{-1} = 0.5$
$2^{-2} = 0.25$
$2^{-3} = 0.125$
$2^{-4} = 0.0625$

Review Questions

- Convert 11.375_{10} to normalized binary scientific notation
  $8+2+1+0.25+0.125$
  $2^3+2^1+2^0+2^{-2}+2^{-3} = 1011.011_2 = 1.011011 \times 2^3$
- What is the value encoded by the following floating point number?

  0b 0 | 1000 0000 | 110 0000 0000 0000 0000 0000 (S, E, M)
  - bias = $2^{w-1}-1$ $= 2^7-1 = 127$
  - exponent = E – bias $= 2^7 - 127 = 128-127 = 1$
  - mantissa = 1.M $= 1.110...0_2$

  $(-1)^0 \times 1.11_2 \times 2^1 = 11.1_2 = +3.5$

Number Representation Revisited

- What can we represent in one word?
  - Signed and Unsigned Integers
  - Characters (ASCII)
  - Addresses

- How do we encode the following:
  - Real numbers (*e.g.*, 3.14159)
  - Very large numbers (*e.g.*, 6.02×10^{23})
  - Very small numbers (*e.g.*, 6.626×10^{-34})
  - Special numbers (*e.g.*, ∞, NaN)

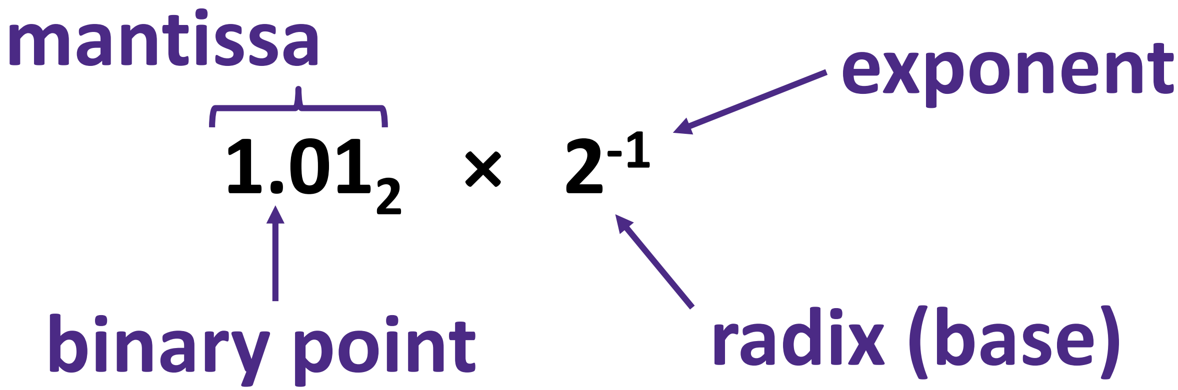
Floating Point

Floating Point Topics

- **IEEE floating-point standard**
- Floating-point operations and rounding
- Floating-point in C

- There are many more details that we won't cover
  - It's a 58-page standard…

Binary Scientific Notation (Review)



- *Normalized form*: exactly one digit (non-zero) to left of binary point

- Computer arithmetic that supports this called floating point due to the "floating" of the binary point
  - Declare such variable in C as `float` (or `double`)

IEEE Floating Point

- IEEE 754 (established in 1985)
  - Standard to make numerically-sensitive programs portable
  - Specifies two things: *representation scheme* and result of *floating point operations*
  - Supported by all major CPUs
- Driven by numerical concerns
  - **Scientists**/numerical analysts want them to be as **real** as possible
  - **Engineers** want them to be **easy to implement** and **fast**

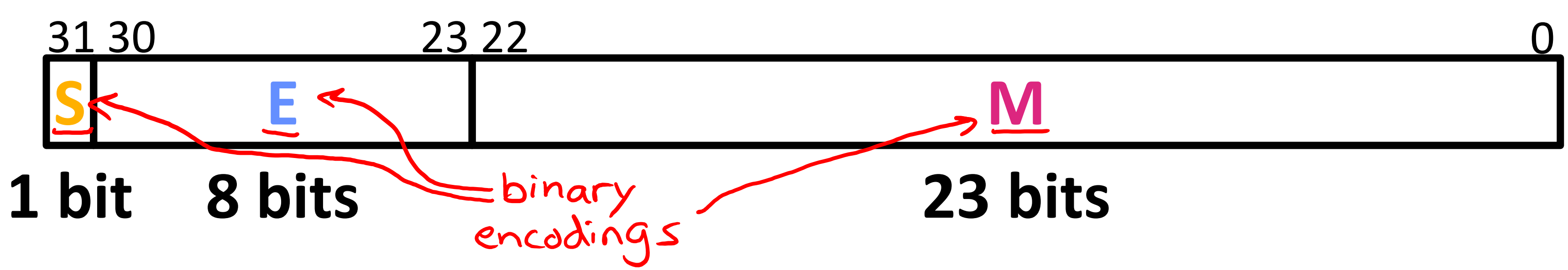
    competing goals!
  - Scientists mostly won out:
    - Nice standards for rounding, overflow, underflow, but...
    - Hard to make fast in hardware
    - **Float operations can be an order of magnitude slower than integer ops**

      FLOPs — used in computer benchmarks

Floating Point Encoding (Review)

- Use normalized, base 2 scientific notation:
  - Value: $\pm 1 \times \text{Mantissa} \times 2^{\text{Exponent}}$
  - Bit Fields: $(-1)^{S} \times 1.M \times 2^{(E-\text{bias})}$
- Representation Scheme: (3 separate fields within 32 bits)
  - Sign bit (0 is positive, 1 is negative)
  - Mantissa (a.k.a. significand) is the fractional part of the number in normalized form and encoded in bit vector **M**
  - Exponent weights the value by a (possibly negative) power of 2 and encoded in the bit vector **E**

values

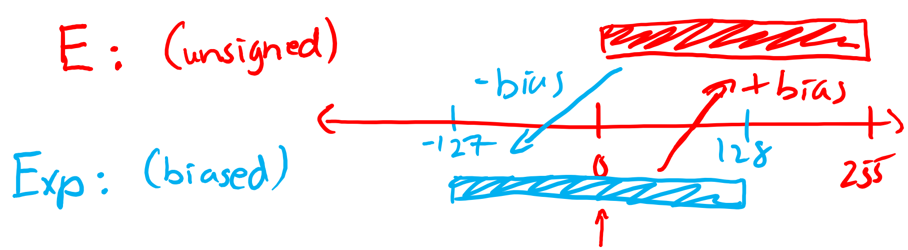


The Exponent Field (Review)

- Use biased notation

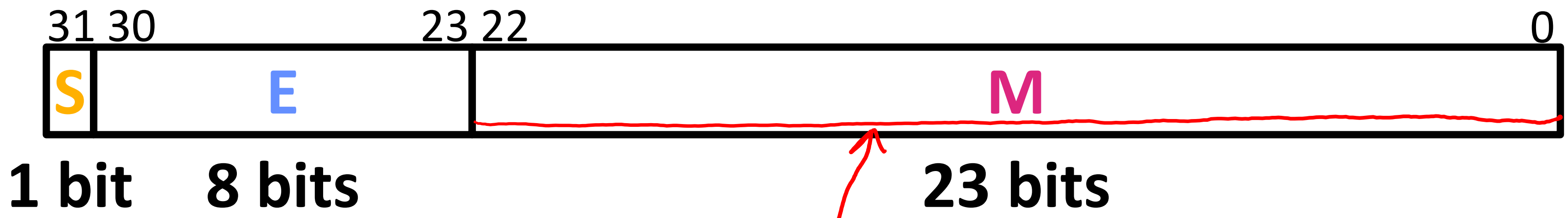
  w=8, can encode 2^8=256 exponents

  - Read exponent as unsigned, but with *bias* of $2^{w-1}-1$ = 127
  - Representable exponents roughly ½ positive and ½ negative
  - Exp = E – bias ↔ E = Exp + bias
    - Exponent 0 (Exp = 0) is represented as E = 0b 0111 1111 $= 2^7 - 1$



- Why biased?
  - Now it's a sign-and-magnitude representation!
  - Makes floating point arithmetic easier (somewhat compatible with two's complement hardware)

The Mantissa (Fraction) Field (Review)



$$(-1)^{S} \times (1\,.\,M) \times 2^{(E-\text{bias})}$$

- Note the implicit leading 1 in front of the M bit vector
  - Example: 0b 0011 1111 1100 0000 0000 0000 0000 0000 is read as $1.1_2 = 1.5_{10}$, *not* $0.1_2 = 0.5_{10}$
  - Gives us an extra bit of *precision*
- Mantissa "limits"
  - Low values near M = 0b0…0 are close to 2^{Exp}
  - High values near M = 0b1…1 are close to 2^{Exp+1}

Normalized Floating Point Conversions

- FP → Decimal
  1. Append the bits of M to implicit leading 1 to form the mantissa.
  2. Multiply the mantissa by $2^{E - \text{bias}}$.
  3. Multiply the sign $(-1)^S$.
  4. Multiply out the exponent by shifting the binary point.
  5. Convert from binary to decimal.

- Decimal → FP
  1. Convert decimal to binary.
  2. Convert binary to normalized scientific notation.
  3. Encode sign as S (0/1).
  4. Add the bias to exponent and encode E as unsigned.
  5. The first bits after the leading 1 that fit are encoded into M.

Practice Question

- Convert the decimal number **-7.375 = -1.11011 x 2^2** into floating point representation.

S = 1 , E = 2+127 = 129 = 0b 1000 0001 , M = 0b 110110...0

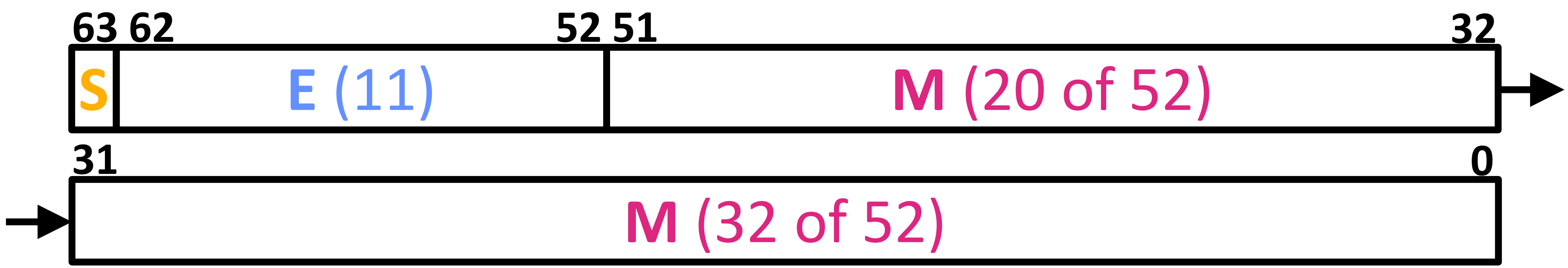
0b 1100 0000 1110 1100 0...0 = 0x C0EC 0000

Precision and Accuracy

- Accuracy is a measure of the difference between the *actual value of a number* and its computer representation
- Precision is a count of the number of bits in a computer word used to represent a value
  - Capacity for accuracy
- *High precision permits high accuracy but doesn't guarantee it*
  - **Example:** `float pi = 3.14;` will be represented using all 24 bits of the mantissa (highly precise), but is only an approximation (not accurate)

Need Greater Precision?

- Double Precision (vs. Single Precision) in 64 bits



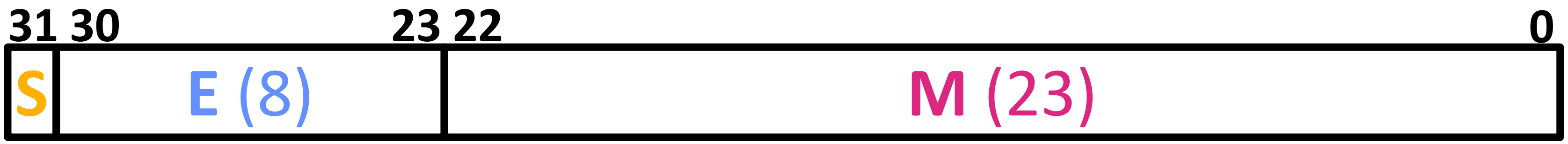
  - C variable declared as `double`
  - Exponent bias is now $2^{10}–1 = 1023$, bias $= 2^{w-1}-1$
  - **Advantages:** greater precision (larger mantissa), greater range (larger exponent)
  - **Disadvantages:** more bits used, slower to manipulate

Current Limitations

- Largest magnitude we can represent? E=0b1111 1111, M=0b1…1 (Exp=128)
- Smallest magnitude we can represent? E=0b0000 0000, M=0b0…0 (Exp=-127)
  - Limited ***range*** due to width of E field
- What happens if we try to represent $2^0 + 2^{-30}$? = 1.0…01 (29 zeros; M stores first 23 zeros)
  - Rounding due to limited ***precision***: stores 2^0
- There is a need for *special cases*
  - How do we represent the value zero? $0 \neq \pm 1.M \times 2^{E-bias}$
  - What about ∞ and NaN? ???

Summary

- Floating point approximates real numbers:



  - Handles large numbers, small numbers, special numbers
  - Exponent in biased notation (bias = $2^{w-1} - 1$)
    - Size of exponent field determines our representable *range*
    - Outside of representable exponents is *overflow* and *underflow*
  - Mantissa approximates fractional portion of binary point
    - Size of mantissa field determines our representable *precision*
    - Implicit leading 1 (normalized) except in special cases
    - Exceeding length causes *rounding*

Preview Question

- Find the sum of the following binary numbers in normalized scientific binary notation:

$$1.01_2 \times 2^0 + 1.11_2 \times 2^2$$

① match exponents
② sum mantissas
③ normalize

0.0101×2^2
$+ 1.11 \times 2^2$
$10.0001 \times 2^2 = 1.00001 \times 2^3$