Floating Point II

CSE 351 Autumn 2020

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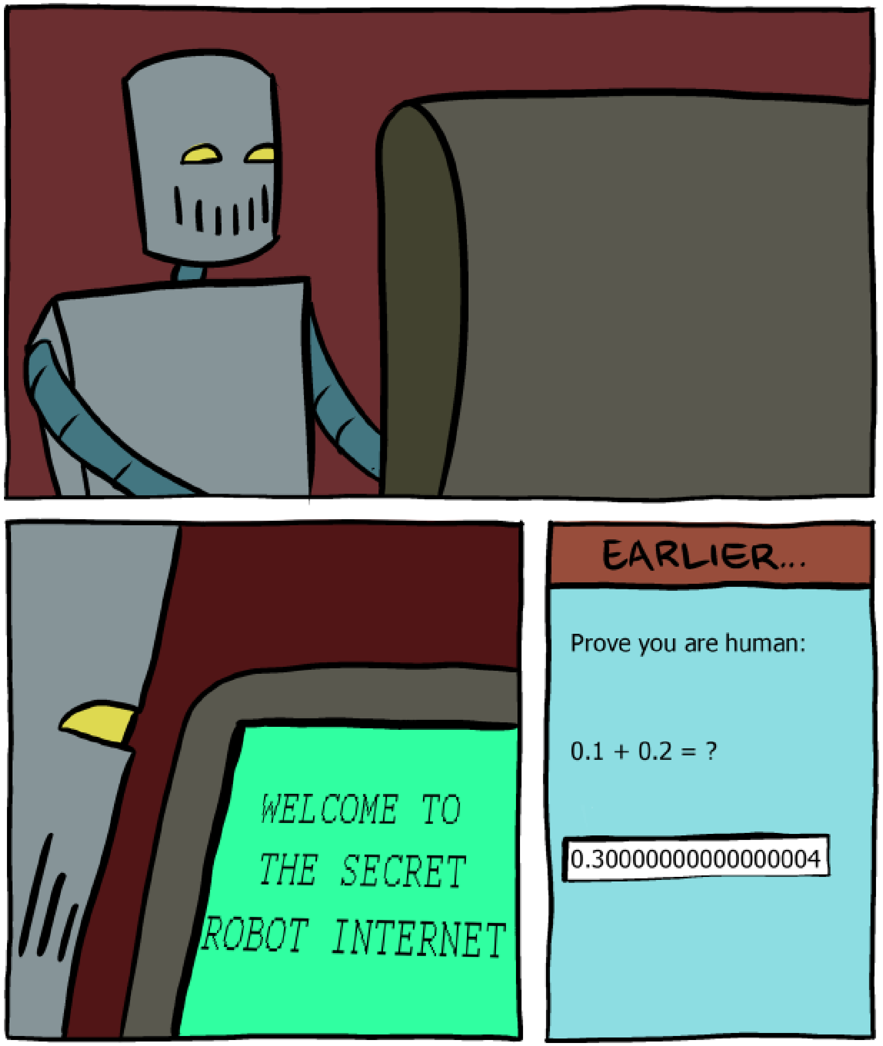

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Administrivia

- hw6 due Friday, hw7 due Monday
- Lab 1b due Friday (1/22)
  - Submit `aisle_manager.c`, `store_client.c`, and `lab1Breflect.txt`
- Section tomorrow on Integers and Floating Point
- Study Guide 1 released today, due Friday 1/29
  - https://courses.cs.washington.edu/courses/cse351/21wi/guides/
  - Task 1 -> group work allowed
  - Tasks 2 and 3 -> individual

Reading Review

- Terminology:
  - Special cases
    - Denormalized numbers
    - $\pm\infty$
    - Not-a-Number (NaN)
  - Limits of representation
    - Overflow
    - Underflow
    - Rounding

- Questions from the Reading?

Review Questions

- What is the value of the following floats?
  - 0x00000000
  - 0xFF800000

- For the following code, what is the smallest value of n that will encounter a limit of representation?

```
float f = 1.0;  // 2^0
for (int i = 0; i < n; ++i)
    f *= 1024;  // 1024 = 2^10
printf("f = %f\n", f);
```

Floating Point Encoding Summary

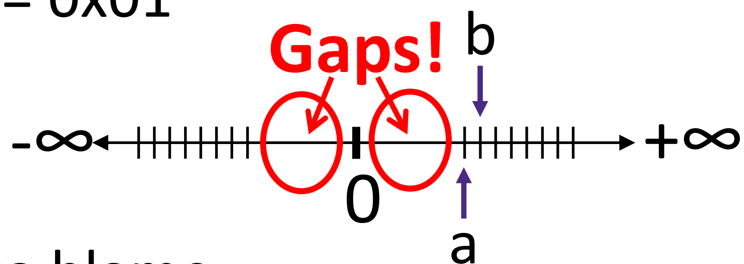
E	M	Interpretation
0x00	0	± 0
0x00	non-zero	± denorm num
0x01 – 0xFE	anything	± norm num
0xFF	0	± ∞
0xFF	non-zero	NaN

Special Cases

- But wait… what happened to zero?
  - *Special case:* E and M all zeros = 0
  - Two zeros! But at least 0x00000000 = 0 like integers
- E = 0xFF, M = 0: ± ∞
  - *e.g.*, division by 0
  - Still work in comparisons!
- E = 0xFF, M ≠ 0: Not a Number (NaN)
  - *e.g.*, square root of negative number, 0/0, ∞–∞
  - NaN propagates through computations
  - Value of M can be useful in debugging

New Representation Limits

- New largest value (besides ∞)?
  - E = 0xFF has now been taken!
  - E = 0xFE has largest: $1.1\ldots1_2 \times 2^{127} = 2^{128} - 2^{104}$
- New numbers closest to 0:
  - E = 0x00 taken; next smallest is E = 0x01
  - $a = 1.0\ldots0_2 \times 2^{-126} = 2^{-126}$
  - $b = 1.0\ldots01_2 \times 2^{-126} = 2^{-126} + 2^{-149}$
  - Normalization and implicit 1 are to blame
  - *Special case:* E = 0, M ≠ 0 are denormalized numbers



Denorm Numbers

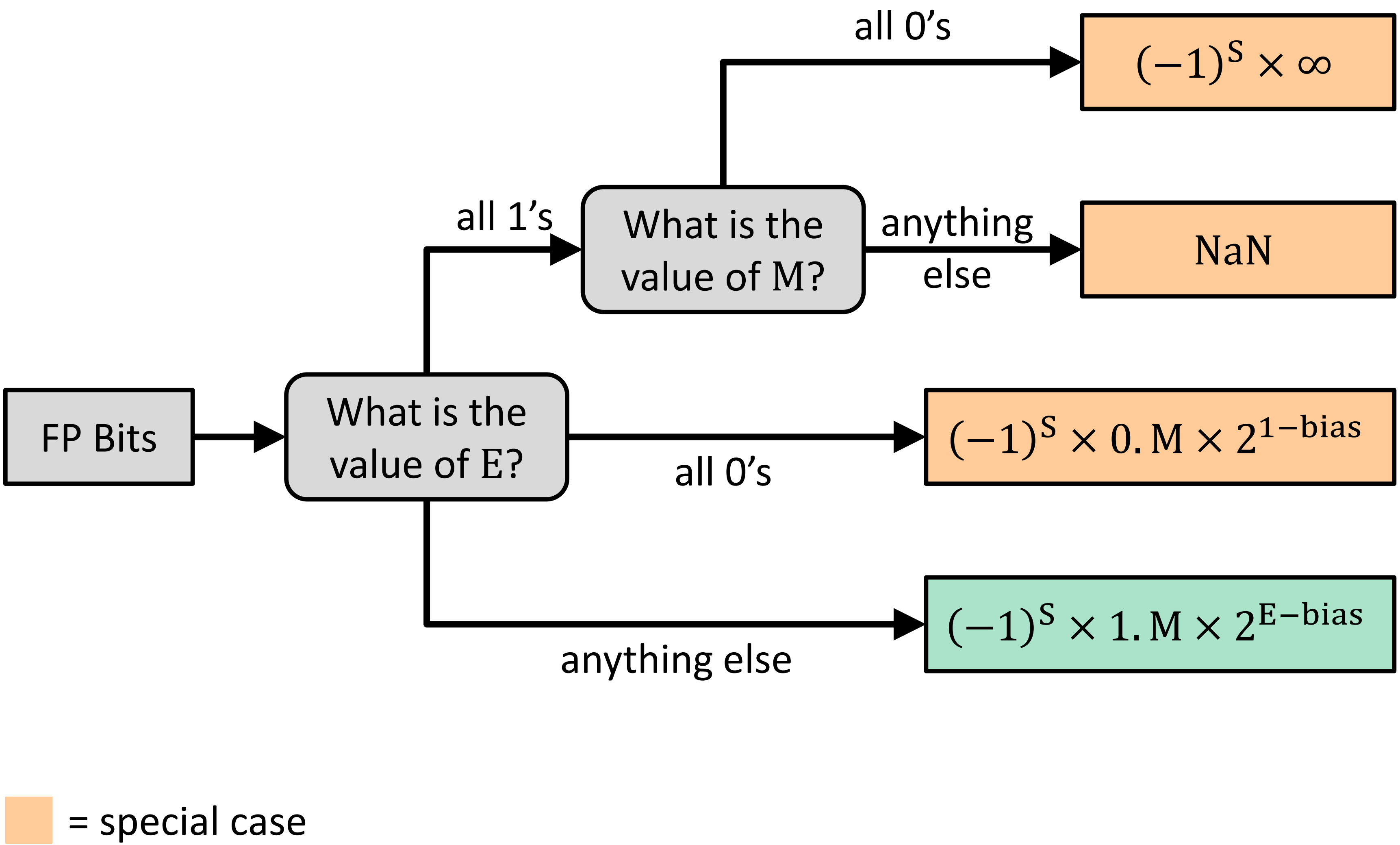
This is extra (non-testable) material

- Denormalized numbers
  - No leading 1
  - Uses implicit exponent of –126 even though E = 0x00

- Denormalized numbers close the gap between zero and the smallest normalized number
  - Smallest norm: $\pm\ 1.0\ldots0_{\text{two}}\times2^{-126} = \pm\ 2^{-126}$
  - Smallest denorm: $\pm\ 0.0\ldots01_{\text{two}}\times2^{-126} = \pm\ 2^{-149}$
    - There is still a gap between zero and the smallest denormalized number

So much closer to 0

Floating Point Interpretation Flow Chart



Floating point topics

- Fractional binary numbers
- IEEE floating-point standard
- **Floating-point operations and rounding**
- **Floating-point in C**

- There are many more details that we won't cover
  - It's a 58-page standard…

Tiny Floating Point Representation

- We will use the following **8-bit** floating point representation to illustrate some key points:

S	E	M
1	4	3

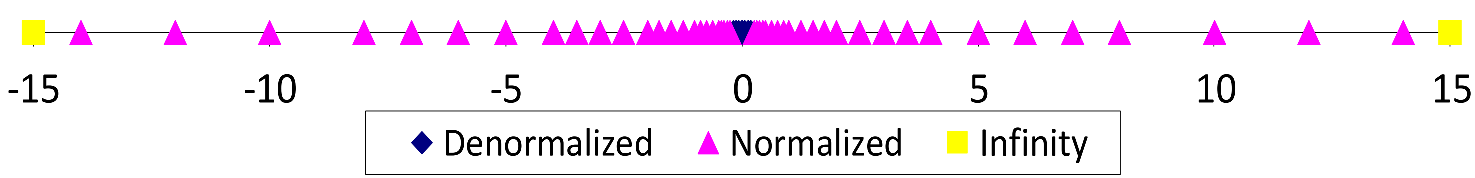
- Assume that it has the same properties as IEEE floating point:
  - bias =
  - encoding of -0 =
  - encoding of $+\infty$ =
  - encoding of the largest (+) normalized # =
  - encoding of the smallest (+) normalized # =

Distribution of Values

- What ranges are NOT representable?
  - Between largest norm and infinity **Overflow** (Exp too large)
  - Between zero and smallest denorm **Underflow** (Exp too small)
  - Between norm numbers? **Rounding**
- Given a FP number, what's the next largest representable number?
  - What is this "step" when Exp = 0?
  - What is this "step" when Exp = 100?

if M = 0b0…00, then $2^{\text{Exp}} \times 1.0$
if M = 0b0…01, then $2^{\text{Exp}} \times (1 + 2^{-23})$
diff = $2^{\text{Exp}-23}$

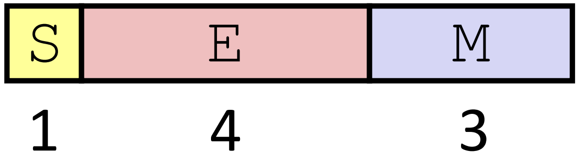
- Distribution of values is denser toward zero



Floating Point Rounding

This is extra (non-testable) material

- The IEEE 754 standard actually specifies different rounding modes:
  - Round to nearest, ties to nearest even digit
  - Round toward $+\infty$ (round up)
  - Round toward $-\infty$ (round down)
  - Round toward 0 (truncation)



- In our tiny example:
  - Man = 1.001 01 rounded to M = 0b001
  - Man = 1.001 11 rounded to M = 0b010
  - Man = 1.001 10 rounded to M = 0b010
  - Man = 1.000 10 rounded to M = 0b000

Floating Point Operations: Basic Idea

$$\text{Value} = (-1)^{S} \times \text{Mantissa} \times 2^{\text{Exponent}}$$

S	E	M

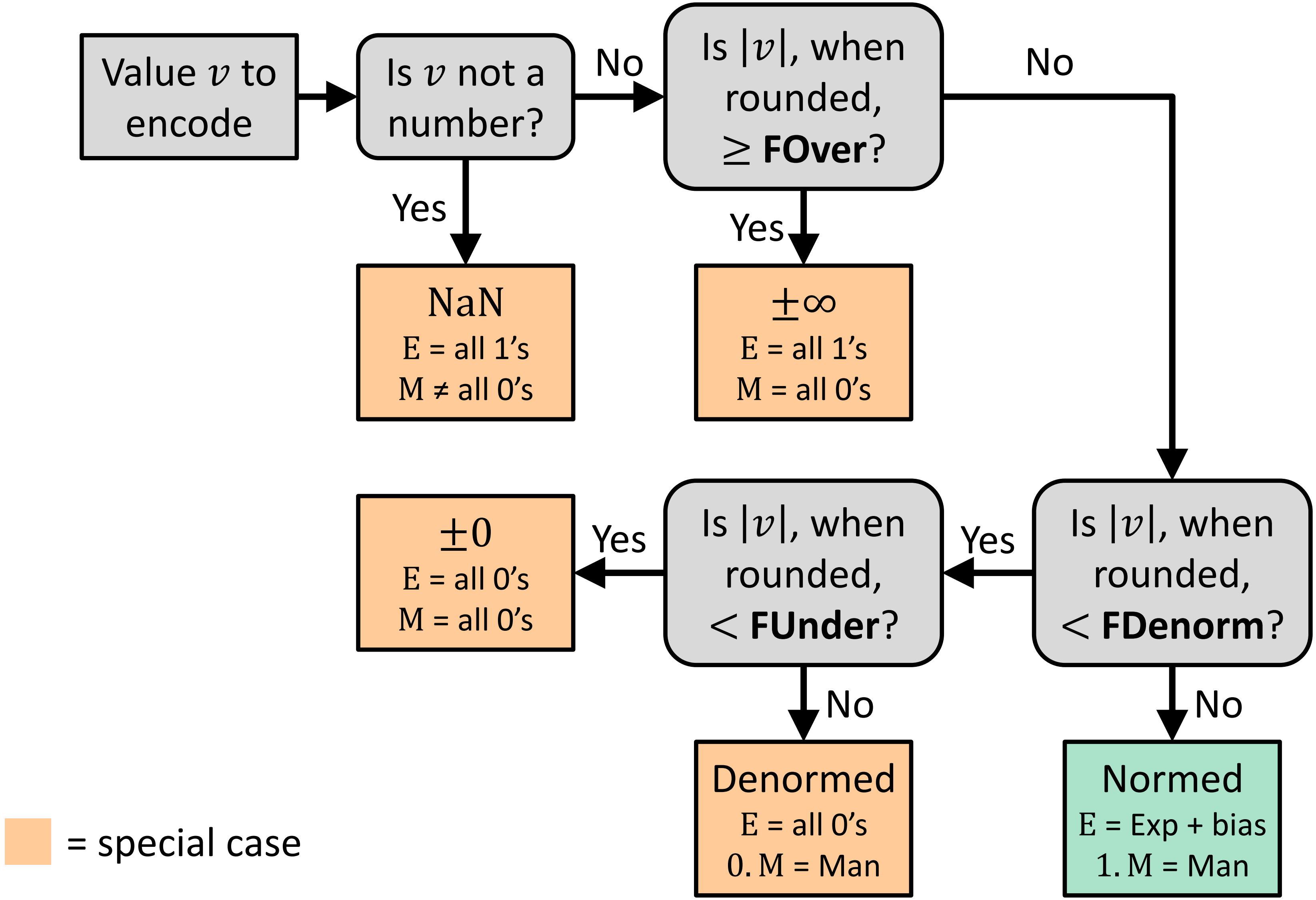
- `x +f y = Round(x + y)` ($x +_f y = \text{Round}(x + y)$)
- `x *f y = Round(x * y)` ($x *_f y = \text{Round}(x * y)$)

- Basic idea for floating point operations:
  - First, compute the exact result
  - Then *round* the result to make it fit into the specified precision (width of M)
    - Possibly over/underflow if exponent outside of range

Mathematical Properties of FP Operations

- Overflow yields $\pm\infty$ and underflow yields 0
- Floats with value $\pm\infty$ and NaN can be used in operations
  - Result usually still $\pm\infty$ or NaN, but not always intuitive
- Floating point operations do not work like real math, due to rounding
  - Not associative: `(3.14+1e100)-1e100 != 3.14+(1e100-1e100)`
    `0` **`3.14`**
  - Not distributive: `100*(0.1+0.2) != 100*0.1+100*0.2`
    `30.000000000000003553` **`30`**
  - Not cumulative
    - Repeatedly adding a very small number to a large one may do nothing

Floating Point Encoding Flow Chart



Limits of Interest

This is extra (non-testable) material

- The following thresholds will help give you a sense of when certain outcomes come into play, but don't worry about the specifics:
  - **FOver** = $2^{\text{bias}+1}$ = 2^8
    - This is just larger than the largest representable normalized number
  - **FDenorm** = $2^{1-\text{bias}}$ = 2^{-6}
    - This is the smallest representable normalized number
  - **FUnder** = $2^{1-\text{bias}-m}$ = 2^{-9}
    - m is the width of the mantissa field
    - This is the smallest representable denormalized number

Polling Question 1

- Using our **8-bit** representation, what value gets stored when we try to encode **2.625** = $2^{1} + 2^{-1} + 2^{-3}$?

S	E	M
1	4	3

A. **+ 2.5**

B. **+ 2.625**

C. **+ 2.75**

D. **+ 3.25**

E. **We're lost…**

Polling Question 2

- Using our **8-bit** representation, what value gets stored when we try to encode **384** = $2^8 + 2^7$?

S	E	M
1	4	3

A. + 256

B. + 384

C. + ∞

D. NaN

E. We're lost…

Floating Point in C

!!!

- Two common levels of precision:

`float`	`1.0f`	single precision (32-bit)
`double`	`1.0`	double precision (64-bit)

- `#include <math.h>` to get `INFINITY` and `NAN` constants
- `#include <float.h>` for additional constants

- Equality (==) comparisons between floating point numbers are tricky, and often return unexpected results, so just avoid them!

Floating Point Conversions in C

!!!

- Casting between `int`, `float`, and `double` **changes** the bit representation
  - `int` → `float`
    - May be rounded (not enough bits in mantissa: 23)
    - Overflow impossible
  - `int` or `float` → `double`
    - Exact conversion (all 32-bit `int`s are representable)
  - `long` → `double`
    - Depends on word size (32-bit is exact, 64-bit may be rounded)
  - `double` or `float` → `int`
    - Truncates fractional part (rounded toward zero)
    - "Not defined" when out of range or NaN: generally sets to TMin (even if the value is a very big positive)

Challenge Question

- We execute the following code in C. How many bytes are the same (value and position) between `i` and `f`?
  - No voting

```
int i = 384;  // 2^8 + 2^7
float f = (float) i;
```

A. 0 bytes

B. 1 byte

C. 2 bytes

D. 3 bytes

E. We're lost…

Floating Point Summary

- Floats also suffer from the fixed number of bits available to represent them
  - Can get overflow/underflow
  - "Gaps" produced in representable numbers means we can lose precision, unlike `int`s
    - Some "simple fractions" have no exact representation (*e.g.*, 0.2)
    - "Every operation gets a slightly wrong result"
- Floating point arithmetic not associative or distributive
  - Mathematically equivalent ways of writing an expression may compute different results
- Never test floating point values for equality!
- Careful when converting between `int`s and `float`s!

Number Representation Really Matters

- **1991:** Patriot missile targeting error
  - clock skew due to conversion from integer to floating point
- **1996:** Ariane 5 rocket exploded ($1 billion)
  - overflow converting 64-bit floating point to 16-bit integer
- **2000:** Y2K problem
  - limited (decimal) representation: overflow, wrap-around
- **2038:** Unix epoch rollover
  - Unix epoch = seconds since 12am, January 1, 1970
  - signed 32-bit integer representation rolls over to TMin in 2038
- **Other related bugs:**
  - 1982: Vancouver Stock Exchange 10% error in less than 2 years
  - 1994: Intel Pentium FDIV (floating point division) HW bug ($475 million)
  - 1997: USS Yorktown "smart" warship stranded: divide by zero
  - 1998: Mars Climate Orbiter crashed: unit mismatch ($193 million)

Summary

E	M	Meaning
0x00	0	± 0
0x00	non-zero	± denorm num
0x01 – 0xFE	anything	± norm num
0xFF	0	± ∞
0xFF	non-zero	NaN

- Floating point encoding has many limitations
  - Overflow, underflow, rounding
  - Rounding is a HUGE issue due to limited mantissa bits and gaps that are scaled by the value of the exponent
  - Floating point arithmetic is NOT associative or distributive
- Converting between integral and floating point data types *does* change the bits

BONUS SLIDES

An example that applies the IEEE Floating Point concepts to a smaller (8-bit) representation scheme. These slides expand on material covered today, so while you don't need to read these, the information is "fair game."

Tiny Floating Point Example

S	E	M
1	4	3

- 8-bit Floating Point Representation
  - The sign bit is in the most significant bit (MSB)
  - The next four bits are the exponent, with a bias of $2^{4-1}–1 = 7$
  - The last three bits are the mantissa

- Same general form as IEEE Format
  - Normalized binary scientific point notation
  - Similar special cases for 0, denormalized numbers, NaN, ∞

Dynamic Range (Positive Only)

	S	E	M	Exp	Value	
Denormalized numbers	0	0000	000	-6	0	
	0	0000	001	-6	1/8*1/64 = 1/512	closest to zero
	0	0000	010	-6	2/8*1/64 = 2/512	
	…					
	0	0000	110	-6	6/8*1/64 = 6/512	
	0	0000	111	-6	7/8*1/64 = 7/512	largest denorm
Normalized numbers	0	0001	000	-6	8/8*1/64 = 8/512	smallest norm
	0	0001	001	-6	9/8*1/64 = 9/512	
	…					
	0	0110	110	-1	14/8*1/2 = 14/16	
	0	0110	111	-1	15/8*1/2 = 15/16	closest to 1 below
	0	0111	000	0	8/8*1 = 1	
	0	0111	001	0	9/8*1 = 9/8	closest to 1 above
	0	0111	010	0	10/8*1 = 10/8	
	…					
	0	1110	110	7	14/8*128 = 224	
	0	1110	111	7	15/8*128 = 240	largest norm
	0	1111	000	n/a	inf	

Special Properties of Encoding

- Floating point zero (0^+) exactly the same bits as integer zero
  - All bits = 0
- Can (Almost) Use Unsigned Integer Comparison
  - Must first compare sign bits
  - Must consider $0^- = 0^+ = 0$
  - NaNs problematic
    - Will be greater than any other values
    - What should comparison yield?
  - Otherwise OK
    - Denorm vs. normalized
    - Normalized vs. infinity