Floating Point I

CSE 351 Spring 2021

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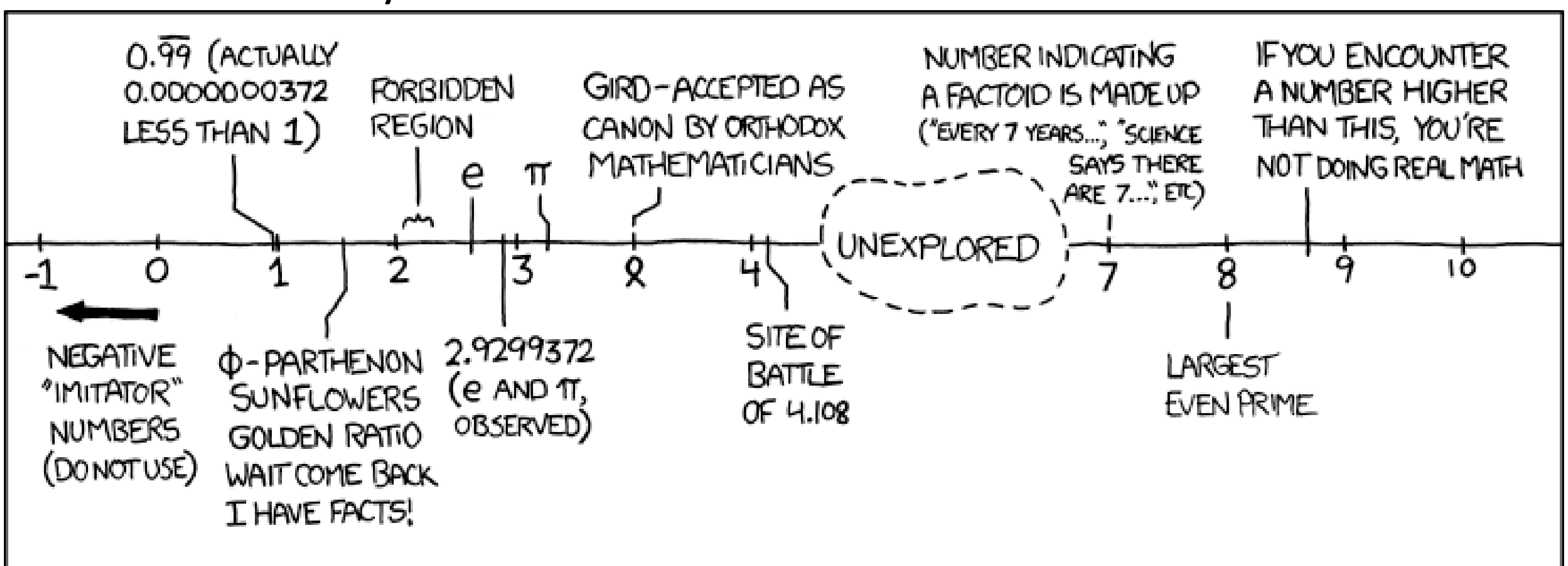


http://xkcd.com/899/

Administrivia

- hw4 due Friday (4/09) @ 11:59 pm
- hw5 due Monday (4/12) @ 11:59 pm
- Lab 1a due Monday (4/12) @ 11:59 pm
  - Submit `pointer.c` and `lab1Areflect.txt`
  - Make sure you submit *something* to Gradescope before the deadline and that the file names are correct
  - Can use late day tokens to submit up until Wed 11:59 pm
- Lab 1b, due 4/19
  - Submit `aisle_manager.c`, `store_client.c`, and `lab1Breflect.txt`
- **Questions Docs**: Use @uw google account to access!!
  - https://tinyurl.com/CSE351-21sp-Questions

Reading Review

- Terminology:
  - normalized scientific binary notation
  - trailing zeros
  - sign, mantissa, exponent ↔ bit fields S, M, and E
  - `float`, `double`
  - biased notation (exponent), implicit leading one (mantissa)
  - rounding errors

Review Questions

- Convert 11.375_{10} to normalized binary scientific notation

- What is the correct value encoded by the following floating point number?

  0b 0 | 1000 0000 | 110 0000 0000 0000 0000 0000

  - bias = $2^{w-1}-1$
  - exponent = E – bias
  - mantissa = 1.M

Number Representation Revisited

- What can we represent in one word?
  - Signed and Unsigned Integers
  - Characters (ASCII)
  - Addresses

- How do we encode the following:
  - Real numbers (*e.g.*, 3.14159)
  - Very large numbers (*e.g.*, 6.02×10^{23})
  - Very small numbers (*e.g.*, 6.626×10^{-34})
  - Special numbers (*e.g.*, ∞, NaN)

Floating Point

Floating Point Topics

- **Fractional binary numbers**
- **IEEE floating-point standard**
- Floating-point operations and rounding
- Floating-point in C

- There are many more details that we won't cover
  - It's a 58-page standard…

Representation of Fractions

- "Binary Point," like decimal point, signifies boundary between integer and fractional parts:

  Example 6-bit representation: xx.yyyy

  2^{1} 2^{0} 2^{-1} 2^{-2} 2^{-3} 2^{-4}

- Example: $10.1010_2 = 1\times2^1 + 1\times2^{-1} + 1\times2^{-3} = 2.625_{10}$

Representation of Fractions

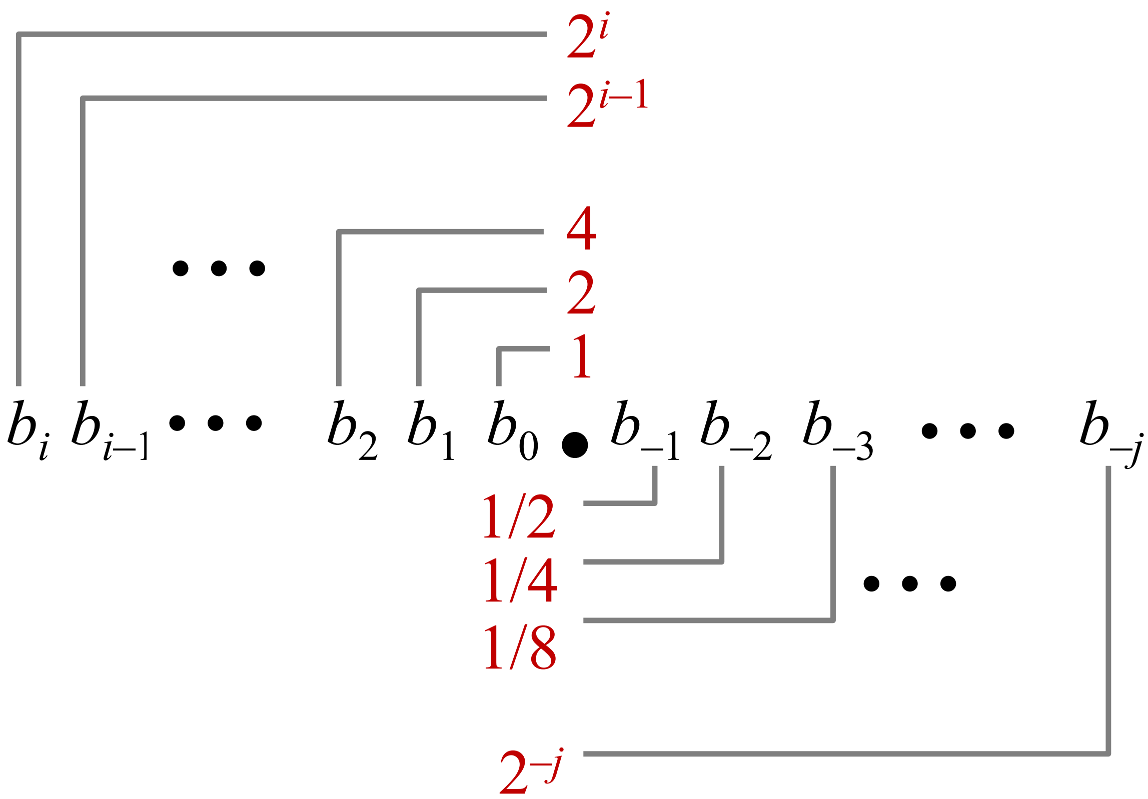
- "Binary Point," like decimal point, signifies boundary between integer and fractional parts:

Example 6-bit representation: **xx.yyyy**

2^{1} 2^{0} 2^{-1} 2^{-2} 2^{-3} 2^{-4}

- In this 6-bit representation:
  - What is the encoding and value of the smallest (most negative) number?
  - What is the encoding and value of the largest (most positive) number?
  - What is the smallest number greater than 2 that we can represent?

Fractional Binary Numbers



- **Representation**
  - Bits to right of "binary point" represent fractional powers of 2
  - Represents rational number: $$\sum_{k=-j}^{i} b_k \cdot 2^k$$

Fractional Binary Numbers

- Value Representation
  - 5 and 3/4 `101.11`$_2$
  - 2 and 7/8 `10.111`$_2$
  - 47/64 `0.101111`$_2$
- Observations
  - Shift left = multiply by power of 2
  - Shift right = divide by power of 2
  - Numbers of the form `0.111111`…$_2$ are just below 1.0
    - $1/2 + 1/4 + 1/8 + \ldots + 1/2^i + \ldots \rightarrow 1.0$
    - Use notation $1.0 - \varepsilon$

Limits of Representation

- Limitations:
  - Even given an arbitrary number of bits, can only **exactly** represent numbers of the form $x * 2^y$ (y can be negative)
  - Other rational numbers have repeating bit representations

Value:	**Binary Representation:**
• 1/3 = $0.333333\ldots_{10}$ =	$0.01010101[01]\ldots_2$
• 1/5 =	$0.001100110011[0011\ ]\ldots_2$
• 1/10 =	$0.0001100110011[0011\ ]\ldots_2$

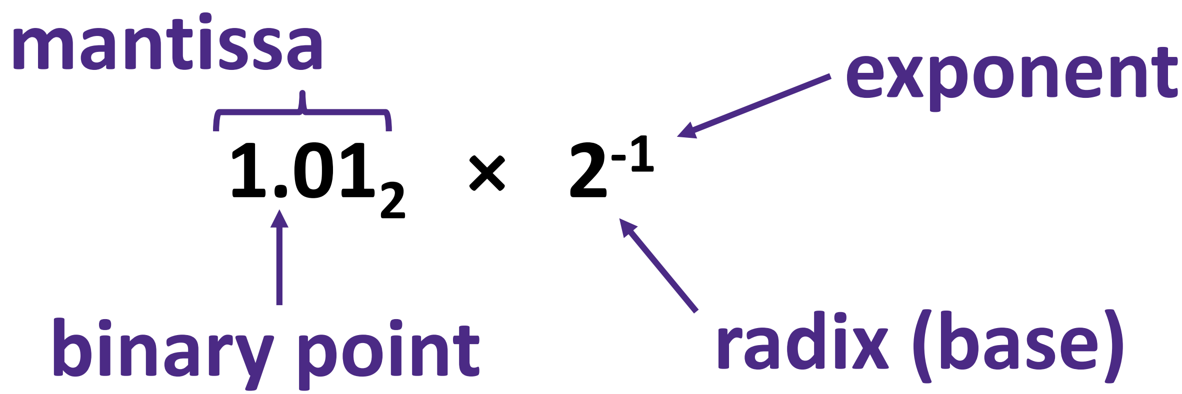
Fixed Point Representation

- Implied binary point. Two example schemes:
  - #1: the binary point is between bits 2 and 3
    - $b_7\ b_6\ b_5\ b_4\ b_3$ [.] $b_2\ b_1\ b_0$
  - #2: the binary point is between bits 4 and 5
    - $b_7\ b_6\ b_5$ [.] $b_4\ b_3\ b_2\ b_1\ b_0$
- Which scheme is best?

Floating Point Representation

- Analogous to scientific notation
  - In Decimal:
    - Not 12000000, but 1.2×10^{7} In C: 1.2e7
    - Not 0.0000012, but 1.2×10^{-6} In C: 1.2e-6
  - In Binary:
    - Not 11000.000, but 1.1×2^{4}
    - Not 0.000101, but 1.01×2^{-4}
- We have to divvy up the bits we have (e.g., 32) among:
  - the sign (1 bit)
  - the mantissa (significand)
  - the exponent

Scientific Notation (Binary)



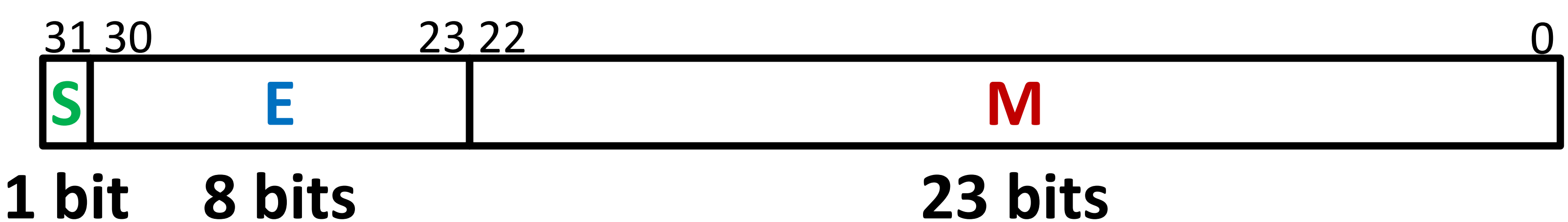
- *Normalized form*: exactly one digit (non-zero) to left of binary point
- Computer arithmetic that supports this called floating point due to the "floating" of the binary point
  - Declare such variable in C as `float` (or `double`)

IEEE Floating Point

- IEEE 754 (established in 1985)
  - Standard to make numerically-sensitive programs portable
  - Specifies two things: *representation scheme* and result of *floating point operations*
  - Supported by all major CPUs
- Driven by numerical concerns
  - **Scientists**/numerical analysts want them to be as **real** as possible
  - **Engineers** want them to be **easy to implement** and **fast**
  - Scientists mostly won out:
    - Nice standards for rounding, overflow, underflow, but...
    - Hard to make fast in hardware
    - **Float operations can be an order of magnitude slower than integer ops**

Floating Point Encoding

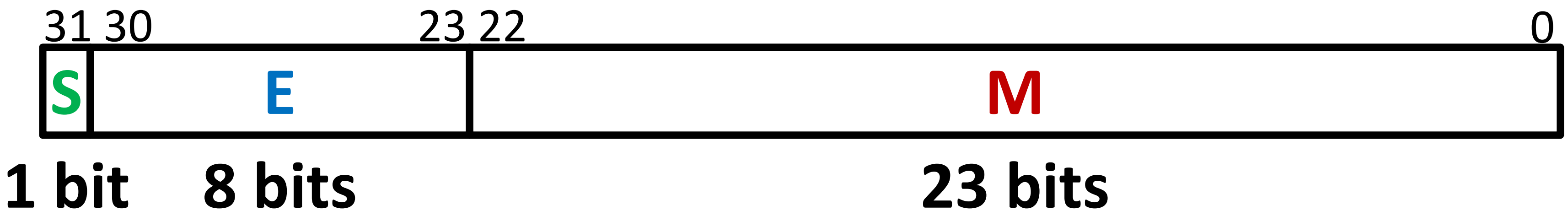
- Use normalized, base 2 scientific notation:
  - Value: $\pm 1 \times \text{Mantissa} \times 2^{\text{Exponent}}$
  - Bit Fields: $(-1)^{S} \times 1.M \times 2^{(E-\text{bias})}$
- Representation Scheme:
  - Sign bit (0 is positive, 1 is negative)
  - Mantissa (a.k.a. significand) is the fractional part of the number in normalized form and encoded in bit vector **M**
  - Exponent weights the value by a (possibly negative) power of 2 and encoded in the bit vector **E**



The Exponent Field

- Use biased notation
  - Read exponent as unsigned, but with *bias* of $2^{w-1}-1 = 127$
  - Representable exponents roughly ½ positive and ½ negative
  - Exp = E – bias ↔ E = Exp + bias
    - Exponent 0 (Exp = 0) is represented as E = 0b 0111 1111
- Why biased?
  - Makes floating point arithmetic easier
  - Makes somewhat compatible with two's complement hardware

The Mantissa (Fraction) Field



$$(-1)^{S} \times (1\,.\,M) \times 2^{(E-\text{bias})}$$

- Note the implicit 1 in front of the M bit vector
  - Example: 0b 0011 1111 1100 0000 0000 0000 0000 0000 is read as $1.1_2 = 1.5_{10}$, *not* $0.1_2 = 0.5_{10}$
  - Gives us an extra bit of *precision*
- Mantissa "limits"
  - Low values near M = 0b0…0 are close to 2^{Exp}
  - High values near M = 0b1…1 are close to 2^{Exp+1}

Normalized Floating Point Conversions

- FP → Decimal
  1. Append the bits of M to implicit leading 1 to form the mantissa.
  2. Multiply the mantissa by $2^{E - \text{bias}}$.
  3. Multiply the sign $(-1)^S$.
  4. Multiply out the exponent by shifting the binary point.
  5. Convert from binary to decimal.

- Decimal → FP
  1. Convert decimal to binary.
  2. Convert binary to normalized scientific notation.
  3. Encode sign as S (0/1).
  4. Add the bias to exponent and encode E as unsigned.
  5. The first bits after the leading 1 that fit are encoded into M.

Practice Question

- Convert the decimal number **-7.375** into floating point representation

Challenge Question

- Find the sum of the following binary numbers in normalized scientific binary notation:

$$1.01_2 \times 2^0 + 1.11_2 \times 2^2$$

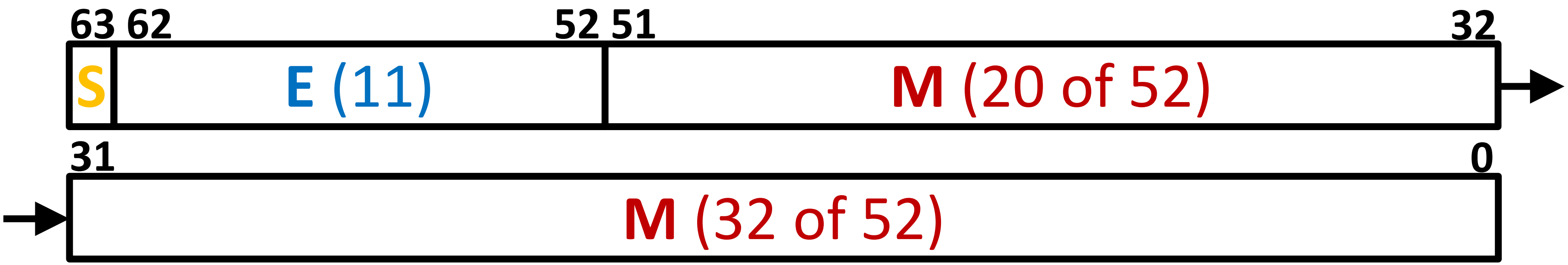
Precision and Accuracy

- Precision is a count of the number of bits in a computer word used to represent a value
  - Capacity for accuracy
- Accuracy is a measure of the difference between the *actual value of a number* and its computer representation

  - *High precision permits high accuracy but doesn't guarantee it. It is possible to have high precision but low accuracy.*
  - **Example:** `float pi = 3.14;`
    - `pi` will be represented using all 24 bits of the mantissa (highly precise), but is only an approximation (not accurate)

Need Greater Precision?

- Double Precision (vs. Single Precision) in 64 bits



  - C variable declared as `double`
  - Exponent bias is now $2^{10}–1 = 1023$
  - **Advantages:** greater precision (larger mantissa), greater range (larger exponent)
  - **Disadvantages:** more bits used, slower to manipulate

Current Limitations

- Largest magnitude we can represent?
- Smallest magnitude we can represent?
  - Limited *range* due to width of E field

- What happens if we try to represent $2^0 + 2^{-30}$?
  - Rounding due to limited *precision*: stores 2^0

- There is a need for *special cases*
  - How do we represent the value zero?
  - What about ∞ and NaN?

Summary

- Floating point approximates real numbers:

31	30 … 23	22 … 0
S	E (8)	M (23)

  - Handles large numbers, small numbers, special numbers
  - Exponent in biased notation (bias = $2^{w-1}–1$)
    - Size of exponent field determines our representable *range*
    - Outside of representable exponents is *overflow* and *underflow*
  - Mantissa approximates fractional portion of binary point
    - Size of mantissa field determines our representable *precision*
    - Implicit leading 1 (normalized) except in special cases
    - Exceeding length causes *rounding*