Floating Point I

CSE 351 Autumn 2021

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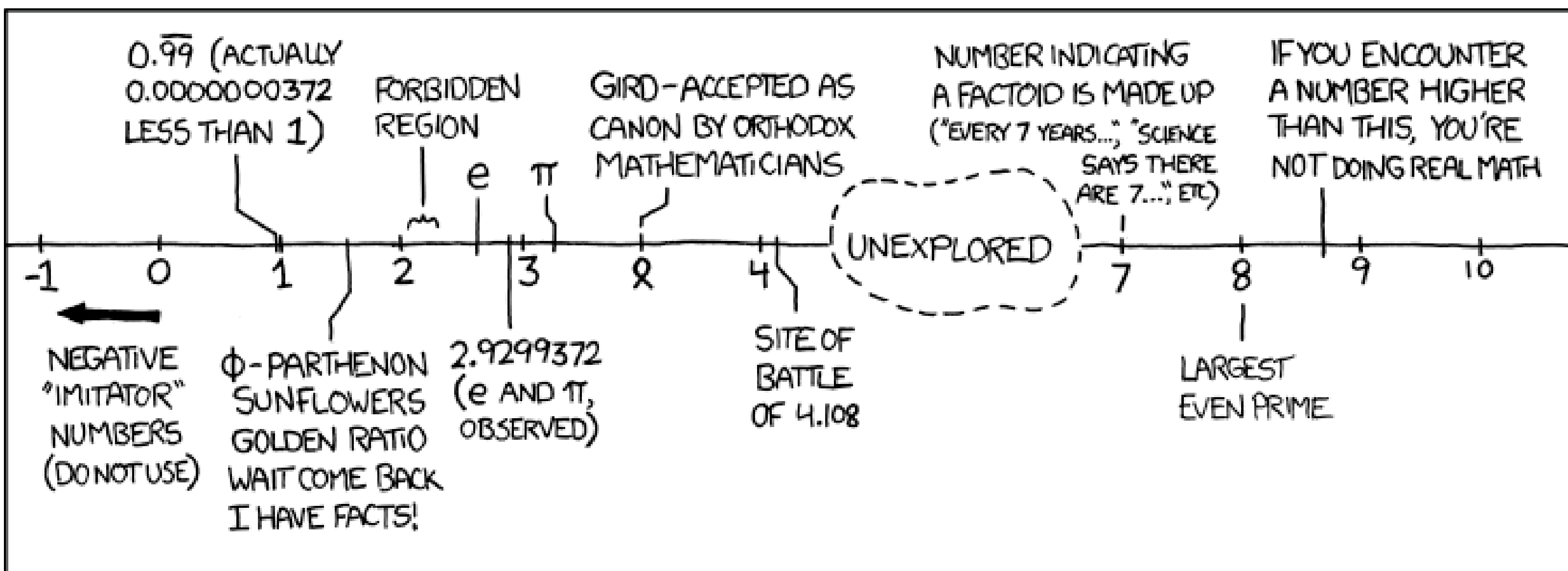


http://xkcd.com/899/

Relevant Course Information

- hw5 due Wednesday, hw6 due Friday

- Lab 1a due tonight at 11:59 pm
  - Submit `pointer.c` and `lab1Asynthesis.txt`
    - Make sure there are no lingering `printf` statements in your code!
  - Make sure you submit *something* to Gradescope before the deadline and that the file names are correct
  - Can use late day tokens to submit up until Wed 11:59 pm

- Lab 1b due next Monday (10/19)
  - Submit `aisle_manager.c`, `store_client.c`, and `lab1Bsynthesis.txt`

Lab 1b Aside: C Macros

- C macros basics:
  - Basic syntax is of the form: `#define NAME expression`
  - Allows you to use "`NAME`" instead of "`expression`" in code
    - Does naïve copy and replace *before* compilation – everywhere the characters "NAME" appear in the code, the characters "expression" will now appear instead
    - NOT the same as a Java constant
  - Useful to help with readability/factoring in code

- You'll use C macros in Lab 1b for defining bit masks
  - See Lab 1b starter code and Lecture 4 slides (card operations) for examples

Reading Review

- Terminology:
  - normalized scientific binary notation
  - trailing zeros
  - sign, mantissa, exponent ↔ bit fields S, M, and E
  - `float`, `double`
  - biased notation (exponent), implicit leading one (mantissa)
  - rounding errors
- Questions from the Reading?

$2^{-1} = 0.5$
$2^{-2} = 0.25$
$2^{-3} = 0.125$
$2^{-4} = 0.0625$

Review Questions

- Convert 11.375_{10} to normalized binary scientific notation

$8+2+1+0.25+0.125$

$2^3+2^1+2^0+2^{-2}+2^{-3} = 1011.011_2 = \boxed{1.011011 \times 2^3}$

- What is the value encoded by the following floating point number?

0b 0 | 1000 0000 | 110 0000 0000 0000 0000 0000

S E M

  - bias = $2^{w-1}-1$ $= 2^7-1 = 127$
  - exponent = E – bias $= 2^7 - 127 = 128-127 = 1$
  - mantissa = 1.M $= 1.110\ldots0_2$

$(-1)^0 \times 1.11_2 \times 2^1 = 11.1_2 = \boxed{+3.5}$

Number Representation Revisited

- What can we represent in one word?
  - Signed and Unsigned Integers
  - Characters (ASCII)
  - Addresses

- How do we encode the following:
  - Real numbers (*e.g.*, 3.14159)
  - Very large numbers (*e.g.*, 6.02×10^{23})
  - Very small numbers (*e.g.*, 6.626×10^{-34})
  - Special numbers (*e.g.*, ∞, NaN)

Floating Point

Floating Point Topics

- **Fractional binary numbers**
- **IEEE floating-point standard**
- Floating-point operations and rounding
- Floating-point in C

- There are many more details that we won't cover
  - It's a 58-page standard…

Representation of Fractions

- "Binary Point," like decimal point, signifies boundary between integer and fractional parts:

  Example 6-bit representation: "leading zeros" 0 . . . 0 xx.yyyy 0 . . . 0 "trailing zeros"

  $$\begin{array}{cccccc} 1 & 0 & & -1 & -2 & -3 \\ x & x & . & y & y & y & y \\ 2^{1} & 2^{0} & & 2^{-1} & 2^{-2} & 2^{-3} & 2^{-4} \end{array}$$

- Example: $10.1010_2 = 1 \times 2^1 + 1 \times 2^{-1} + 1 \times 2^{-3} = 2.625_{10}$

Representation of Fractions

- "Binary Point," like decimal point, signifies boundary between integer and fractional parts:

Example 6-bit representation: **xx.yyyy**

2^1 2^0 2^{-1} 2^{-2} 2^{-3} 2^{-4}

0.5 0.25 0.125 0.0625

- In this 6-bit representation:
  - What is the encoding and value of the smallest (most negative) number?
  - What is the encoding and value of the largest (most positive) number?
  - What is the smallest number greater than 2 that we can represent?

$00.0000_2 = 0$

$11.111\underbrace{1}_{2^{-4}} = 4 - 2^{-4}$

$2 = 10.0000_2$

$10.0001 = 2 + 2^{-4}$

can't represent anything in-between! ☹

Binary Scientific Notation (Review)

exponent

$$\overbrace{1.01_2}^{\text{mantissa}} \times 2^{-1}$$

binary point

radix (base)

- *Normalized form*: exactly one digit (non-zero) to left of binary point
- Computer arithmetic that supports this called floating point due to the "floating" of the binary point
  - Declare such variable in C as `float` (or `double`)

IEEE Floating Point

- IEEE 754 (established in 1985)
  - Standard to make numerically-sensitive programs portable
  - Specifies two things: *representation scheme* and result of *floating point operations*
  - Supported by all major CPUs
- Driven by numerical concerns
  - **Scientists**/numerical analysts want them to be as **real** as possible
  - **Engineers** want them to be **easy to implement** and **fast**

    competing goals!
  - Scientists mostly won out:
    - Nice standards for rounding, overflow, underflow, but...
    - Hard to make fast in hardware
    - **Float operations can be an order of magnitude slower than integer ops**

      FLOPs

      used in computer benchmarks

Floating Point Encoding (Review)

- Use normalized, base 2 scientific notation:
  - Value: $\pm 1 \times \text{Mantissa} \times 2^{\text{Exponent}}$
  - Bit Fields: $(-1)^S \times 1.M \times 2^{(E-\text{bias})}$
- Representation Scheme: (3 separate fields within 32 bits)
  - Sign bit (0 is positive, 1 is negative)
  - Mantissa (a.k.a. significand) is the fractional part of the number in normalized form and encoded in bit vector **M**
  - Exponent weights the value by a (possibly negative) power of 2 and encoded in the bit vector **E**

values

31	30 … 23	22 … 0
S	E	M
1 bit	8 bits	23 bits

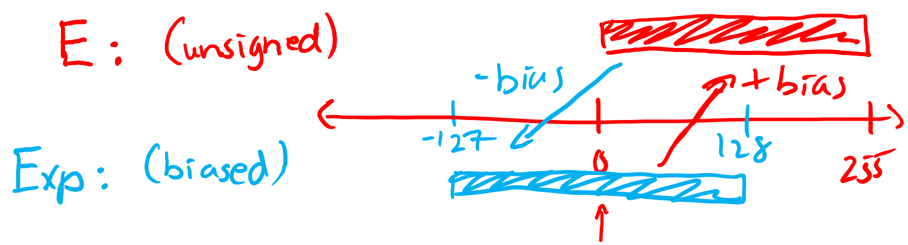
binary encodings

The Exponent Field (Review)

- Use biased notation

  $w=8$, can encode $2^8=256$ exponents

  - Read exponent as unsigned, but with *bias* of $2^{w-1}-1$ = 127
  - Representable exponents roughly ½ positive and ½ negative
  - Exp = E – bias ↔ E = Exp + bias
    - Exponent 0 (Exp = 0) is represented as E = 0b 0111 1111 $= 2^7-1$



- Why biased?
  - Makes floating point arithmetic easier
  - Makes somewhat compatible with two's complement hardware

The Mantissa (Fraction) Field (Review)

31	30 … 23	22 … 0
S	E	M
1 bit	8 bits	23 bits

$$(-1)^{S} \times (1\,.\,M) \times 2^{(E-\text{bias})}$$

- Note the implicit leading 1 in front of the M bit vector
  - ⊕, Exp = 0, Man = 1.10…0
  - Example: 0b 0011 1111 1100 0000 0000 0000 0000 0000 is read as $1.1_2 = 1.5_{10}$, *not* $0.1_2 = 0.5_{10}$
  - Gives us an extra bit of *precision*
- Mantissa "limits"
  - $2^{Exp} \times 1.0...0 = 2^{Exp}$
  - Low values near M = 0b0…0 are close to 2^{Exp}
  - High values near M = 0b1…1 are close to 2^{Exp+1}
  - $2^{Exp} \times 1.1...1 = 2^{Exp}(2 - 2^{-23}) = 2^{Exp+1} - 2^{Exp-23}$

Normalized Floating Point Conversions

- FP → Decimal
  1. Append the bits of M to implicit leading 1 to form the mantissa.
  2. Multiply the mantissa by $2^{\mathrm{E} - \text{bias}}$.
  3. Multiply the sign $(-1)^{\mathrm{S}}$.
  4. Multiply out the exponent by shifting the binary point.
  5. Convert from binary to decimal.

- Decimal → FP
  1. Convert decimal to binary.
  2. Convert binary to normalized scientific notation.
  3. Encode sign as S (0/1).
  4. Add the bias to exponent and encode E as unsigned.
  5. The first bits after the leading 1 that fit are encoded into M.

Practice Question

- Convert the decimal number **-7.375** into floating point representation

$-7.375 = -(4+2+1+0.25+0.125) = -(2^2+2^1+2^0+2^{-2}+2^{-3}) = -111.011_2 = -1.11011_2 \times 2^2$

$S = 1$, $E = 2+127 = 129 = 0b\,1000\,0001$, $M = 0b\,110110\ldots0$

$0b\,1100\,0000\,1110\,1100\,0\ldots0 = \boxed{0x\ C0EC\ 0000}$

Exploration Question

- Find the sum of the following binary numbers in normalized scientific binary notation:

① match exponents
② sum mantissas
③ normalize

$$.0\,1.01_2 \times 2^{\cancel{0}\,2} + 1.11_2 \times 2^2$$

$$\begin{array}{rl} & 0.0101 \times 2^2 \\ + & 1.11 \times 2^2 \\ \hline & 10.0001 \times 2^2 \end{array} = \boxed{1.00001 \times 2^3}$$

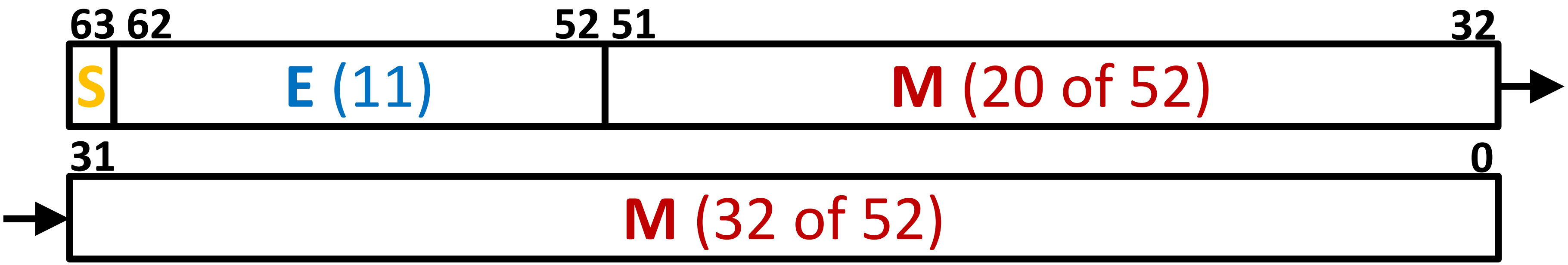
Precision and Accuracy

- Precision is a count of the number of bits in a computer word used to represent a value
  - Capacity for accuracy
- Accuracy is a measure of the difference between the *actual value of a number* and its computer representation

  - *High precision permits high accuracy but doesn't guarantee it. It is possible to have high precision but low accuracy.*
  - **Example:** `float pi = 3.14;`
    - `pi` will be represented using all 24 bits of the mantissa (highly precise), but is only an approximation (not accurate)

Need Greater Precision?

- Double Precision (vs. Single Precision) in 64 bits



  - C variable declared as `double`
  - Exponent bias is now $2^{10}-1 = 1023$, bias = $2^{w-1}-1$
  - **Advantages:** greater precision (larger mantissa), greater range (larger exponent)
  - **Disadvantages:** more bits used, slower to manipulate

Current Limitations

- Largest magnitude we can represent? $E = 0b1111\ 1111$, $M = 0b1\ldots1$ → Exp = 128
- Smallest magnitude we can represent? $E = 0b0000\ 0000$, $M = 0b0\ldots0$ → Exp = -127
  - Limited *range* due to width of E field
- What happens if we try to represent $2^0 + 2^{-30}$? $= 1.\underbrace{0\ldots0}_{\text{29 zeros}}\cancel{1}$
  - Rounding due to limited *precision*: stores 2^0

  M stores first 23 zeros

- There is a need for *special cases*
  - How do we represent the value zero? $0 \neq \pm 1.M \times 2^{E-\text{bias}}$
  - What about ∞ and NaN? ???

Summary

- Floating point approximates real numbers:

31	30 … 23	22 … 0
S	E (8)	M (23)

  - Handles large numbers, small numbers, special numbers
  - Exponent in biased notation (bias = $2^{w-1} - 1$)
    - Size of exponent field determines our representable *range*
    - Outside of representable exponents is *overflow* and *underflow*
  - Mantissa approximates fractional portion of binary point
    - Size of mantissa field determines our representable *precision*
    - Implicit leading 1 (normalized) except in special cases
    - Exceeding length causes *rounding*