# Floating Point I

CSE 351 Winter 2020


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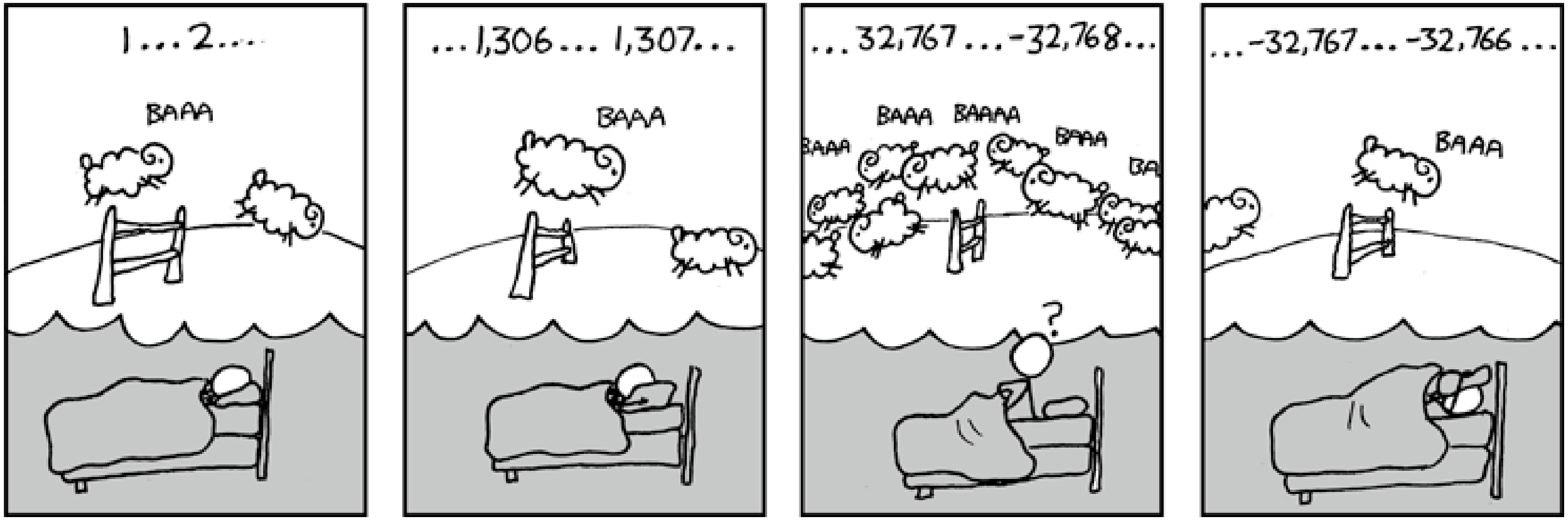



http://xkcd.com/571/

# Administrivia

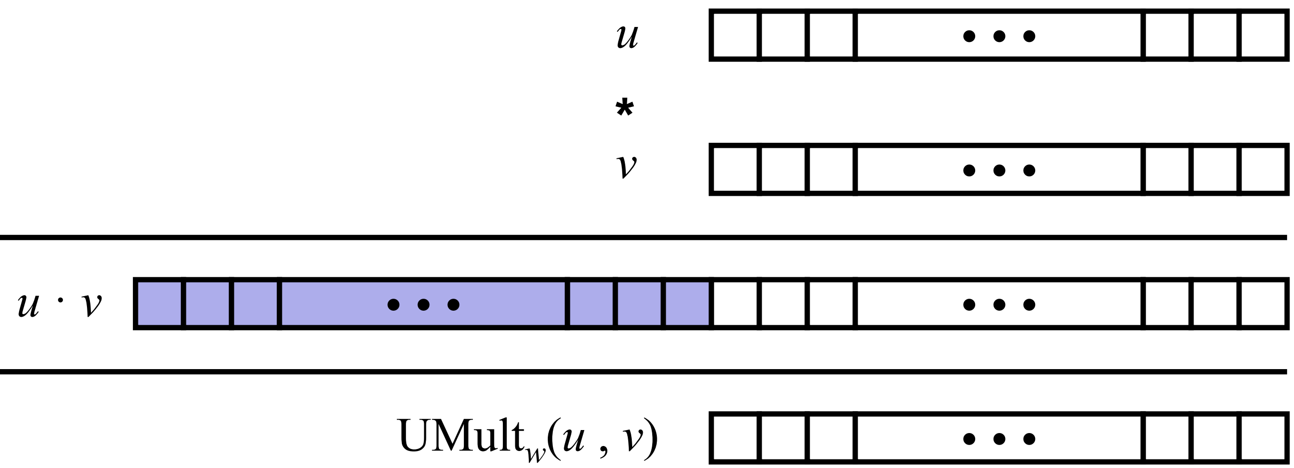
- Lab 1a due ~~tonight~~ Tues 1/21 at 11:59 pm
  - Submit `pointer.c` and `lab1Areflect.txt`
  - Make sure you submit *something* before the deadline and that the file names are correct
- hw5 due Wednesday, hw6 due Friday
- Lab 1b due next Monday (1/27)
  - Submit `bits.c` and `lab1Breflect.txt`

# Unsigned Multiplication in C

*Operands:*
***w*** **bits**

*True Product:*
**2*w*** **bits**

*Discard w bits:*
***w*** **bits**




- Standard Multiplication Function
  - Ignores high order *w* bits
- Implements Modular Arithmetic
  - $\text{UMult}_w(u, v) = u \cdot v \mod 2^w$

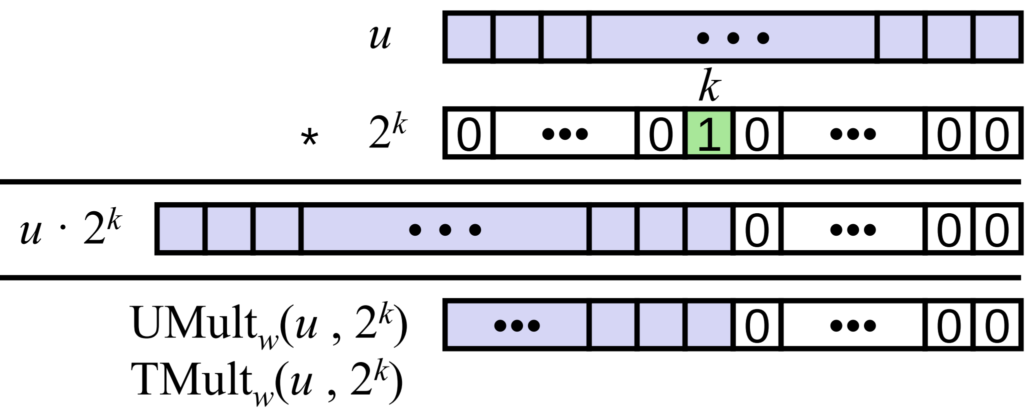
# Multiplication with shift and add

- Operation `u<<k` gives `u*`$2^k$
  - Both signed and unsigned

*Operands:* ***w*** **bits**

*True Product:* ***w*** **+** ***k*** **bits**

*Discard k bits:* ***w*** **bits**




- Examples:
  - `u<<3          ==  u * 8`
  - `u<<5 - u<<3   ==  u * 24`
  - Most machines shift and add faster than multiply
    - ***Compiler generates this code automatically***

# Number Representation Revisited

- What can we represent in one word?
  - Signed and Unsigned Integers
  - Characters (ASCII)
  - Addresses

- How do we encode the following:
  - Real numbers (*e.g.* 3.14159)
  - Very large numbers (*e.g.* $6.02\times10^{23}$)
  - Very small numbers (*e.g.* $6.626\times10^{-34}$)
  - Special numbers (*e.g.* ∞, NaN)

**Floating Point**

# Floating Point Topics

- **Fractional binary numbers**
- IEEE floating-point standard
- Floating-point operations and rounding
- Floating-point in C


- There are many more details that we won't cover
  - It's a 58-page standard…

# Floating Point Summary

- Floats also suffer from the fixed number of bits available to represent them
  - Can get overflow/underflow, just like `int`s
  - "Gaps" produced in representable numbers means we can lose precision, unlike `int`s
    - Some "simple fractions" have no exact representation (*e.g.* 0.2)
    - "Every operation gets a slightly wrong result"
- Floating point arithmetic not associative or distributive
  - *Mathematically* equivalent ways of writing an expression may compute different results
- Never test floating point values for equality!
- Careful when converting between `int`s and `float`s!

# Representation of Fractions

- "Binary Point," like decimal point, signifies boundary between integer and fractional parts:

Example 6-bit representation: **xx.yyyy**

$2^{1}$ $2^{0}$ $2^{-1}$ $2^{-2}$ $2^{-3}$ $2^{-4}$

- <u>Example</u>: $10.1010_2 = 1\times2^{1} + 1\times2^{-1} + 1\times2^{-3} = 2.625_{10}$

# Representation of Fractions

- "Binary Point," like decimal point, signifies boundary between integer and fractional parts:

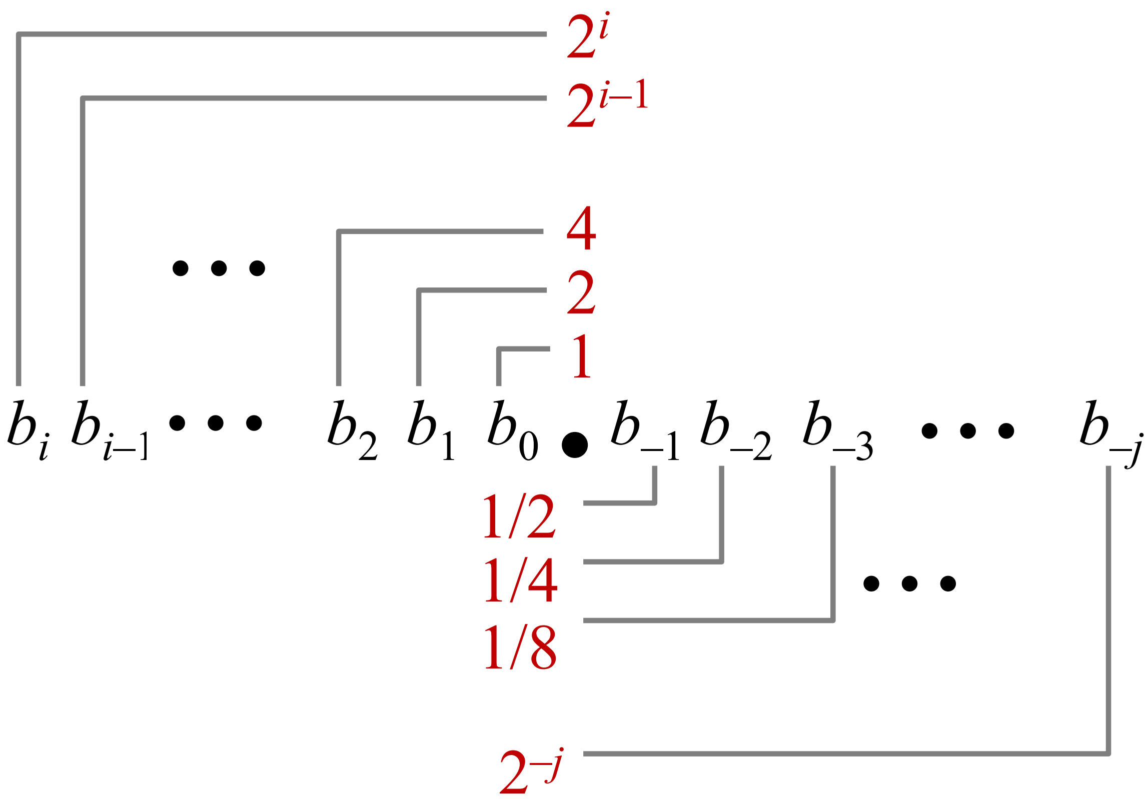
  Example 6-bit representation:

  xx.yyyy

  $2^{1}$ $2^{0}$ $2^{-1}$ $2^{-2}$ $2^{-3}$ $2^{-4}$

- In this 6-bit representation:
  - What is the encoding and value of the smallest (most negative) number?
  - What is the encoding and value of the largest (most positive) number?
  - What is the smallest number greater than 2 that we can represent?

# Fractional Binary Numbers




- **Representation**
  - Bits to right of "binary point" represent fractional powers of 2
  - Represents rational number: $$\sum_{k=-j}^{i} b_k \cdot 2^k$$

# Fractional Binary Numbers

- Value Representation
  - 5 and 3/4 `101.11`$_2$
  - 2 and 7/8 `10.111`$_2$
  - 47/64 `0.101111`$_2$
- Observations
  - Shift left = multiply by power of 2
  - Shift right = divide by power of 2
  - Numbers of the form `0.111111`…$_2$ are just below 1.0
    - $1/2 + 1/4 + 1/8 + \ldots + 1/2^i + \ldots \rightarrow 1.0$
    - Use notation $1.0 - \varepsilon$

# Limits of Representation

- Limitations:
  - Even given an arbitrary number of bits, can only **exactly** represent numbers of the form $x * 2^y$ (y can be negative)
  - Other rational numbers have repeating bit representations

| Value: | Binary Representation: |
|---|---|
| • 1/3 = $0.333333\ldots_{10}$ = | $0.01010101[01]\ldots_2$ |
| • 1/5 = | $0.001100110011[0011\ ]\ldots_2$ |
| • 1/10 = | $0.0001100110011[0011\ ]\ldots_2$ |

# Fixed Point Representation

- Implied binary point. Two example schemes:

  #1: the binary point is between bits 2 and 3

  $b_7\ b_6\ b_5\ b_4\ b_3\ \ [.]\ b_2\ b_1\ b_0$

  #2: the binary point is between bits 4 and 5

  $b_7\ b_6\ b_5\ [.]\ b_4\ b_3\ b_2\ b_1\ b_0$

- Wherever we put the binary point, with fixed point representations there is a trade off between the amount of range and precision we have
- Fixed point = fixed *range* and fixed *precision*
  - range: difference between largest and smallest numbers possible
  - precision: smallest possible difference between any two numbers
- Hard to pick how much you need of each!

# Floating Point Representation

- Analogous to scientific notation
  - In Decimal:
    - Not 12000000, but $1.2 \times 10^{7}$ In C: 1.2e7
    - Not 0.0000012, but $1.2 \times 10^{-6}$ In C: 1.2e-6
  - In Binary:
    - Not 11000.000, but $1.1 \times 2^{4}$
    - Not 0.000101, but $1.01 \times 2^{-4}$
- We have to divvy up the bits we have (e.g., 32) among:
  - the sign (1 bit)
  - the mantissa (significand)
  - the exponent

# Scientific Notation (Decimal)

$$\overbrace{6.02}^{\text{mantissa}}{}_{10} \times 10^{23}$$

exponent

decimal point

radix (base)

- *Normalized form*: exactly one digit (non-zero) to left of decimal point
- Alternatives to representing 1/1,000,000,000
  - Normalized: $1.0 \times 10^{-9}$
  - Not normalized: $0.1 \times 10^{-8}, 10.0 \times 10^{-10}$

# Scientific Notation (Binary)

exponent

$$\overbrace{1.01}^{\text{mantissa}}{}_2 \times 2^{-1}$$

binary point

radix (base)

- Computer arithmetic that supports this called floating point due to the “floating” of the binary point
  - Declare such variable in C as `float` (or `double`)

# Scientific Notation Translation

- Convert from scientific notation to binary point
  - Perform the multiplication by shifting the decimal until the exponent disappears
    - Example: $1.011_2 \times 2^4 = 10110_2 = 22_{10}$
    - Example: $1.011_2 \times 2^{-2} = 0.01011_2 = 0.34375_{10}$
- Convert from binary point to *normalized* scientific notation
  - Distribute out exponents until binary point is to the right of a single digit
    - Example: $1101.001_2 = 1.101001_2 \times 2^3$
- **Practice:** Convert $11.375_{10}$ to normalized binary scientific notation

# Floating Point Topics

- Fractional binary numbers
- **IEEE floating-point standard**
- Floating-point operations and rounding
- Floating-point in C

- There are many more details that we won't cover
  - It's a 58-page standard…

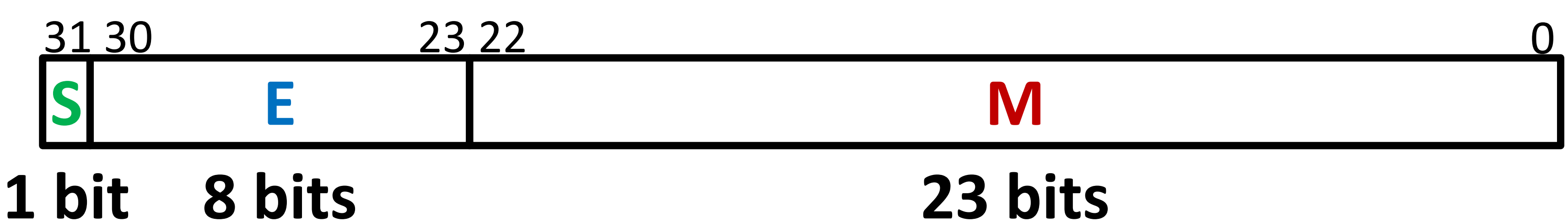
# IEEE Floating Point

- IEEE 754
  - Established in 1985 as uniform standard for floating point arithmetic
  - Main idea: make numerically sensitive programs portable
  - Specifies two things: representation and result of floating operations
  - Now supported by all major CPUs

- Driven by numerical concerns
  - **Scientists**/numerical analysts want them to be as **real** as possible
  - **Engineers** want them to be **easy to implement** and **fast**
  - In the end:
    - Scientists mostly won out
    - Nice standards for rounding, overflow, underflow, but...
    - Hard to make fast in hardware
    - **Float operations can be an order of magnitude slower than integer ops**

# Floating Point Encoding

- Use normalized, base 2 scientific notation:
  - Value: $\pm 1 \times \text{Mantissa} \times 2^{\text{Exponent}}$
  - Bit Fields: $(-1)^{S} \times 1.M \times 2^{(E-\text{bias})}$
- Representation Scheme:
  - Sign bit (0 is positive, 1 is negative)
  - Mantissa (a.k.a. significand) is the fractional part of the number in normalized form and encoded in bit vector **M**
  - Exponent weights the value by a (possibly negative) power of 2 and encoded in the bit vector **E**



# The Exponent Field

- Use biased notation
  - Read exponent as unsigned, but with *bias* of $2^{w-1}-1 = 127$
  - Representable exponents roughly ½ positive and ½ negative
  - Exponent 0 (Exp = 0) is represented as E = 0b 0111 1111
- Why biased?
  - Makes floating point arithmetic easier
  - Makes somewhat compatible with two's complement
- **Practice:** To encode in biased notation, add the bias then encode in unsigned:
  - Exp = 1 → → E = 0b
  - Exp = 127 → → E = 0b
  - Exp = -63 → → E = 0b

# The Mantissa (Fraction) Field

| 31 | 30 … 23 | 22 … 0 |
|---|---|---|
| S | E | M |
| 1 bit | 8 bits | 23 bits |

$$(-1)^{S} \times (1\,.\,M) \times 2^{(E-\text{bias})}$$

- Note the implicit 1 in front of the M bit vector
  - Example: 0b 0011 1111 1100 0000 0000 0000 0000 0000 is read as $1.1_2 = 1.5_{10}$, *not* $0.1_2 = 0.5_{10}$
  - Gives us an extra bit of *precision*
- Mantissa "limits"
  - Low values near M = 0b0…0 are close to $2^{Exp}$
  - High values near M = 0b1…1 are close to $2^{Exp+1}$

# Polling Question

- What is the correct value encoded by the following floating point number?
  - 0b 0 10000000 11000000000000000000000
  - Vote at http://pollev.com/rea

A. + 0.75

B. + 1.5

C. + 2.75

D. + 3.5

E. We're lost…

# Normalized Floating Point Conversions

- FP → Decimal
  1. Append the bits of `M` to implicit leading 1 to form the mantissa.
  2. Multiply the mantissa by $2^{\mathtt{E} - \text{bias}}$.
  3. Multiply the sign $(-1)^{\mathtt{S}}$.
  4. Multiply out the exponent by shifting the binary point.
  5. Convert from binary to decimal.

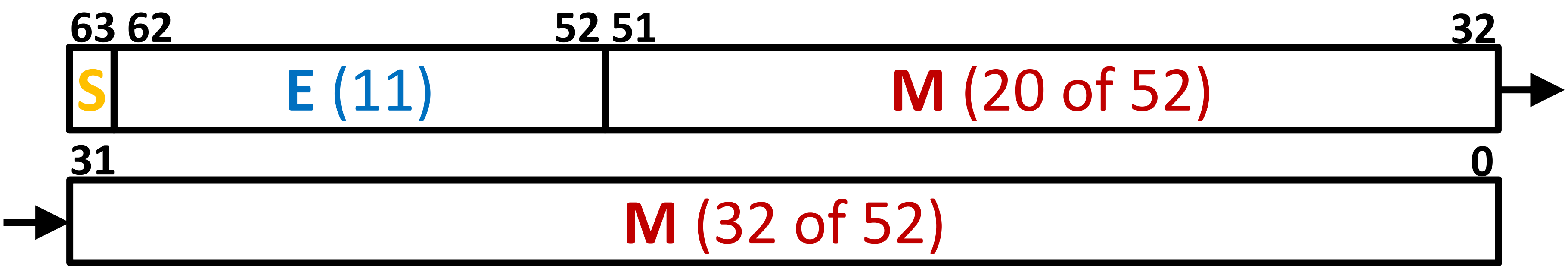
- Decimal → FP
  1. Convert decimal to binary.
  2. Convert binary to normalized scientific notation.
  3. Encode sign as `S` (0/1).
  4. Add the bias to exponent and encode `E` as unsigned.
  5. The first bits after the leading 1 that fit are encoded into `M`.

# Precision and Accuracy

- Precision is a count of the number of bits in a computer word used to represent a value
  - Capacity for accuracy
- Accuracy is a measure of the difference between the *actual value of a number* and its computer representation
  - *High precision permits high accuracy but doesn't guarantee it. It is possible to have high precision but low accuracy.*
  - **Example:** `float pi = 3.14;`
    - `pi` will be represented using all 24 bits of the mantissa (highly precise), but is only an approximation (not accurate)

# Need Greater Precision?

- Double Precision (vs. Single Precision) in 64 bits

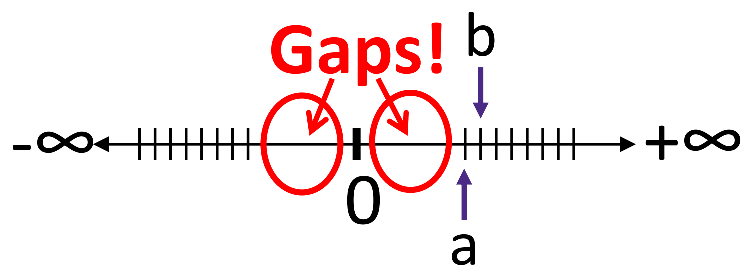



  - C variable declared as `double`
  - Exponent bias is now $2^{10}–1 = 1023$
  - **Advantages:** greater precision (larger mantissa), greater range (larger exponent)
  - **Disadvantages:** more bits used, slower to manipulate

# Representing Very Small Numbers

- But wait… what happened to zero?
  - Using standard encoding 0x00000000 =
  - *Special case:* E and M all zeros = 0
    - Two zeros! But at least 0x00000000 = 0 like integers

- New numbers closest to 0:
  - $a = 1.0…0_2 \times 2^{-126} = 2^{-126}$
  - $b = 1.0…01_2 \times 2^{-126} = 2^{-126} + 2^{-149}$
  - Normalization and implicit 1 are to blame
  - *Special case:* E = 0, M ≠ 0 are denormalized numbers



# Denorm Numbers

This is extra (non-testable) material

- Denormalized numbers
  - No leading 1
  - Uses implicit exponent of –126 even though E = 0x00

- Denormalized numbers close the gap between zero and the smallest normalized number
  - Smallest norm: $\pm\ 1.0\ldots0_{two}\times2^{-126} = \pm\ 2^{-126}$
  - Smallest denorm: $\pm\ 0.0\ldots01_{two}\times2^{-126} = \pm\ 2^{-149}$
    - There is still a gap between zero and the smallest denormalized number

**So much closer to 0**

# Summary

- Floating point approximates real numbers:


  - Handles large numbers, small numbers, special numbers
  - Exponent in biased notation (bias = $2^{w-1}-1$)
    - Size of exponent field determines our representable *range*
    - Outside of representable exponents is *overflow* and *underflow*
  - Mantissa approximates fractional portion of binary point
    - Size of mantissa field determines our representable *precision*
    - Implicit leading 1 (normalized) except in special cases
    - Exceeding length causes *rounding*

# BONUS SLIDES

An example that applies the IEEE Floating Point concepts to a smaller (8-bit) representation scheme.

These slides expand on material covered today, so while you don't need to read these, the information is "fair game."

# Tiny Floating Point Example

| S | E | M |
|---|---|---|
| 1 | 4 | 3 |

- 8-bit Floating Point Representation
  - The sign bit is in the most significant bit (MSB)
  - The next four bits are the exponent, with a bias of $2^{4-1}–1 = 7$
  - The last three bits are the mantissa

- Same general form as IEEE Format
  - Normalized binary scientific point notation
  - Similar special cases for 0, denormalized numbers, NaN, ∞

# Dynamic Range (Positive Only)

| | S E M | Exp | Value | |
|---|---|---|---|---|
| Denormalized numbers | 0 0000 000 | -6 | 0 | |
| | 0 0000 001 | -6 | 1/8*1/64 = 1/512 | closest to zero |
| | 0 0000 010 | -6 | 2/8*1/64 = 2/512 | |
| | … | | | |
| | 0 0000 110 | -6 | 6/8*1/64 = 6/512 | |
| | 0 0000 111 | -6 | 7/8*1/64 = 7/512 | largest denorm |
| Normalized numbers | 0 0001 000 | -6 | 8/8*1/64 = 8/512 | smallest norm |
| | 0 0001 001 | -6 | 9/8*1/64 = 9/512 | |
| | … | | | |
| | 0 0110 110 | -1 | 14/8*1/2 = 14/16 | |
| | 0 0110 111 | -1 | 15/8*1/2 = 15/16 | closest to 1 below |
| | 0 0111 000 | 0 | 8/8*1 = 1 | |
| | 0 0111 001 | 0 | 9/8*1 = 9/8 | closest to 1 above |
| | 0 0111 010 | 0 | 10/8*1 = 10/8 | |
| | … | | | |
| | 0 1110 110 | 7 | 14/8*128 = 224 | |
| | 0 1110 111 | 7 | 15/8*128 = 240 | largest norm |
| | 0 1111 000 | n/a | inf | |

# Special Properties of Encoding

- Floating point zero ($0^+$) exactly the same bits as integer zero
  - All bits = 0
- Can (Almost) Use Unsigned Integer Comparison
  - Must first compare sign bits
  - Must consider $0^- = 0^+ = 0$
  - NaNs problematic
    - Will be greater than any other values
    - What should comparison yield?
  - Otherwise OK
    - Denorm vs. normalized
    - Normalized vs. infinity