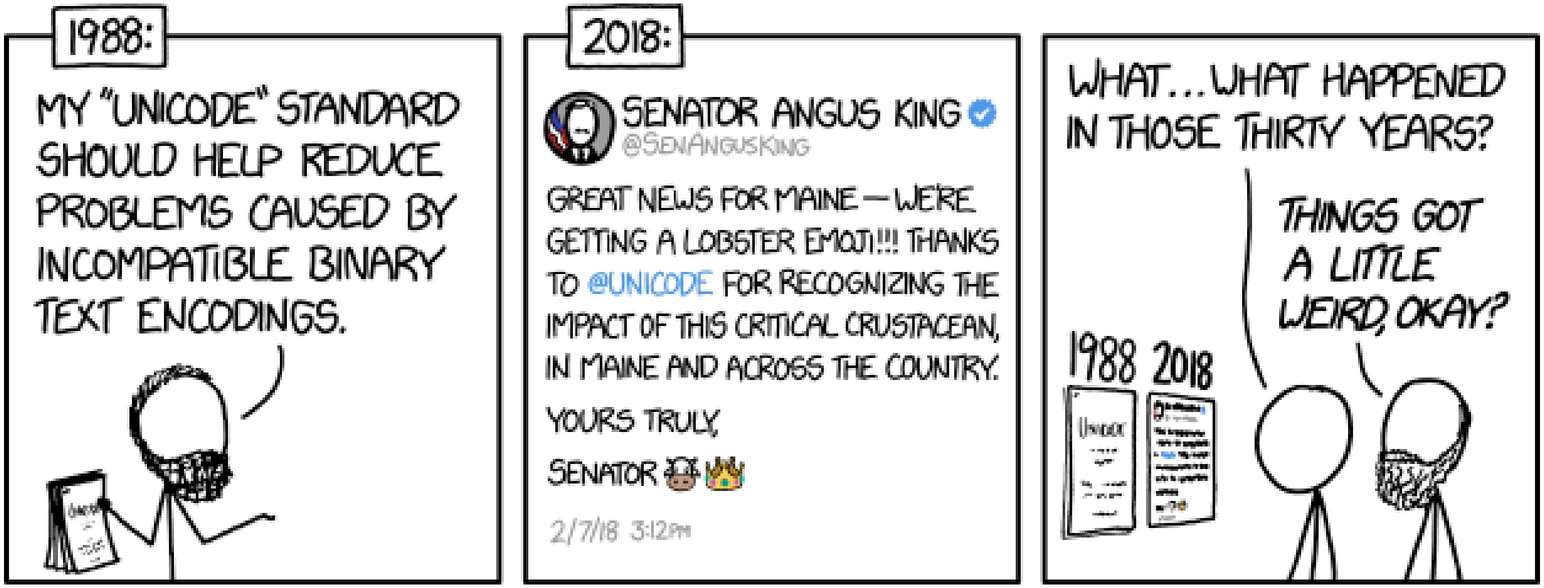
Integers II

CSE 351 Winter 2020

Instructor:	**Teaching Assistants:**		
Ruth Anderson	Jonathan Chen	Justin Johnson	Porter Jones
	Josie Lee	Jeffery Tian	Callum Walker
	Eddy (Tianyi) Zhou		



http://xkcd.com/1953/

Administrivia

- hw4 due 1/17, hw5 due 1/22
- Lab 1a due ~~Friday (1/17)~~ Tues
  - Submit `pointer.c` and `lab1Areflect.txt` to Gradescope
- Lab 1b coming soon, due 1/27
  - Bit puzzles on number representation
  - Can start after today’s lecture, but floating point will be introduced next week
  - Section worksheet from yesterday has helpful examples, too
  - Bonus slides at the end of today’s lecture have relevant examples

Extra Credit

- All labs starting with Lab 1b have extra credit portions
    - These are meant to be fun extensions to the labs

- Extra credit points *don't* affect your lab grades
    - From the course policies: “they will be accumulated over the course and will be used to bump up borderline grades at the end of the quarter.”
    - Make sure you finish the rest of the lab before attempting any extra credit

Integers

- **Binary representation of integers**
  - **Unsigned and signed**
  - **Casting in C**
- Consequences of finite width representations
  - Overflow, sign extension
- Shifting and arithmetic operations

Two's Complement Arithmetic

MSB has negative weight: -8 +4 +2 +1

0b _ _ _ _

- The same addition procedure works for both unsigned and two's complement integers
  - **Simplifies hardware:** only one algorithm for addition
  - **Algorithm:** simple addition, discard the highest carry bit
    - Called modular addition: result is sum *modulo* 2^w

- **4-bit Examples:**

HW	TC
0100	+4
+0011	+3
= 0111	+7 ☑

HW	TC
1100	-4
+0011	+3
= 1111	-1 ☑

HW	TC
0100	+4
+1101	-3
= ~~1~~0001	+1 ☑

Why Does Two's Complement Work?

- For all representable positive integers x, we want:

$$\text{additive inverse}\left\{\begin{array}{r} \textit{bit representation of}\ \ x \\ +\ \textit{bit representation of}\ {-x} \\ \hline 0 \end{array}\right.$$

(ignoring the carry-out bit)

- What are the 8-bit negative encodings for the following?

$$\begin{array}{r} 00000001 \\ +\ \overset{11111111}{????????} \\ \hline \cancel{1}\ 00000000 \end{array} \qquad \begin{array}{r} 00000010 \\ +\ \overset{11111110}{????????} \\ \hline \cancel{1}\ 00000000 \end{array} \qquad \begin{array}{r} \overset{1111111}{11000011} \\ +\ \overset{00111101}{????????} \\ \hline \cancel{1}\ 00000000 \end{array}$$

Why Does Two's Complement Work?

- For all representable positive integers x, we want:

$$\begin{array}{r} \textit{bit representation of} \quad x \\ + \textit{ bit representation of} -x \\ \hline 0 \end{array}$$

(ignoring the carry-out bit)

0b1… 1

$x+(\sim x)=$

$x+(\sim x) = -1$

$x+(\sim x+1) = 0$

$\boxed{-x = \sim x+1}$

- What are the 8-bit negative encodings for the following?

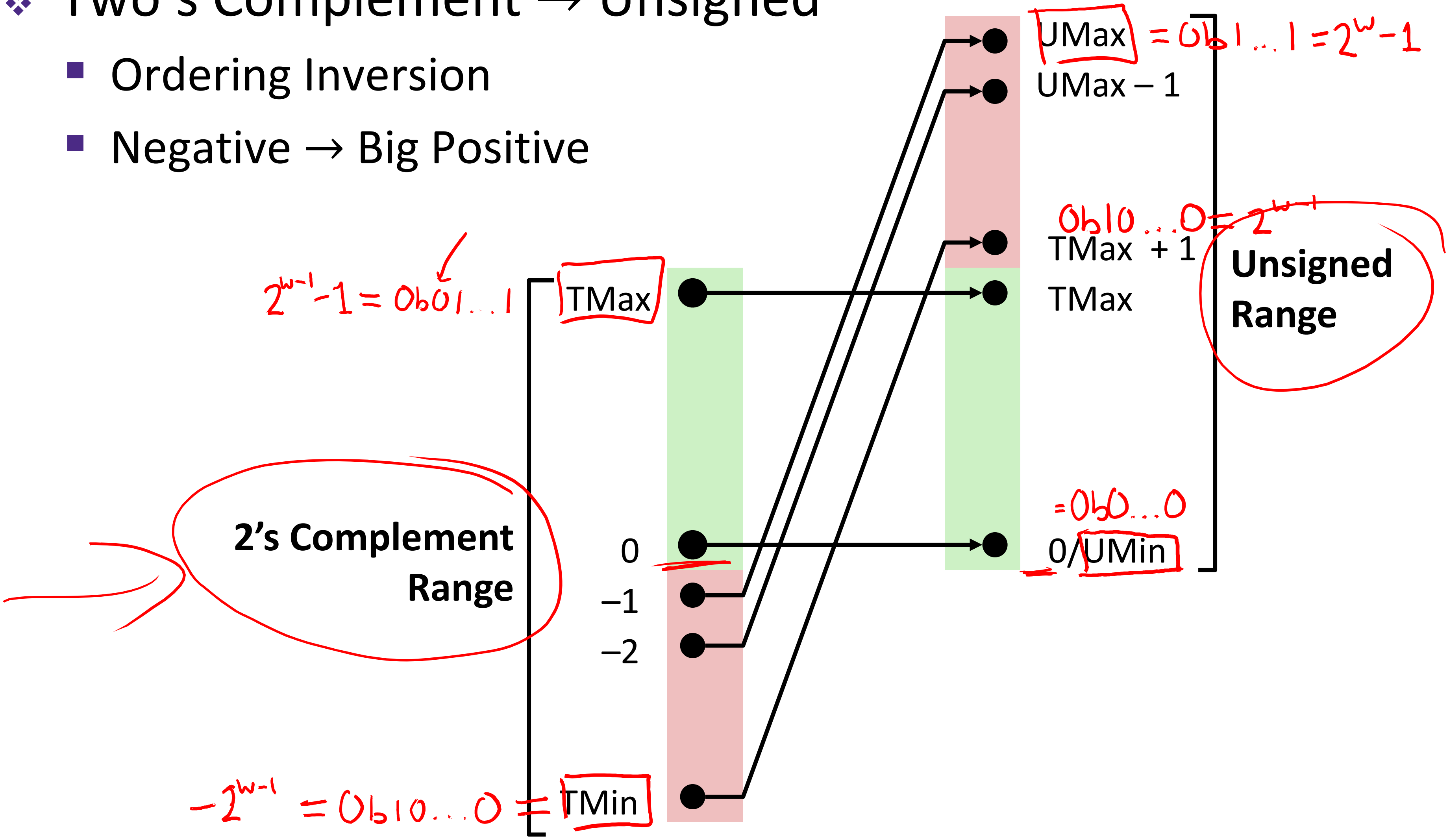
$$\begin{array}{r} 00000001 \\ +\ 11111111 \\ \hline \cancel{1}00000000 \end{array} \qquad \begin{array}{r} 00000010 \\ +\ 11111110 \\ \hline \cancel{1}00000000 \end{array} \qquad \begin{array}{r} 11000011 \\ +\ 00111101 \\ \hline \cancel{1}00000000 \end{array}$$

These are the bitwise complement plus 1!

`-x == ~x + 1`

Signed/Unsigned Conversion Visualized

- Two's Complement → Unsigned
  - Ordering Inversion
  - Negative → Big Positive



Values To Remember

- Unsigned Values
  - UMin = `0b00…0`
    = 0
  - UMax = `0b11…1`
    = $2^w - 1$
- Two's Complement Values
  - TMin = `0b10…0`
    = -2^{w-1}
  - TMax = `0b01…1`
    = $2^{w-1} - 1$
  - -1 = `0b11…1`
- **Example:** Values for $w = 64$

0111

	Decimal	Hex
UMax	18,446,744,073,709,551,615	FF FF FF FF FF FF FF FF
TMax	9,223,372,036,854,775,807	7F FF FF FF FF FF FF FF
TMin	-9,223,372,036,854,775,808	80 00 00 00 00 00 00 00
-1	-1	FF FF FF FF FF FF FF FF
0	0	00 00 00 00 00 00 00 00

In C: Signed vs. Unsigned

- Casting
  - Bits are unchanged, just interpreted differently!
    - **int** tx, ty;
    - **unsigned int** ux, uy;
  - *Explicit* casting
    - tx = (**int**) ux;
    - uy = (**unsigned int**) ty;

(new_type) expression

  - *Implicit* casting can occur during assignments or function calls

cast to target variable/parameter type

    - tx = ux;
    - uy = ty;

(also implicitly occurs with printf format specifiers)

Casting Surprises

!!!

- Integer literals (constants)
  - By default, integer constants are considered *signed* integers
    - Hex constants already have an explicit binary representation
  - Use “U” (or “u”) suffix to explicitly force *unsigned*
    - **Examples:** 0U, 4294967259u

- Expression Evaluation
  - When you mixed unsigned and signed in a single expression, then **signed values are implicitly cast to <u>unsigned</u>**
  - Including comparison operators <, >, ==, <=, >=

(unsigned "dominates")

Casting Surprises

!!!

- 32-bit examples:
  - TMin = -2,147,483,648, TMax = 2,147,483,647

Left Constant	Order	Right Constant	Interpretation
0 0000 0000 0000 0000 0000 0000 0000 0000	==	**0U** 0000 0000 0000 0000 0000 0000 0000 0000	unsigned
-1 1111 1111 1111 1111 1111 1111 1111 1111	<	**0** 0000 0000 0000 0000 0000 0000 0000 0000	signed
-1 1111 1111 1111 1111 1111 1111 1111 1111	>	**0U** 0000 0000 0000 0000 0000 0000 0000 0000	unsigned
2147483647 0111 1111 1111 1111 1111 1111 1111 1111	>	**-2147483648** 1000 0000 0000 0000 0000 0000 0000 0000	signed
2147483647U 0111 1111 1111 1111 1111 1111 1111 1111	<	**-2147483648** 1000 0000 0000 0000 0000 0000 0000 0000	unsigned
-1 1111 1111 1111 1111 1111 1111 1111 1111	>	**-2** 1111 1111 1111 1111 1111 1111 1111 1110	signed
(unsigned) -1 1111 1111 1111 1111 1111 1111 1111 1111	>	**-2** 1111 1111 1111 1111 1111 1111 1111 1110	unsigned
2147483647 0111 1111 1111 1111 1111 1111 1111 1111	<	**2147483648U** 1000 0000 0000 0000 0000 0000 0000 0000	unsigned
2147483647 0111 1111 1111 1111 1111 1111 1111 1111	>	**(int) 2147483648U** 1000 0000 0000 0000 0000 0000 0000 0000	signed

Integers

- Binary representation of integers
  - Unsigned and signed
  - Casting in C
- **Consequences of finite width representations**
  - **Overflow, sign extension**
- Shifting and arithmetic operations

Arithmetic Overflow

Bits	Unsigned	Signed
0000	0 UMin	0
0001	1	1
0010	2	2
0011	3	3
0100	4	4
0101	5	5
0110	6	6
0111	7	7 TMax
1000	8	-8 TMin
1001	9	-7
1010	10	-6
1011	11	-5
1100	12	-4
1101	13	-3
1110	14	-2
1111	15	-1

UMax

- When a calculation produces a result that can't be represented in the current encoding scheme
  - Integer range limited by fixed width UMin - UMax TMin - TMax
  - Can occur in both the positive and negative directions
- C and Java ignore overflow exceptions
  - You end up with a bad value in your program and no warning/indication… oops!

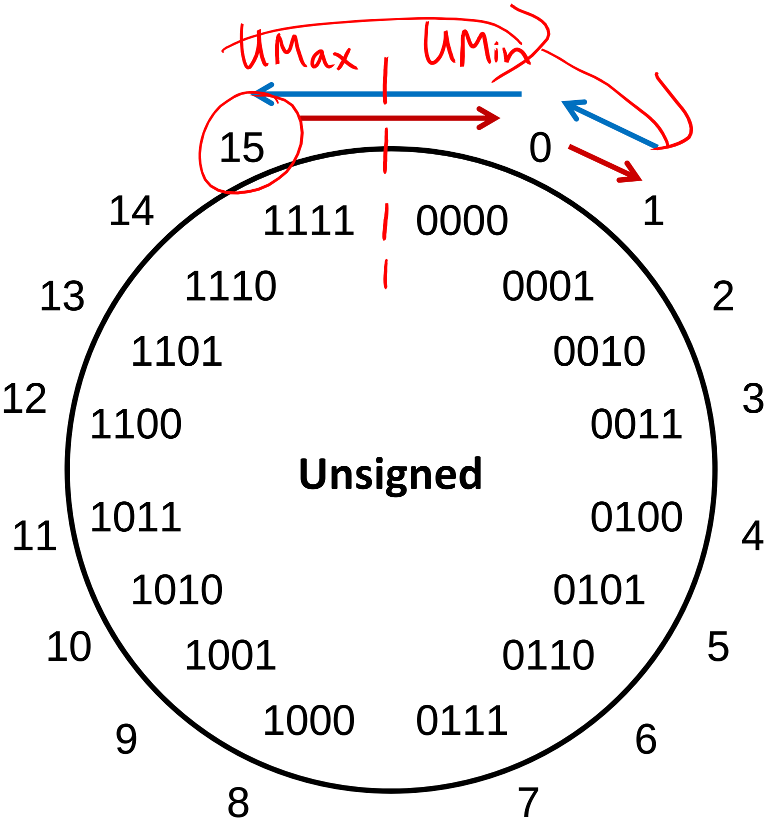
Overflow: Unsigned

- **Addition:** drop carry bit (-2^N)

	1 1
15	1111
+ 2	+ 0010
~~17~~	~~1~~0001
1	

- **Subtraction:** borrow ($+2^N$)

	1 1 ~~2~~ ~~2~~ 2
1	~~1~~0001
- 2	- 0010
~~-1~~	1111
15	



$\pm 2^N$ because of modular arithmetic

$2^4 = 16$

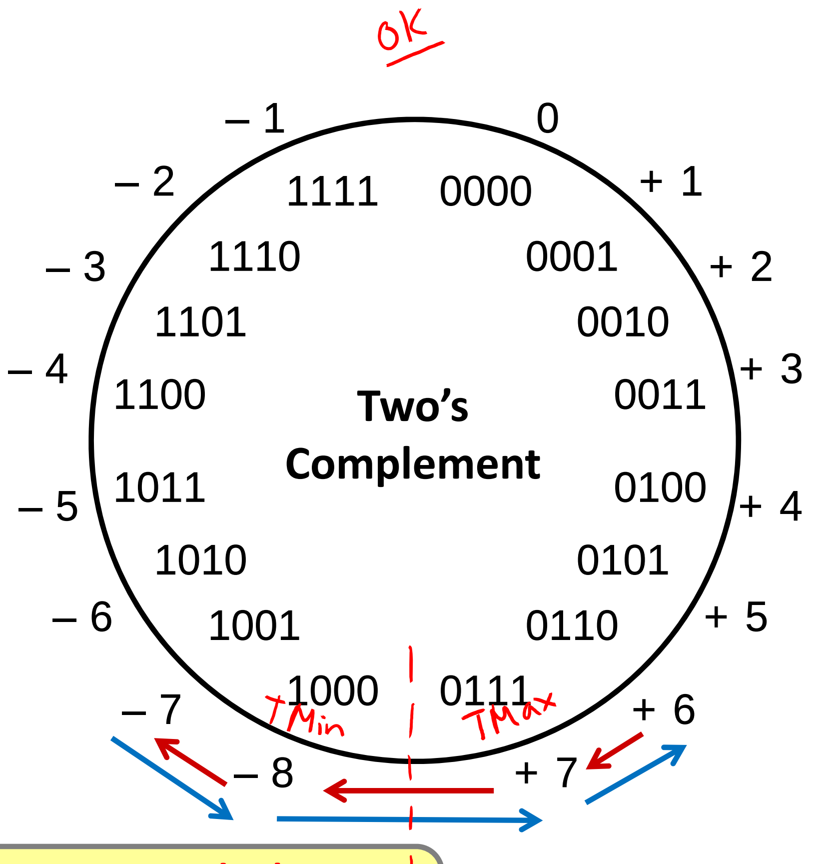
Overflow: Two's Complement

- **Addition:** (+) + (+) = (–) result?

$$\begin{array}{r} 6 \\ +\ \ 3 \\ \hline \cancel{9} \\ -7 \end{array} \qquad \begin{array}{r} 0110 \\ +\ \ 0011 \\ \hline 1001 \end{array}$$

- **Subtraction:** (–) + (–) = (+)?

$$\begin{array}{r} -7 \\ -\ \ 3 \\ \hline \cancel{-10} \\ 6 \end{array} \qquad \begin{array}{r} 1001 \\ -\ \ 0011 \\ \hline 0110 \end{array}$$



For signed: overflow if operands have same sign and result's sign is different

Sign Extension

- What happens if you convert a *signed* integral data type to a larger one?
  - *e.g.* $\overset{\text{1 byte}}{\texttt{char}} \rightarrow \overset{\text{2 bytes}}{\texttt{short}} \rightarrow \overset{\text{4 bytes}}{\texttt{int}} \rightarrow \overset{\text{8 bytes}}{\texttt{long}}$
- **4-bit → 8-bit Example:**
  - Positive Case
    - ✓ Add 0's?

4-bit:	`0010`	=	+2
8-bit:	**`00000010`**	=	**+2**

  - Negative Case?

Polling Question

- Which of the following 8-bit numbers has the same *signed* value as the 4-bit number **0b$\underline{1}$100**?
  - Underlined digit = MSB
  - Vote at http://pollev.com/rea

-8 4 2 1

$-8+4 = -4$

$$\begin{array}{r} -x = 0\,0\,1\,1 \\ +1 \\ \hline 0\,1\,0\,0 = 4 \end{array} \Rightarrow \boxed{x=-4}$$

~~A.~~ **0b $\underline{0}$000 1100** (add zeros) positive!

~~B.~~ **0b $\underline{1}$000 1100** (add leading 1) much too negative: $-2^7+2^3+2^2 = -116$

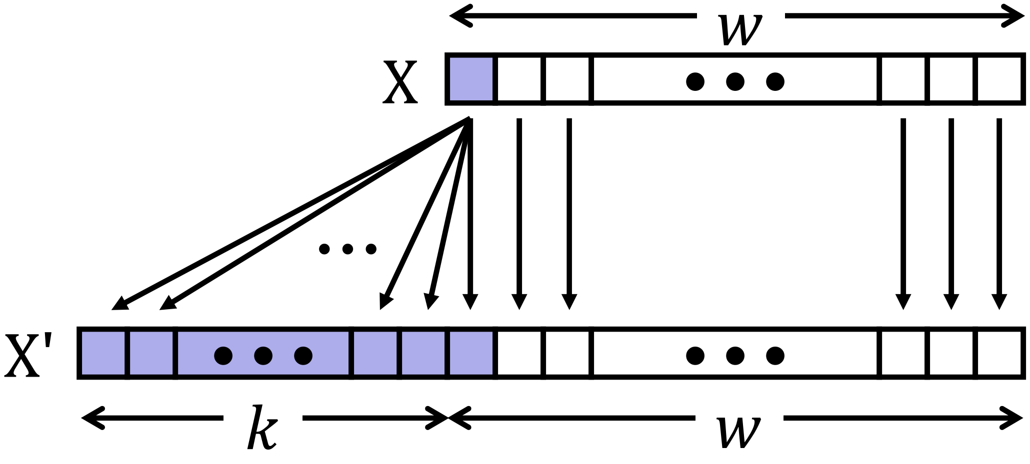
C. **0b $\underline{1}$111 1100** (add ones) correct! $-y = 0b\ 0000\ 0011 + 1 = 4$, $y = -4$

~~D.~~ **0b $\underline{1}$100 1100** (duplicate) $-2^7+2^6+2^3+2^2 = -52$

E. **We're lost…**

Sign Extension

- **Task:** Given a w-bit signed integer X, convert it to w+k-bit signed integer X' *with the same value*
- **Rule:** Add k copies of sign bit
  - Let x_i be the i-th digit of X in binary
  - $\mathrm{X}' = \underbrace{x_{w-1}, \dots, x_{w-1}}_{k \textbf{ copies of MSB}}, \underbrace{x_{w-1}, x_{w-2}, \dots, x_1, x_0}_{\textbf{original } \mathrm{X}}$



Sign Extension Example

- Convert from smaller to larger integral data types
- C automatically performs sign extension
    - Java too

```
short int x =   12345;
int      ix = (int) x;
short int y =  -12345;
int      iy = (int) y;
```

0b 0011

Var	Decimal	Hex	Binary
x	12345	30 39	00110000 00111001
ix	12345	00 00 30 39	00000000 00000000 00110000 00111001
y	-12345	CF C7	11001111 11000111
iy	-12345	FF FF CF C7	11111111 11111111 11001111 11000111

0b 1100

Integers

- Binary representation of integers
  - Unsigned and signed
  - Casting in C
- Consequences of finite width representations
  - Overflow, sign extension
- **Shifting and arithmetic operations**

Shift Operations

- Left shift (`x<<n`) bit vector `x` by `n` positions
  - Throw away (drop) extra bits on left
  - Fill with `0`s on right
- Right shift (`x>>n`) bit-vector `x` by `n` positions
  - Throw away (drop) extra bits on right
  - Logical shift (for unsigned values)
    - Fill with `0`s on left
  - Arithmetic shift (for signed values)
    - Replicate most significant bit on left
    - Maintains sign of `x`

Shift Operations

- Left shift (x<<n)
  - Fill with 0s on right
- Right shift (x>>n)
  - Logical shift (for unsigned values)
    - Fill with 0s on left
  - Arithmetic shift (for signed values)
    - Replicate most significant bit on left

8-bit example:

	x	0010 0010
	x<<3	0001 0000
logical:	x>>2	0000 1000
arithmetic:	x>>2	0000 1000

	x	1010 0010
	x<<3	0001 0000
logical:	x>>2	0010 1000
arithmetic:	x>>2	1110 1000

- Notes:
  - Shifts by n<0 or n≥w (w is bit width of x) are *undefined*

    behavior not guaranteed
  - **In C:** behavior of >> is determined by compiler
    - In gcc / C lang, depends on data type of x (signed/unsigned)

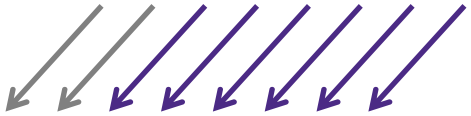
      arithmetic / logical
  - **In Java:** logical shift is >>> and arithmetic shift is >>

Shifting Arithmetic?

- What are the following computing?
  - x>>n
    - $\texttt{0b}\ \overset{4}{\texttt{0100}} \texttt{ >> 1 = 0b } \overset{2}{\texttt{0010}}$
    - $\texttt{0b 0100 >> }\underline{\texttt{2}}\texttt{ = 0b }\underbrace{\texttt{0001}}_{1}$
    - Divide by 2^n
  - x<<n
    - $\texttt{0b}\ \overset{1}{\texttt{0001}} \texttt{ << 1 = 0b } \overset{2}{\texttt{0010}}$
    - $\texttt{0b 0001 }\underline{\texttt{<< 2}}\texttt{ = 0b }\underbrace{\texttt{0100}}_{4}$
    - Multiply by 2^n
- Shifting is faster than general multiply and divide operations

Left Shifting Arithmetic 8-bit Example

- No difference in left shift operation for unsigned and signed numbers (just manipulates bits)
    - Difference comes during interpretation: x*2^n?



		Signed	Unsigned
`x = 25;`	00011001 =	25	25
`L1=x<<2;`	~~00~~01100100 =	100	100
`L2=x<<3;`	~~000~~11001000 =	-56	200
`L3=x<<4;`	~~0001~~10010000 =	-112	144

$$200 - 256 \rightarrow 2^8 = -56$$

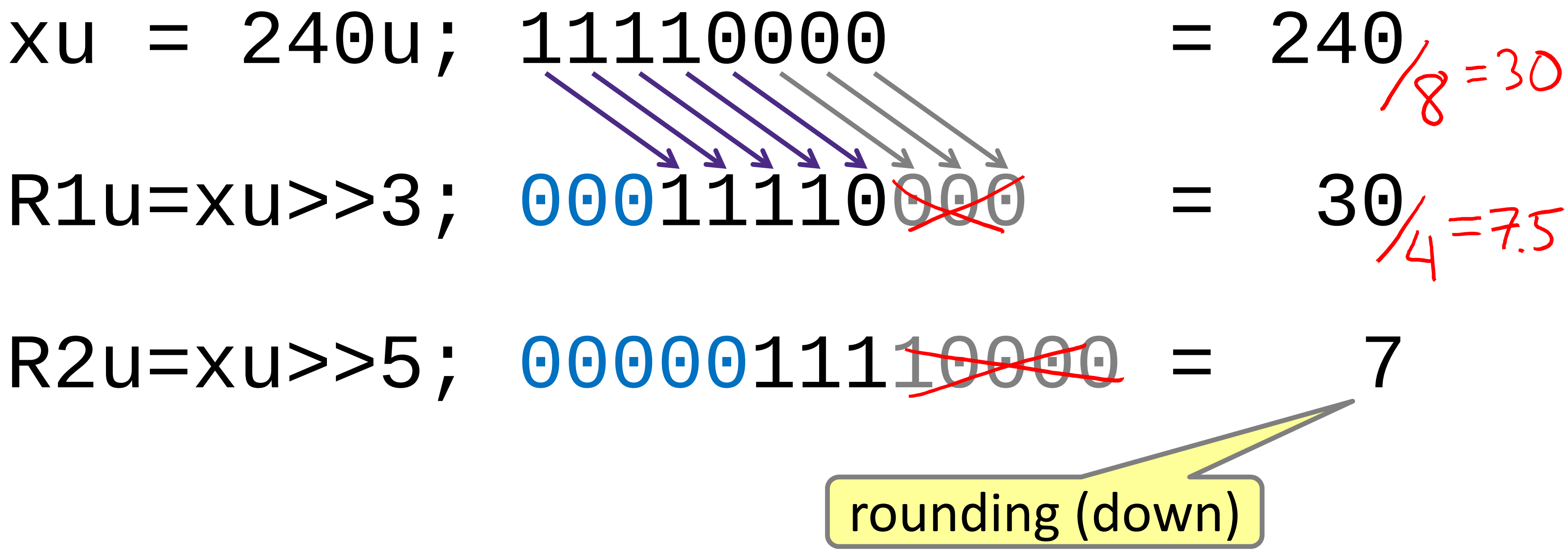
$$400 - 256 \rightarrow 2^8 = 144$$

signed overflow

unsigned overflow

Right Shifting Arithmetic 8-bit Examples

- **Reminder:** C operator >> does *logical* shift on unsigned values and *arithmetic* shift on signed values
  - Logical Shift: `x/2`^{n}?



xu = 240u; 11110000 = 240 /8 = 30

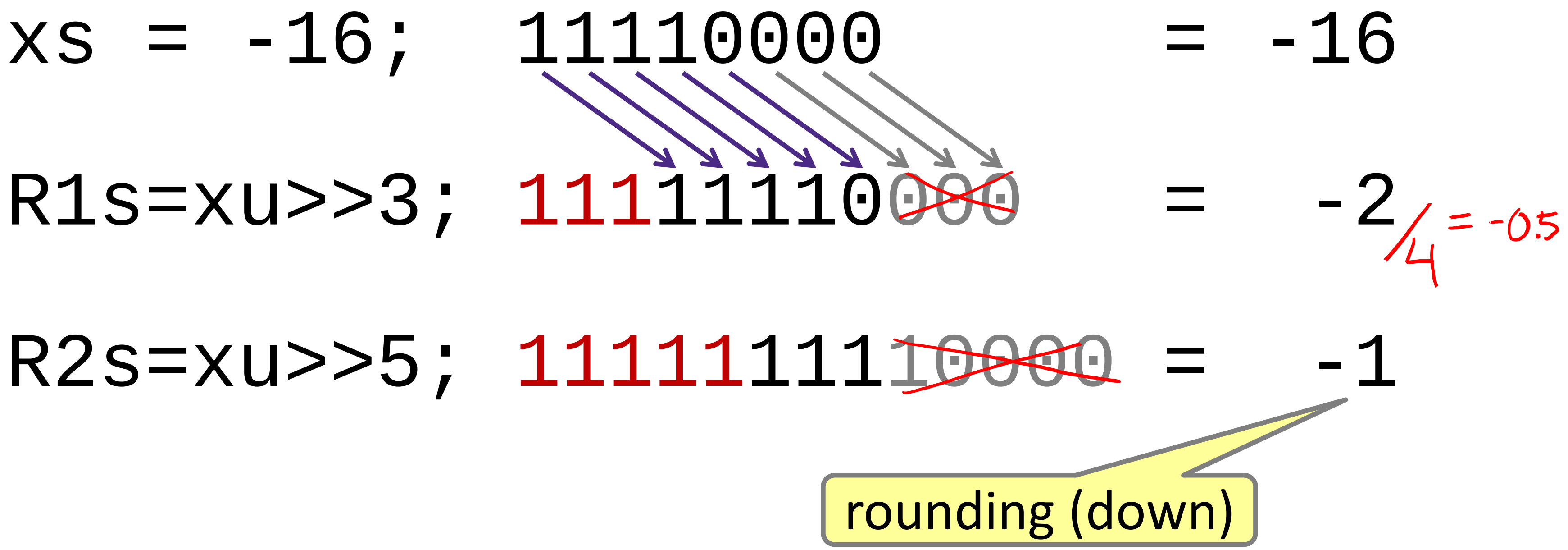
R1u=xu>>3; 00011110~~000~~ = 30 /4 = 7.5

R2u=xu>>5; 00000111~~10000~~ = 7

rounding (down)

Right Shifting Arithmetic 8-bit Examples

- **Reminder:** C operator >> does *logical* shift on unsigned values and *arithmetic* shift on signed values
  - Arithmetic Shift: x/2^n?



xs = -16;   11110000   = -16

R1s=xu>>3; 11111110~~000~~   = -2 /4 = -0.5

R2s=xu>>5; 11111111~~10000~~ = -1

rounding (down)

Practice Question

UMin = 0, UMax = 255
8-bits, so TMin = -128, TMax = 127

For the following expressions, find a value of **`signed char`** `x`, if there exists one, that makes the expression TRUE. Compare with your neighbor(s)!

- Assume we are using 8-bit arithmetic:

	Example:	All solutions:
`x` $\overset{\text{unsigned}}{==}$ `(`**`unsigned char`**`) x`	x= 0	works for all x
`x` $\overset{\text{unsigned}}{>=}$ $\underset{\text{0b1000 0000}}{\texttt{128U}}$	x= -1	any x < 0
`x != (x>>2)<<2`	x = 3	any x where lowest two bits are not 0b00
`x == -x` • Hint: there are two solutions	x=0	① x=0b0...0 = 0 ② x=0b10...0 = -128
`(x < 128U) && (x > 0x3F)`		any x where upper two bits are exactly 0b01

Summary

- Sign and unsigned variables in C
  - Bit pattern remains the same, just *interpreted* differently
  - Strange things can happen with our arithmetic when we convert/cast between sign and unsigned numbers
    - Type of variables affects behavior of operators (shifting, comparison)
- We can only represent so many numbers in *w* bits
  - When we exceed the limits, *arithmetic overflow* occurs
  - *Sign extension* tries to preserve value when expanding
- Shifting is a useful bitwise operator
  - Right shifting can be arithmetic (sign) or logical (0)
  - Can be used in multiplication with constant or bit masking

BONUS SLIDES

Some examples of using shift operators in combination with bitmasks, which you may find helpful for Lab 1.

- Extract the 2nd most significant byte of an `int`
- Extract the sign bit of a signed `int`
- Conditionals as Boolean expressions

Using Shifts and Masks

- Extract the 2nd most significant *byte* of an `int`:
  - First shift, then mask: `(x>>16) & 0xFF`

`x`	`00000001 00000010 00000011 00000100`
`x>>16`	`00000000 00000000 00000001 00000010`
`0xFF`	`00000000 00000000 00000000 11111111`
`(x>>16) & 0xFF`	`00000000 00000000 00000000 00000010`

  - Or first mask, then shift: `(x & 0xFF0000)>>16`

`x`	`00000001 00000010 00000011 00000100`
`0xFF0000`	`00000000 11111111 00000000 00000000`
`x & 0xFF0000`	`00000000 00000010 00000000 00000000`
`(x&0xFF0000)>>16`	`00000000 00000000 00000000 00000010`

Using Shifts and Masks

- Extract the *sign bit* of a signed `int`:
  - First shift, then mask: `(x>>31) & 0x1`
    - Assuming arithmetic shift here, but this works in either case
    - Need mask to clear 1s possibly shifted in

`x`	`00000001 00000010 00000011 00000100`
`x>>31`	`00000000 00000000 00000000 00000000`
`0x1`	`00000000 00000000 00000000 00000001`
`(x>>31) & 0x1`	`00000000 00000000 00000000 00000000`

`x`	`10000001 00000010 00000011 00000100`
`x>>31`	`11111111 11111111 11111111 11111111`
`0x1`	`00000000 00000000 00000000 00000001`
`(x>>31) & 0x1`	`00000000 00000000 00000000 00000001`

Using Shifts and Masks

- Conditionals as Boolean expressions
  - For **`int`** `x`, what does `(x<<31)>>31` do?

`x=!!123`	`00000000 00000000 00000000 00000001`
`x<<31`	`10000000 00000000 00000000 00000000`
`(x<<31)>>31`	`11111111 11111111 11111111 11111111`
`!x`	`00000000 00000000 00000000 00000000`
`!x<<31`	`00000000 00000000 00000000 00000000`
`(!x<<31)>>31`	`00000000 00000000 00000000 00000000`

  - Can use in place of conditional:
    - In C: `if(x) {a=y;} else {a=z;}` equivalent to `a=x?y:z;`
    - `a=(((x<<31)>>31)&y) | (((!x<<31)>>31)&z);`