Floating Point II

CSE 351 Spring 2020

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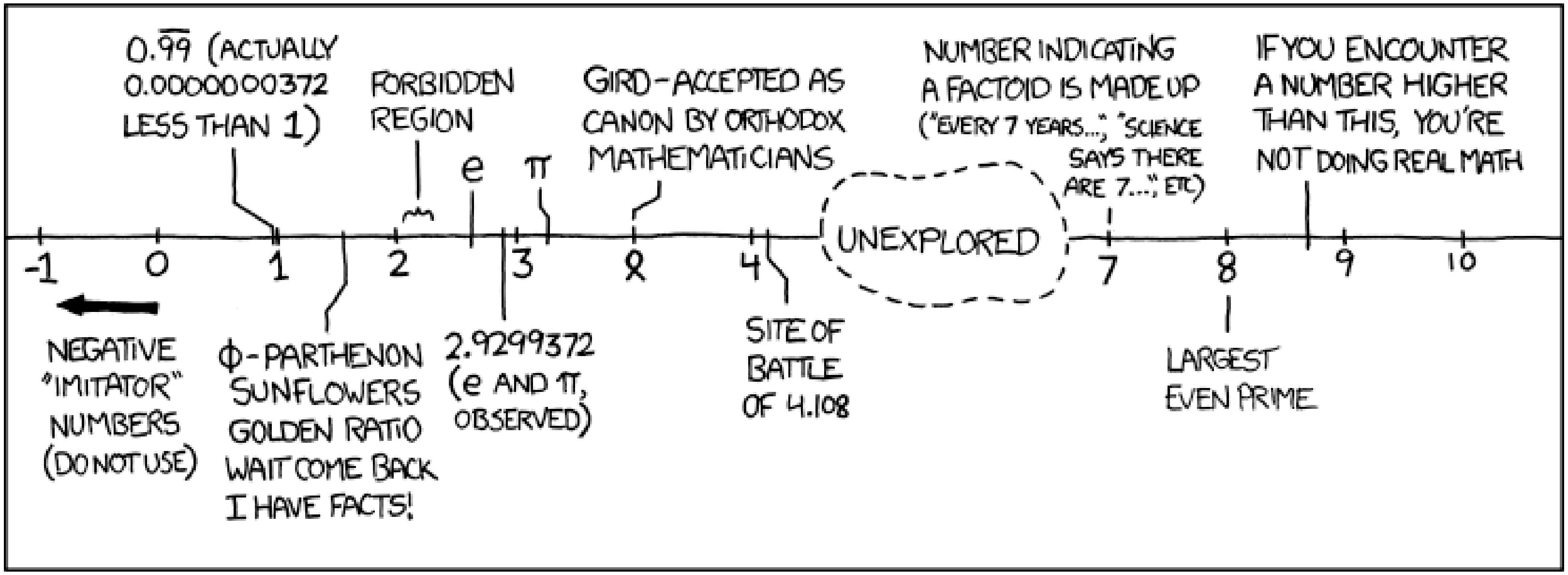


http://xkcd.com/899/

Administrivia

- Lab 1a due TONIGHT (4/13) at 11:59 pm
  - Submit `pointer.c` and `lab1Areflect.txt`
- hw6 due Wednesday – 11am
- Lab 1b due Monday (4/20)
  - Submit `bits.c` and `lab1Breflect.txt`
- **You must log on with your @uw google account to access!!**
  - **Google doc** for 11:30 Lecture: https://tinyurl.com/351-04-13A
  - **Google doc** for 2:30 Lecture: https://tinyurl.com/351-04-13B
- Week 2 Feedback Survey
  - https://catalyst.uw.edu/webq/survey/rea2000/388285

Other Special Cases

- E = 0xFF, M = 0: ± ∞
  - *e.g.* division by 0
  - Still work in comparisons!

all ones

- E = 0xFF, M ≠ 0: Not a Number (NaN)
  - *e.g.* square root of negative number, 0/0, ∞–∞
  - NaN propagates through computations
  - Value of M can be useful in debugging (tells you cause of NaN)
- New largest value (besides ∞)?
  - E = 0xFF has now been taken!
  - E = 0xFE has largest: $1.\overbrace{1\ldots1}^{23\ \text{ones}}{}_2\times 2^{127} = 2^{128} - 2^{104}$

    ↳ 254 - bias

Floating Point Encoding Summary

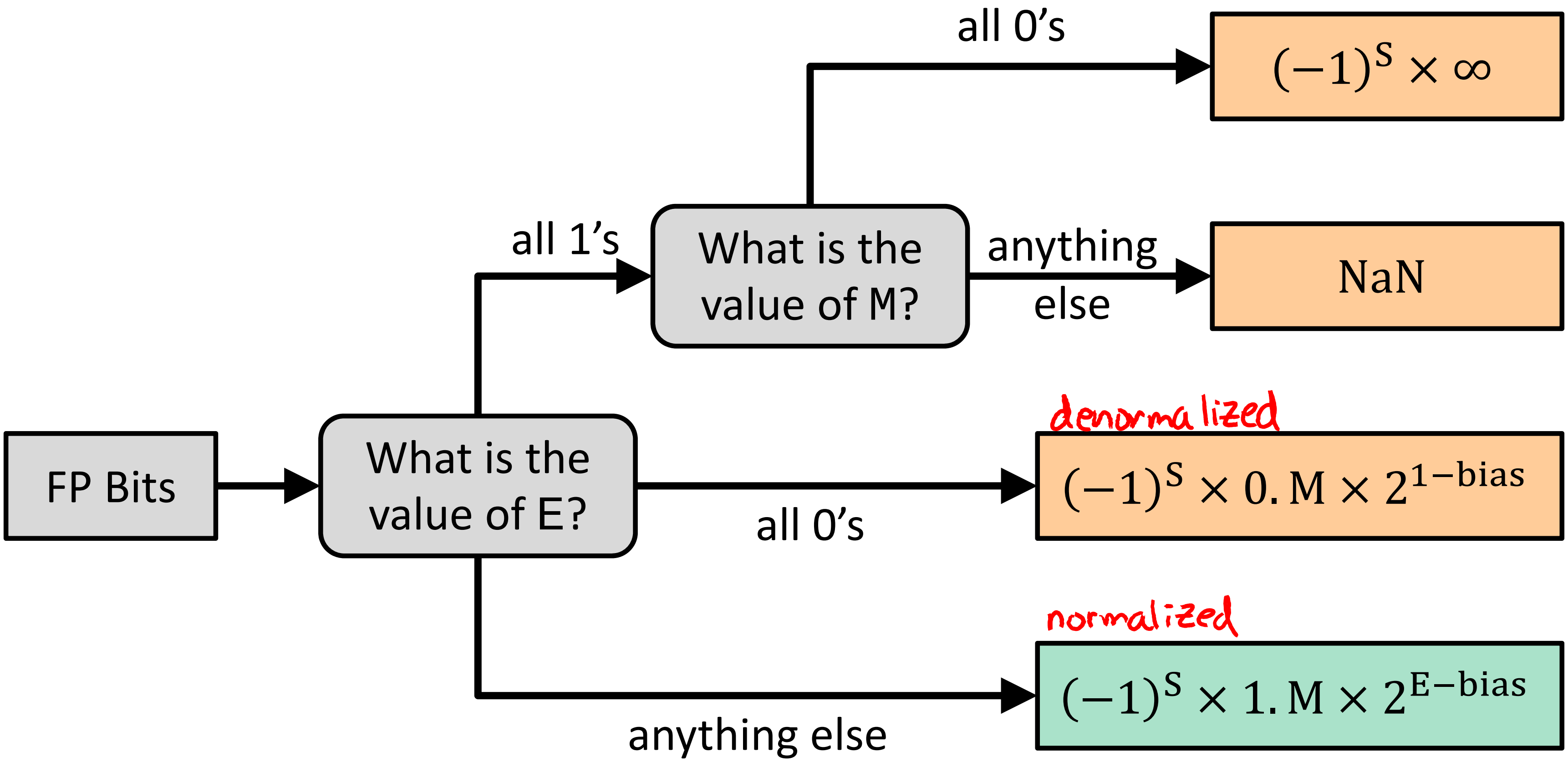
smallest E (all 0's)

everything else

largest E (all 1's)

E	M	Meaning
0x00	0	± 0
0x00	non-zero	± denorm num
0x01 – 0xFE	anything	± norm num
0xFF	0	± ∞
0xFF	non-zero	NaN

Floating Point Interpretation Flow Chart



FP Bits

What is the value of E?

all 1’s

What is the value of M?

all 0’s

$(-1)^S \times \infty$

anything else

NaN

denormalized

all 0’s

$(-1)^S \times 0.M \times 2^{1-\text{bias}}$

normalized

anything else

$(-1)^S \times 1.M \times 2^{E-\text{bias}}$

🟧 = special case

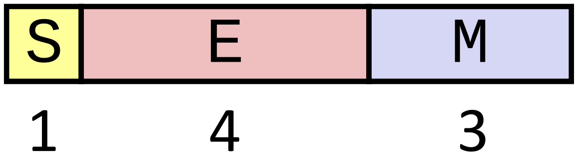
Floating point topics

- Fractional binary numbers
- IEEE floating-point standard
- **Floating-point operations and rounding**
- Floating-point in C

- There are many more details that we won't cover
  - It's a 58-page standard…

Tiny Floating Point Representation

- We will use the following **8-bit** floating point representation to illustrate some key points:



- Assume that it has the same properties as IEEE floating point:
  - bias = $2^{w-1}-1 = 2^{4-1}-1 = 7$
  - encoding of –0 = 0b 1 0000 000 = 0x80
  - encoding of +∞ = 0b 0 1111 000 = 0x78
  - encoding of the largest (+) normalized # = 0b 0 1110 111 = 0x77
  - encoding of the smallest (+) normalized # = 0b 0 0001 000 = 0x08

$1.111_2 \times 2^{14-7}$

$1.0_2 \times 2^{1-7}$

Distribution of Values

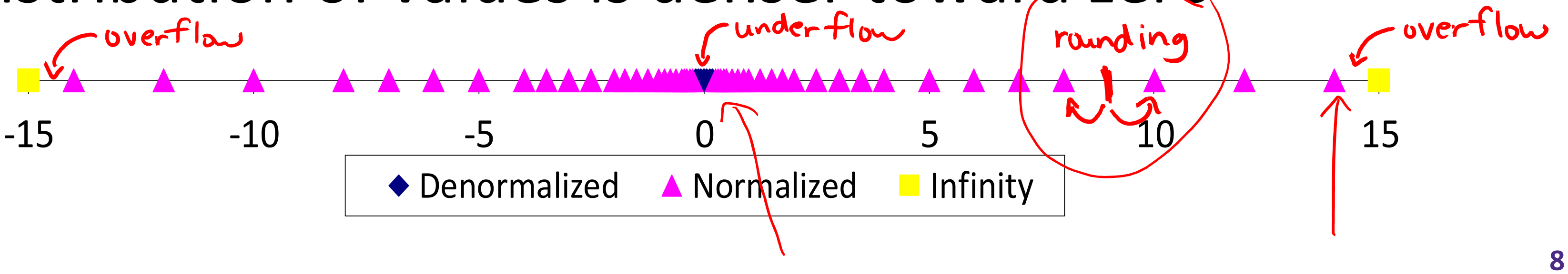
- What ranges are NOT representable?
  - Between largest norm and infinity **Overflow** (Exp too large)
  - Between zero and smallest denorm **Underflow** (Exp too small)
  - Between norm numbers? **Rounding**
- Given a FP number, what's the bit pattern of the next largest representable number?
  - What is this "step" when Exp = 0? 2^{-23}
  - What is this "step" when Exp = 100? 2^{77}

if M=0b0...00, then $2^{Exp} \times 1.0$

if M=0b0...01, then $2^{Exp} \times (1+2^{-23})$

diff = 2^{Exp-23}

- Distribution of values is denser toward zero



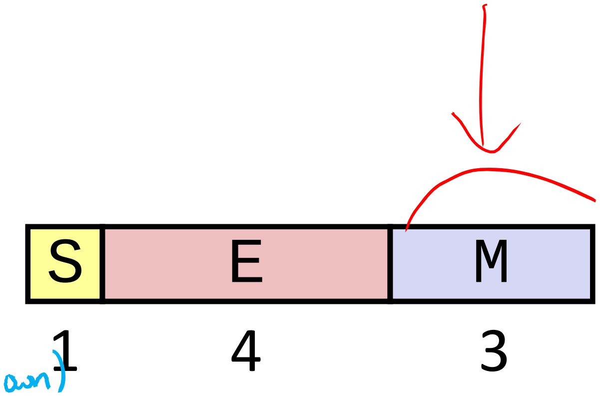
Floating Point Rounding

This is extra (non-testable) material

- The IEEE 754 standard actually specifies different rounding modes:
  - Round to nearest, ties to nearest even digit
  - Round toward +∞ (round up)
  - Round toward −∞ (round down)
  - Round toward 0 (truncation)

- In our tiny example:
  - Man = 1.001/01 rounded to M = 0b001 (down) — < half
  - Man = 1.001/11 rounded to M = 0b010 (up) — > half
  - Man = 1.001/10 rounded to M = 0b010 (up) — == half — even digit

Man = 1.000/10 rounded to M = 0b000 (down)



Floating Point Operations: Basic Idea

$$\text{Value} = (-1)^{S} \times \text{Mantissa} \times 2^{\text{Exponent}}$$

- $x +_f y = \text{Round}(x + y)$
- $x *_f y = \text{Round}(x * y)$

- Basic idea for floating point operations:
  - First, compute the exact result
  - Then *round* the result to make it fit into the specified precision (width of M)
    - Possibly over/underflow if exponent outside of range

Mathematical Properties of FP Operations

- **Overflow** yields ±∞ and **underflow** yields 0
- Floats with value ±∞ and NaN can be used in operations
  - Result usually still ±∞ or NaN, but not always intuitive
- Floating point operations do not work like real math, due to **rounding**
  - Not associative:

    10^{100} 10^{100} 10^{100} 10^{100}

    `(3.14+1e100)–1e100 != 3.14+(1e100–1e100)`

    0 **3.14**

  - Not distributive: `100*(0.1+0.2) != 100*0.1+100*0.2`

    30.000000000000003553 **30**

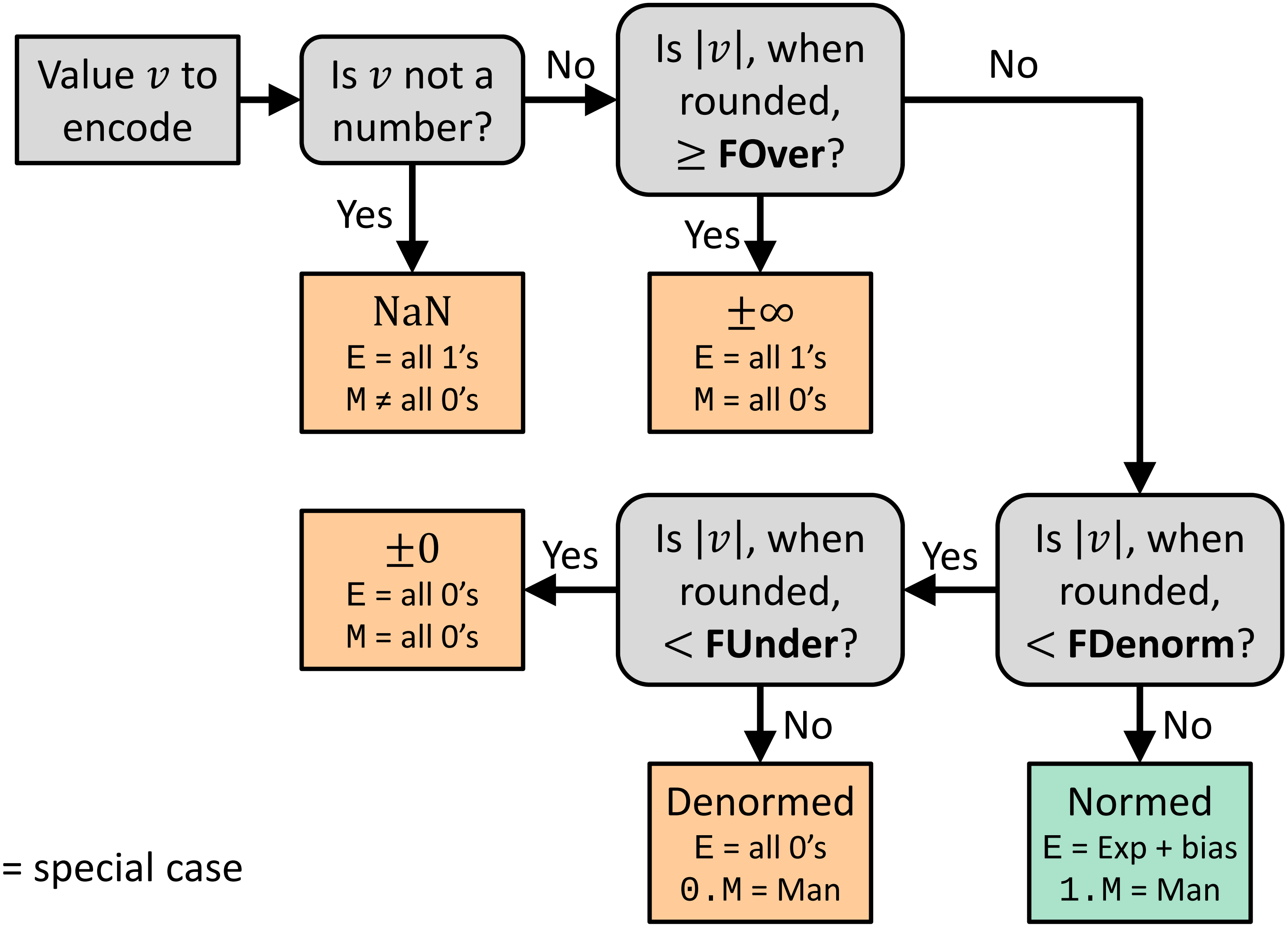
  - Not cumulative
    - Repeatedly adding a very small number to a large one may do nothing

Aside: Limits of Interest

This is extra (non-testable) material

- The following thresholds will help give you a sense of when certain outcomes come into play, but don't worry about the specifics:
  - **FOver** = $2^{\text{bias}+1}$ = 2^8
    - This is just larger than the largest representable normalized number
  - **FDenorm** = $2^{1-\text{bias}}$ = 2^{-6}
    - This is the smallest representable normalized number
  - **FUnder** = $2^{1-\text{bias}-m}$ = 2^{-9}
    - m is the width of the mantissa field
    - This is the smallest representable denormalized number

Floating Point Encoding Flow Chart



Value v to encode

Is v not a number?

Yes

NaN
E = all 1's
M ≠ all 0's

No

Is $|v|$, when rounded, $\geq$ **FOver**?

Yes

$\pm\infty$
E = all 1's
M = all 0's

No

Is $|v|$, when rounded, $<$ **FDenorm**?

No

Normed
E = Exp + bias
`1.M` = Man

Yes

Is $|v|$, when rounded, $<$ **FUnder**?

No

Denormed
E = all 0's
`0.M` = Man

Yes

± 0
E = all 0's
M = all 0's

🟧 = special case

Example Question [FP II - a]

- Using our **8-bit** representation, what value gets stored when we try to encode **384** = $2^8 + 2^7$? $= 2^8 (1 + 2^{-1})$

$= 2^8 \times 1.1_2$

S	E	M
1	4	3

$S = 0$

$E = Exp + bias$

$= 8 + 7 = 15$

$= 0b\underline{1}\underline{1}\underline{1}\underline{1}$

this falls outside of the normalized exponent range!

  - No voting

A. + 256

B. + 384

[✓ C. + ∞]

D. NaN

E. We're lost…

this number is too large, so we store

$\boxed{+\infty \longleftrightarrow 0b0\ \underline{1}\underline{1}\underline{1}\underline{1}\ 000}$

instead

Polling Question [FP II - b]

- Using our **8-bit** representation, what value gets stored when we try to encode **2.625** = $2^1 + 2^{-1} + 2^{-3}$?

S	E	M
1	4	3

  - Vote at http://pollev.com/rea

A. + 2.5

B. + 2.625

C. + 2.75

D. + 3.25

E. We're lost…

$= 2^1 (1 + 2^{-2} + 2^{-4})$

$= 2^1 \times 1.\underline{0101}_2$

$S = 0$

$E = \text{Exp} + \text{bias}$

$= 1 + 7 = 8$

$= 0b\ 1000$

$M = 0b\ \underline{010}/1$

can only store 3 bits!

stored as: $\boxed{0b\ 0\ 1000\ 010 = 2.5}$

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- **Floating-point in C**

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Floating Point in C

!!!

- Two common levels of precision:

`float`	`1.0f`	single precision (32-bit)
`double`	`1.0`	double precision (64-bit)

- `#include <math.h>` to get `INFINITY` and `NAN` constants

<float.h> for additional constants

- Equality (==) comparisons between floating point numbers are tricky, and often return unexpected results, so just avoid them!

instead use $\text{abs}(f1 - f2) < 2^{-20}$

some arbitrary threshold

Floating Point Conversions in C

!!!

- Casting between `int`, `float`, and `double` **changes** the bit representation
  - `int` → `float`
    - May be rounded (not enough bits in mantissa: 23)
    - Overflow impossible
  - `int` or `float` → `double`
    - Exact conversion (all 32-bit `int`s representable)
  - `long` → `double`
    - Depends on word size (32-bit is exact, 64-bit may be rounded)
  - `double` or `float` → `int`
    - Truncates fractional part (rounded toward zero)
    - "Not defined" when out of range or NaN: generally sets to `Tmin` (even if the value is a very big positive)

Polling Question [FP II - c]

- We execute the following code in C. How many bytes are the same (value and position) between `i` and `f`?
  - Vote at http://pollev.com/rea

```c
int i = 384;  // 2^8 + 2^7
float f = (float) i;
```

8 7 6 5 4 3 2 1 0

= 0b 1/1 000/0000

$= 1.1_2 \times 2^8$

S = 0

E = 8 + 127 = 135

= 0b1000 0111

M = 0b 10...0

0b 0 1000/ 0111/ 100/...0

[✓ **A.** 0 bytes]

B. 1 byte

C. 2 bytes

D. 3 bytes

E. We're lost…

i stored as 0x 00 00 01 80

f stored as 0x 43 C0 00 00

Floating Point and the Programmer

1.0 × 2^0 → S=0, E = 0111 1111, M = 0 … 0

f1 = 0b 0/011 1111 1/000 0000 0000 0000 0000 0000 = 0x3F800000

```
#include <stdio.h>

int main(int argc, char* argv[]) {
  float f1 = 1.0;
  float f2 = 0.0;
  int i;
  for (i = 0; i < 10; i++)
    f2 += 1.0/10.0;
```

specify float constant

f2 should == 10 × $\frac{1}{10}$ = 1

```
  printf("0x%08x  0x%08x\n", *(int*)&f1, *(int*)&f2);
  printf("f1 = %10.9f\n", f1);
  printf("f2 = %10.9f\n\n", f2);

  f1 = 1E30;
  f2 = 1E-30;
  float f3 = f1 + f2;
  printf("f1 == f3? %s\n", f1 == f3 ? "yes" : "no" );
```

10^{30}

10^{-30}

$10^{30} == 10^{30} + 10^{-30}$

```
  return 0;
}
```

```
$ ./a.out
0x3f800000  0x3f800001
f1 = 1.000000000
f2 = 1.000000119

f1 == f3? yes
```

f2

see float.c

Floating Point Summary

- Floats also suffer from the fixed number of bits available to represent them
  - Can get overflow/underflow
  - "Gaps" produced in representable numbers means we can lose precision, unlike `ints`
    - Some "simple fractions" have no exact representation (*e.g.* 0.2)
    - "Every operation gets a slightly wrong result"
- Floating point arithmetic not associative or distributive
  - Mathematically equivalent ways of writing an expression may compute different results
- Never test floating point values for equality!
- Careful when converting between `ints` and `floats`!

Number Representation Really Matters

- **1991:** Patriot missile targeting error
  - clock skew due to conversion from integer to floating point
- **1996:** Ariane 5 rocket exploded ($1 billion)
  - overflow converting 64-bit floating point to 16-bit integer
- **2000:** Y2K problem
  - limited (decimal) representation: overflow, wrap-around
- **2038:** Unix epoch rollover
  - Unix epoch = seconds since 12am, January 1, 1970
  - signed 32-bit integer representation rolls over to TMin in 2038
- **Other related bugs:**
  - 1982: Vancouver Stock Exchange 10% error in less than 2 years
  - 1994: Intel Pentium FDIV (floating point division) HW bug ($475 million)
  - 1997: USS Yorktown “smart” warship stranded: divide by zero
  - 1998: Mars Climate Orbiter crashed: unit mismatch ($193 million)

Summary

E	M	Meaning
0x00	0	± 0
0x00	non-zero	± denorm num
0x01 – 0xFE	anything	± norm num
0xFF	0	± ∞
0xFF	non-zero	NaN

- Floating point encoding has many limitations
  - Overflow, underflow, rounding
  - Rounding is a HUGE issue due to limited mantissa bits and gaps that are scaled by the value of the exponent
  - Floating point arithmetic is NOT associative or distributive
- Converting between integral and floating point data types *does* change the bits