Floating Point I
CSE 351 Spring 2020

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http://xkcd.com/571/
Administrivia

- hw5 due Monday – 11am

- Lab 1a due Monday (4/13) at 11:59 pm
  - Submit `pointer.c` and `lab1Areflect.txt`

- hw6 due Wednesday – 11am

- Lab 1b due Monday (4/20)
  - Submit `bits.c` and `lab1Breflect.txt`
Questions During Lecture – An Experiment

- Asking too many questions in **chat window** during lecture is very distracting to some students

- **While I am lecturing**
  - If you need to ask a question about content, please use the Google doc
  - Staff will answer your questions in the Google doc during lecture
  - We will reserve the **chat window** for short logistical questions (e.g. “which slide deck?”, “We can’t see your screen”)

- **When I explicitly pause to take questions** - Use **chat window** to type your question, or “**raise hand**” and I will call on you to speak

- We will not be saving the **chat window**. We WILL be saving, and anonymizing the Google doc and sharing with the class.

- **You must log on with your @uw google account to access!!**
  - Google doc for 11:30 Lecture: [https://tinyurl.com/351-04-10A](https://tinyurl.com/351-04-10A)
  - Google doc for 2:30 Lecture: [https://tinyurl.com/351-04-10B](https://tinyurl.com/351-04-10B)
Groups & Feedback

❖ Groups

 Some classes are allowing students to pick people they would like to be in breakout groups with and stick with those same breakout groups for the rest of the quarter.
 We are up for trying this.
 Tell us what you think on this survey!

❖ Week 2 Feedback Survey

 https://catalyst.uw.edu/webq/survey/rea2000/388285
Aside: Unsigned Multiplication in C

Operands:
- \( w \) bits

True Product:
- \( 2w \) bits
  \[ u \cdot v \]

Discard \( w \) bits:
- \( w \) bits

- Standard Multiplication Function
  - Ignores high order \( w \) bits
- Implements Modular Arithmetic
  - \( \text{UMult}_w(u, v) = u \cdot v \mod 2^w \)
Aside: Multiplication with shift and add

- **Operation** $u << k$ gives $u \times 2^k$
  - Both signed and unsigned

<table>
<thead>
<tr>
<th>Operands: $w$ bits</th>
<th>$u$ ( \times 2^k )</th>
<th>True Product: $w + k$ bits</th>
<th>$u \times 2^k$</th>
<th>Discard $k$ bits: $w$ bits</th>
</tr>
</thead>
<tbody>
<tr>
<td>$u$</td>
<td>( \underbrace{\cdots}_{k \text{ bits}} ) ( \underbrace{0 \cdots 0 1 0 \cdots 0 0}_w )</td>
<td>$u \times 2^k$ ( \underbrace{0 \cdots 0 1 0 \cdots 0 0}_w )</td>
<td>$u \times 2^k$ ( \underbrace{0 \cdots 0 1 0 \cdots 0 0}_w )</td>
<td></td>
</tr>
</tbody>
</table>

- **Examples:**
  - $u << 3$ \(== u \times 8\)
  - $u << 5 - u << 3$ \(== u \times 24\)

- Most machines shift and add faster than multiply
  - *Compiler generates this code automatically*
Number Representation Revisited

What can we represent so far?
- Signed and Unsigned Integers
- Characters (ASCII)
- Addresses

How do we encode the following:
- Real numbers (e.g. 3.14159)
- Very large numbers (e.g. $6.02 \times 10^{23}$)
- Very small numbers (e.g. $6.626 \times 10^{-34}$)
- Special numbers (e.g. $\infty$, NaN)
Floating Point Topics

- Fractional binary numbers
- IEEE floating-point standard
- Floating-point operations and rounding
- Floating-point in C

- There are many more details that we won’t cover
  - It’s a 58-page standard...
Floating Point Summary

- Floats also suffer from the fixed number of bits available to represent them
  - Can get overflow/underflow, just like ints
  - “Gaps” produced in representable numbers means we can lose precision, unlike ints
    - Some “simple fractions” have no exact representation (e.g. 0.2)
    - “Every operation gets a slightly wrong result”

- Floating point arithmetic not associative or distributive
  - Mathematically equivalent ways of writing an expression may compute different results

- Never test floating point values for equality!
- Careful when converting between ints and floats!
Representation of Fractions

- “Binary Point,” like decimal point, signifies boundary between integer and fractional parts:

Example 6-bit representation:

\[
\begin{array}{cccc}
  2^1 & 2^0 & 2^{-1} & 2^{-2} & 2^{-3} & 2^{-4} \\
  xx & . & yyyy & & & \\
\end{array}
\]

- **Example:** \(10.1010_2 = 1 \times 2^1 + 1 \times 2^{-1} + 1 \times 2^{-3} = 2.625_{10}\)
Representation of Fractions

- “Binary Point,” like decimal point, signifies boundary between integer and fractional parts:

Example 6-bit representation:

\[ xx \cdot yyyyy \]

- In this 6-bit representation:
  - What is the encoding and value of the smallest (most negative) number?
  - What is the encoding and value of the largest (most positive) number?
  - What is the smallest number greater than 2 that we can represent?
Fractional Binary Numbers

- Bits to right of “binary point” represent fractional powers of 2
- Represents rational number: \[ \sum_{k=-j}^{i} b_k \cdot 2^k \]
Fractional Binary Numbers

- **Value**
  - 5 and 3/4: \(101.11_2\)
  - 2 and 7/8: \(10.111_2\)
  - 47/64: \(0.101111_2\)

- **Observations**
  - Shift left = multiply by power of 2
  - Shift right = divide by power of 2
  - Numbers of the form \(0.111111\ldots_2\) are just below 1.0
    - \(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots + \frac{1}{2^i} + \ldots \rightarrow 1.0\)
    - Use notation \(1.0 - \varepsilon\)
Limits of Representation

- Limitations:
  - Even given an arbitrary number of bits, can only **exactly** represent numbers of the form \( x \times 2^y \) (y can be negative)
  - Other rational numbers have repeating bit representations

<table>
<thead>
<tr>
<th>Value</th>
<th>Binary Representation:</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 1/3 ) = 0.333333... (<em>{10}) = 0.01010101[01]... (</em>{2})</td>
<td></td>
</tr>
<tr>
<td>( 1/5 ) = 0.001100110011[0011]... (_{2})</td>
<td></td>
</tr>
<tr>
<td>( 1/10 ) = 0.0001100110011[0011]... (_{2})</td>
<td></td>
</tr>
</tbody>
</table>
Fixed Point Representation

- Implied binary point. Two example schemes:
  
  #1: the binary point is between bits 2 and 3
  \[ b_7 \ b_6 \ b_5 \ b_4 \ b_3 \ [.] \ b_2 \ b_1 \ b_0 \]
  
  #2: the binary point is between bits 4 and 5
  \[ b_7 \ b_6 \ b_5 \ [.] \ b_4 \ b_3 \ b_2 \ b_1 \ b_0 \]

- Wherever we put the binary point, with fixed point representations there is a trade off between the amount of range and precision we have

- Fixed point = fixed range and fixed precision
  - range: difference between largest and smallest numbers possible
  - precision: smallest possible difference between any two numbers

- Hard to pick how much you need of each!
**Floating Point Representation**

- Analogous to scientific notation
  - **In Decimal:**
    - Not 12000000, but $1.2 \times 10^7$ In C: 1.2e7
    - Not 0.0000012, but $1.2 \times 10^{-6}$ In C: 1.2e-6
  - **In Binary:**
    - Not 11000.000, but $1.1 \times 2^4$
    - Not 0.000101, but $1.01 \times 2^{-4}$

- We have to divvy up the bits we have (e.g., 32) among:
  - the sign (1 bit)
  - the mantissa (significand)
  - the exponent
Scientific Notation (Decimal)

- **Normalized form**: exactly one digit (non-zero) to left of decimal point

- Alternatives to representing $1/1,000,000,000$
  - Normalized: $1.0 \times 10^{-9}$
  - Not normalized: $0.1 \times 10^{-8}, 10.0 \times 10^{-10}$
Scientific Notation (Binary)

- Computer arithmetic that supports this called floating point due to the “floating” of the binary point
  - Declare such variable in C as float (or double)
Scientific Notation Translation

- Convert from scientific notation to binary point
  - Perform the multiplication by shifting the decimal until the exponent disappears
    - Example: \(1.011_2 \times 2^4 = 10110_2 = 22_{10}\)
    - Example: \(1.011_2 \times 2^{-2} = 0.01011_2 = 0.34375_{10}\)

- Convert from binary point to *normalized* scientific notation
  - Distribute out exponents until binary point is to the right of a single digit
    - Example: \(1101.001_2 = 1.101001_2 \times 2^3\)

- Practice: Convert \(11.375_{10}\) to normalized binary scientific notation
Floating Point Topics

- Fractional binary numbers
- IEEE floating-point standard
- Floating-point operations and rounding
- Floating-point in C

- There are many more details that we won’t cover
  - It’s a 58-page standard...
IEEE Floating Point

- **IEEE 754**
  - Established in 1985 as uniform standard for floating point arithmetic
  - Main idea: make numerically sensitive programs portable
  - Specifies two things: representation and result of floating operations
  - Now supported by all major CPUs

- Driven by numerical concerns
  - **Scientists**/numerical analysts want them to be as **real** as possible
  - **Engineers** want them to be **easy to implement** and **fast**
  - In the end:
    - Scientists mostly won out
    - Nice standards for rounding, overflow, underflow, but...
    - Hard to make fast in hardware
    - **Float operations can be an order of magnitude slower than integer ops**
Floating Point Encoding

- Use normalized, base 2 scientific notation:
  - Value: $\pm 1 \times \text{Mantissa} \times 2^{\text{Exponent}}$
  - Bit Fields: $(-1)^S \times 1.M \times 2^{(E-bias)}$

- Representation Scheme:
  - Sign bit (0 is positive, 1 is negative)
  - Mantissa (a.k.a. significand) is the fractional part of the number in normalized form and encoded in bit vector $M$
  - Exponent weights the value by a (possibly negative) power of 2 and encoded in the bit vector $E$
The Exponent Field

- **Use biased notation**
  - Read exponent as unsigned, but with *bias of* $2^{w-1}-1 = 127$
  - Representable exponents roughly ½ positive and ½ negative
  - Exponent 0 (Exp = 0) is represented as $E = 0b\ 0111\ 1111$

- **Why biased?**
  - Makes floating point arithmetic easier
  - Makes somewhat compatible with two’s complement

- **Practice:** To encode in biased notation, add the bias then encode in unsigned:
  - $\text{Exp} = 1 \rightarrow \ E = 0b$
  - $\text{Exp} = 127 \rightarrow \ E = 0b$
  - $\text{Exp} = -63 \rightarrow \ E = 0b$
The Mantissa (Fraction) Field

- Note the implicit 1 in front of the M bit vector
  - Example: 0b 0011 1111 1100 0000 0000 0000 0000 0000 is read as 1.1₂ = 1.5₁₀, not 0.1₂ = 0.5₁₀
  - Gives us an extra bit of precision

- Mantissa “limits”
  - Low values near M = 0b0...0 are close to 2^{Exp}
  - High values near M = 0b1...1 are close to 2^{Exp+1}
Polling Question [FP I – a]

What is the correct value encoded by the following floating point number?

- 0b 0 10000000 110000000000000000000000

Vote at [http://pollev.com/rea](http://pollev.com/rea)

A. + 0.75
B. + 1.5
C. + 2.75
D. + 3.5
E. We’re lost...
Normalized Floating Point Conversions

- FP → Decimal
  1. Append the bits of $M$ to implicit leading 1 to form the mantissa.
  2. Multiply the mantissa by $2^{E - \text{bias}}$.
  3. Multiply the sign $(-1)^S$.
  4. Multiply out the exponent by shifting the binary point.
  5. Convert from binary to decimal.

- Decimal → FP
  1. Convert decimal to binary.
  2. Convert binary to normalized scientific notation.
  3. Encode sign as $S$ (0/1).
  4. Add the bias to exponent and encode $E$ as unsigned.
  5. The first bits after the leading 1 that fit are encoded into $M$. 
Precision and Accuracy

- **Precision** is a count of the number of bits in a computer word used to represent a value
  - Capacity for accuracy
- **Accuracy** is a measure of the difference between the actual value of a number and its computer representation
  - *High precision permits high accuracy but doesn’t guarantee it. It is possible to have high precision but low accuracy.*
  - **Example:** `float pi = 3.14;`
    - `pi` will be represented using all 24 bits of the mantissa (highly precise), but is only an approximation (not accurate)
Need Greater Precision?

- **Double Precision** (vs. Single Precision) in 64 bits

  - C variable declared as `double`
  - Exponent bias is now $2^{10} - 1 = 1023$

  - **Advantages:** greater precision (larger mantissa), greater range (larger exponent)
  - **Disadvantages:** more bits used, slower to manipulate
Representing Very Small Numbers

- But wait... what happened to zero?
  - Using standard encoding 0x00000000 =
  - **Special case**: E and M all zeros = 0
    - Two zeros! But at least 0x00000000 = 0 like integers

- New numbers closest to 0:
  - \( a = 1.0...0 \times 2^{-126} = 2^{-126} \)
  - \( b = 1.0...01 \times 2^{-126} = 2^{-126} + 2^{-149} \)
  - Normalization and implicit 1 are to blame
  - **Special case**: E = 0, M ≠ 0 are denormalized numbers
Denorm Numbers

- Denormalized numbers
  - No leading 1
  - Uses implicit exponent of $-126$ even though $E = 0x00$

- Denormalized numbers close the gap between zero and the smallest normalized number
  - Smallest norm: $\pm 1.0...0_{\text{two}} \times 2^{-126} = \pm 2^{-126}$
  - Smallest denorm: $\pm 0.0...01_{\text{two}} \times 2^{-126} = \pm 2^{-149}$
    - There is still a gap between zero and the smallest denormalized number

This is extra (non-testable) material

So much closer to 0
Summary

- Floating point approximates real numbers:

- Handles large numbers, small numbers, special numbers

- Exponent in biased notation (bias = $2^{w-1} - 1$)
  - Size of exponent field determines our representable range
  - Outside of representable exponents is overflow and underflow

- Mantissa approximates fractional portion of binary point
  - Size of mantissa field determines our representable precision
  - Implicit leading 1 (normalized) except in special cases
  - Exceeding length causes rounding
An example that applies the IEEE Floating Point concepts to a smaller (8-bit) representation scheme. These slides expand on material covered today, so while you don’t need to read these, the information is “fair game.”
Tiny Floating Point Example

- **8-bit Floating Point Representation**
  - The sign bit is in the most significant bit (MSB)
  - The next four bits are the exponent, with a bias of $2^{4-1} - 1 = 7$
  - The last three bits are the mantissa

- **Same general form as IEEE Format**
  - Normalized binary scientific point notation
  - Similar special cases for 0, denormalized numbers, NaN, ∞
## Dynamic Range (Positive Only)

<table>
<thead>
<tr>
<th>S E</th>
<th>M</th>
<th>Exp</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 0000 000</td>
<td>-6</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0 0000 001</td>
<td>-6</td>
<td>1/8*1/64 = 1/512</td>
<td></td>
</tr>
<tr>
<td>0 0000 010</td>
<td>-6</td>
<td>2/8*1/64 = 2/512</td>
<td></td>
</tr>
<tr>
<td></td>
<td>...</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0 0000 110</td>
<td>-6</td>
<td>6/8*1/64 = 6/512</td>
<td></td>
</tr>
<tr>
<td>0 0000 111</td>
<td>-6</td>
<td>7/8*1/64 = 7/512</td>
<td></td>
</tr>
<tr>
<td>0 0001 000</td>
<td>-6</td>
<td>8/8*1/64 = 8/512</td>
<td></td>
</tr>
<tr>
<td>0 0001 001</td>
<td>-6</td>
<td>9/8*1/64 = 9/512</td>
<td></td>
</tr>
<tr>
<td></td>
<td>...</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0 0110 110</td>
<td>-1</td>
<td>14/8*1/2 = 14/16</td>
<td></td>
</tr>
<tr>
<td>0 0110 111</td>
<td>-1</td>
<td>15/8*1/2 = 15/16</td>
<td></td>
</tr>
<tr>
<td>0 0111 000</td>
<td>0</td>
<td>8/8*1 = 1</td>
<td></td>
</tr>
<tr>
<td>0 0111 001</td>
<td>0</td>
<td>9/8*1 = 9/8</td>
<td></td>
</tr>
<tr>
<td>0 0111 010</td>
<td>0</td>
<td>10/8*1 = 10/8</td>
<td></td>
</tr>
<tr>
<td></td>
<td>...</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0 1110 110</td>
<td>7</td>
<td>14/8*128 = 224</td>
<td></td>
</tr>
<tr>
<td>0 1110 111</td>
<td>7</td>
<td>15/8*128 = 240</td>
<td></td>
</tr>
<tr>
<td>0 1111 000</td>
<td>n/a</td>
<td>inf</td>
<td></td>
</tr>
</tbody>
</table>

- **Denormalized numbers**
  - Closest to zero: 0 0000 000
  - Largest denorm: 0 0000 111
  - Smallest norm: 0 0001 000

- **Normalized numbers**
  - Closest to 1 below: 0 0110 110
  - Closest to 1 above: 0 1110 110
  - Largest norm: 0 1111 000
Special Properties of Encoding

- Floating point zero (0\^+) exactly the same bits as integer zero
  - All bits = 0

- Can (Almost) Use Unsigned Integer Comparison
  - Must first compare sign bits
  - Must consider 0^- = 0^+ = 0
  - NaNs problematic
    - Will be greater than any other values
    - What should comparison yield?
  - Otherwise OK
    - Denorm vs. normalized
    - Normalized vs. infinity