Integers II
CSE 351 Spring 2020

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http://xkcd.com/1953/
Administrivia

- hw4 due Friday – 11am
- hw5 due Monday – 11am
- Lab 1a due Monday (4/13)
  - Submit `pointer.c` and `lab1Areflect.txt` to Gradescope
- Lab 1b coming soon, due 4/20
  - Bit puzzles on number representation
  - Can start after today’s lecture, but floating point will be introduced next week
  - Section worksheet for tomorrow has helpful examples
  - Bonus slides at the end of today’s lecture have relevant examples
Extra Credit

- All labs starting with Lab 1b have extra credit portions
  - These are meant to be fun extensions to the labs
- Extra credit points *don't* affect your lab grades
  - From the course policies: “they will be accumulated over the course and will be used to bump up borderline grades at the end of the quarter.”
  - Make sure you finish the rest of the lab before attempting any extra credit
Integers

- Binary representation of integers
  - Unsigned and signed
- Shifting and arithmetic operations – useful for Lab 1a
- In C: Signed, Unsigned and Casting
- Consequences of finite width representations
  - Overflow, sign extension
Two’s Complement Arithmetic

- The same addition procedure works for both unsigned and two’s complement integers
  - **Simplifies hardware:** only one algorithm for addition
  - **Algorithm:** simple addition, discard the highest carry bit
    - Called modular addition: result is sum $\mod 2^w$

**4-bit Examples:**

<table>
<thead>
<tr>
<th>HW</th>
<th>TC</th>
</tr>
</thead>
<tbody>
<tr>
<td>0100</td>
<td>+0011</td>
</tr>
<tr>
<td>=</td>
<td></td>
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</table>
Why Does Two’s Complement Work?

- For all representable positive integers $x$, we want:

  \[
  \text{bit representation of } x \\
  + \text{bit representation of } -x \\
  0 \quad \text{(ignoring the carry-out bit)}
  \]

- What are the 8-bit negative encodings for the following?

  \[
  \begin{align*}
  00000001 + \text{????????} & = 00000000 \\
  00000010 + \text{????????} & = 00000000 \\
  11000011 + \text{????????} & = 00000000
  \end{align*}
  \]
Why Does Two’s Complement Work?

- For all representable positive integers $x$, we want:

  $\begin{align*}
  \text{bit representation of } x \\
  + \text{bit representation of } -x \\
  0 \quad \text{(ignoring the carry-out bit)}
  \end{align*}$

- What are the 8-bit negative encodings for the following?

  $\begin{align*}
  00000001 + 11111111 &= 100000000 \\
  00000010 + 11111110 &= 100000000 \\
  11000011 + 00111101 &= 100000000
  \end{align*}$

  These are the bitwise complement plus 1!

  $-x = \sim x + 1$
Signed/Unsigned Conversion Visualized

- Two’s Complement → Unsigned
  - Ordering Inversion
  - Negative → Big Positive
Values To Remember

- **Unsigned Values**
  - UMin = 0b00...0 = 0
  - UMax = 0b11...1 = \(2^w - 1\)

- **Two’s Complement Values**
  - Tmin = 0b10...0 = \(-2^{w-1}\)
  - Tmax = 0b01...1 = \(2^{w-1} - 1\)
  - -1 = 0b11...1

**Example: Values for w = 64**

<table>
<thead>
<tr>
<th></th>
<th>Decimal</th>
<th>Hex</th>
</tr>
</thead>
<tbody>
<tr>
<td>UMax</td>
<td>18,446,744,073,709,551,615</td>
<td>FF FF FF FF FF FF FF FF FF</td>
</tr>
<tr>
<td>Tmax</td>
<td>9,223,372,036,854,775,807</td>
<td>7F FF FF FF FF FF FF FF FF</td>
</tr>
<tr>
<td>Tmin</td>
<td>-9,223,372,036,854,775,808</td>
<td>80 00 00 00 00 00 00 00</td>
</tr>
<tr>
<td>-1</td>
<td>-1</td>
<td>FF FF FF FF FF FF FF FF FF</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>00 00 00 00 00 00 00 00 00</td>
</tr>
</tbody>
</table>
Integers

- Binary representation of integers
  - Unsigned and signed
- **Shifting and arithmetic operations** – useful for Lab 1a
- In C: Signed, Unsigned and Casting
- Consequences of finite width representations
  - Overflow, sign extension
Shift Operations

- **Left shift** \((x << n)\) bit vector \(x\) by \(n\) positions
  - Throw away (drop) extra bits on left
  - Fill with 0s on right

- **Right shift** \((x >> n)\) bit-vector \(x\) by \(n\) positions
  - Throw away (drop) extra bits on right
  - Logical shift (for **unsigned** values)
    - Fill with 0s on left
  - Arithmetic shift (for **signed** values)
    - Replicate most significant bit on left
    - Maintains sign of \(x\)
Shift Operations

- **Left shift** ($x << n$)
  - Fill with 0s on right

- **Right shift** ($x >> n$)
  - **Logical shift** (for unsigned values)
    - Fill with 0s on left
  - **Arithmetic shift** (for signed values)
    - Replicate most significant bit on left

- **Notes:**
  - Shifts by $n < 0$ or $n \geq w$ ($w$ is bit width of $x$) are *undefined*
  - **In C:** behavior of $>>$ is determined by compiler
    - In gcc / C lang, depends on data type of $x$ (signed/unsigned)
  - **In Java:** logical shift is $>>>$ and arithmetic shift is $>>$
Shifting Arithmetic?

- What are the following computing?
  - $x >> n$
    - $0b\ 0100 >> 1 = 0b\ 0010$
    - $0b\ 0100 >> 2 = 0b\ 0001$
    - **Divide by $2^n$**
  - $x << n$
    - $0b\ 0001 << 1 = 0b\ 0010$
    - $0b\ 0001 << 2 = 0b\ 0100$
    - **Multiply by $2^n$**

- Shifting is faster than general multiply and divide operations
Left Shifting Arithmetic 8-bit Example

- No difference in left shift operation for unsigned and signed numbers (just manipulates bits)
  - Difference comes during interpretation: \( x \times 2^n \)?

\[
\begin{align*}
x &= 25; & \quad 00011001 &= 25 & \text{Signed} & \quad 00011001 &= 25 & \text{Unsigned} \\
L1 &= x \ll 2; & \quad 0001100100 &= 100 & \text{Signed} & \quad 0001100100 &= 100 & \text{Unsigned} \\
L2 &= x \ll 3; & \quad 000110010000 &= -56 & \text{Signed} & \quad 000110010000 &= 200 & \text{Unsigned, signed overflow} \\
L3 &= x \ll 4; & \quad 000110010000 &= -112 & \text{Signed} & \quad 000110010000 &= 144 & \text{Unsigned, unsigned overflow}
\end{align*}
\]
Right Shifting Arithmetic 8-bit Examples

- **Reminder:** C operator `>>` does *logical* shift on unsigned values and *arithmetic* shift on signed values
  - Logical Shift: \( x / 2^n \)?

\[
\begin{align*}
xu &= 240u; \quad 11110000 = 240 \\
\text{R1u} &= xu >> 3; \quad 00011110000 = 30 \\
\text{R2u} &= xu >> 5; \quad 0000011110000 = 7
\end{align*}
\]

rounding (down)
Right Shifting Arithmetic 8-bit Examples

- **Reminder:** C operator $\gg$ does *logical* shift on unsigned values and *arithmetic* shift on signed values

  - **Arithmetic Shift:** $x / 2^n$?

    - $xs = -16; \quad 11110000 \quad = \quad -16$
    - $R1s=xu>>3; \quad 11111110000 \quad = \quad -2$
    - $R2s=xu>>5; \quad 1111111110000 \quad = \quad -1$

  rounding (down)
Integers

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- In C: Signed, Unsigned and Casting
- Consequences of finite width representations
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In C: Signed vs. Unsigned

- Casting
  - Bits are unchanged, just interpreted differently!
    - int tx, ty;
    - unsigned int ux, uy;
  - Explicit casting
    - tx = (int) ux;
    - uy = (unsigned int) ty;
  - Implicit casting can occur during assignments or function calls
    - tx = ux;
    - uy = ty;
Casting Surprises

- Integer literals (constants)
  - By default, integer constants are considered *signed* integers
    - Hex constants already have an explicit binary representation
  - Use “U” (or “u”) suffix to explicitly force *unsigned*
    - Examples: 0U, 4294967259u

- Expression Evaluation
  - When you mixed unsigned and signed in a single expression, then *signed values are implicitly cast to unsigned*
  - Including comparison operators <, >, ==, <=, >=
Casting Surprises

- 32-bit examples:
  - Tmin = -2,147,483,648, Tmax = 2,147,483,647

<table>
<thead>
<tr>
<th>Left Constant</th>
<th>Order</th>
<th>Right Constant</th>
<th>Interpretation</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td></td>
<td>0U</td>
<td>0000 0000 0000 0000 0000 0000 0000 0000</td>
</tr>
<tr>
<td>0000 0000 0000 0000 0000 0000 0000 0000</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>-1</td>
<td></td>
<td>0</td>
<td>0000 0000 0000 0000 0000 0000 0000 0000</td>
</tr>
<tr>
<td>1111 1111 1111 1111 1111 1111 1111 1111</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>-1</td>
<td></td>
<td>0U</td>
<td>0000 0000 0000 0000 0000 0000 0000 0000</td>
</tr>
<tr>
<td>1111 1111 1111 1111 1111 1111 1111 1111</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2147483647</td>
<td></td>
<td>-2147483648</td>
<td>1000 0000 0000 0000 0000 0000 0000 0000</td>
</tr>
<tr>
<td>0111 1111 1111 1111 1111 1111 1111 1111</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2147483647U</td>
<td></td>
<td>-2147483648</td>
<td>1000 0000 0000 0000 0000 0000 0000 0000</td>
</tr>
<tr>
<td>0111 1111 1111 1111 1111 1111 1111 1111</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>-1</td>
<td></td>
<td>-2</td>
<td>1111 1111 1111 1111 1111 1111 1111 1110</td>
</tr>
<tr>
<td>1111 1111 1111 1111 1111 1111 1111 1111</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(unsigned) -1</td>
<td></td>
<td>-2</td>
<td>1111 1111 1111 1111 1111 1111 1111 1110</td>
</tr>
<tr>
<td>1111 1111 1111 1111 1111 1111 1111 1111</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2147483647</td>
<td></td>
<td>2147483648U</td>
<td>1000 0000 0000 0000 0000 0000 0000 0000</td>
</tr>
<tr>
<td>0111 1111 1111 1111 1111 1111 1111 1111</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2147483647</td>
<td></td>
<td>(int) 2147483648U</td>
<td>1000 0000 0000 0000 0000 0000 0000 0000</td>
</tr>
<tr>
<td>0111 1111 1111 1111 1111 1111 1111 1111</td>
<td></td>
<td></td>
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# Arithmetic Overflow

- When a calculation produces a result that can’t be represented in the current encoding scheme
  - Integer range limited by fixed width
  - Can occur in both the positive and negative directions

- C and Java ignore overflow exceptions
  - You end up with a bad value in your program and no warning/indication... oops!

<table>
<thead>
<tr>
<th>Bits</th>
<th>Unsigned</th>
<th>Signed</th>
</tr>
</thead>
<tbody>
<tr>
<td>0000</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0001</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>0010</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>0011</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>0100</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>0101</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>0110</td>
<td>6</td>
<td>6</td>
</tr>
<tr>
<td>0111</td>
<td>7</td>
<td>7</td>
</tr>
<tr>
<td>1000</td>
<td>8</td>
<td>-8</td>
</tr>
<tr>
<td>1001</td>
<td>9</td>
<td>-7</td>
</tr>
<tr>
<td>1010</td>
<td>10</td>
<td>-6</td>
</tr>
<tr>
<td>1011</td>
<td>11</td>
<td>-5</td>
</tr>
<tr>
<td>1100</td>
<td>12</td>
<td>-4</td>
</tr>
<tr>
<td>1101</td>
<td>13</td>
<td>-3</td>
</tr>
<tr>
<td>1110</td>
<td>14</td>
<td>-2</td>
</tr>
<tr>
<td>1111</td>
<td>15</td>
<td>-1</td>
</tr>
</tbody>
</table>
Overflow: Unsigned

- **Addition**: drop carry bit \((-2^N)\)
  
  \[
  \begin{array}{c}
  15 \\
  + 2 \\
  \hline
  \text{17}^{\text{red}}
  \end{array}
  \quad \quad \quad
  \begin{array}{c}
  1111 \\
  + 0010 \\
  \hline
  10001
  \end{array}
  \]
  
  \[
  1
  \]

- **Subtraction**: borrow \((+2^N)\)
  
  \[
  \begin{array}{c}
  1 \\
  - 2 \\
  \hline
  -1^{\text{blue}}
  \end{array}
  \quad \quad \quad
  \begin{array}{c}
  10001 \\
  - 0010 \\
  \hline
  1111
  \end{array}
  \]

\(\pm 2^N\) because of modular arithmetic
Overflow: Two’s Complement

- **Addition:** $(+) + (+) = (−)$ result?

  $$
  \begin{array}{c}
  6 \\
  + 3 \\
  \hline
  9
  \end{array}
  \quad 0110
  \quad +
  \begin{array}{c}
  + 0011 \\
  \hline
  1001
  \end{array}
  \quad -7
  $$

- **Subtraction:** $(−) + (−) = (+)$?

  $$
  \begin{array}{c}
  -7 \\
  - 3 \\
  \hline
  -10
  \end{array}
  \quad 1001
  \quad -
  \begin{array}{c}
  - 0011 \\
  \hline
  0110
  \end{array}
  \quad 6
  $$

**For signed:** overflow if operands have same sign and result’s sign is different
Sign Extension

- What happens if you convert a signed integral data type to a larger one?
  - *e.g.* char → short → int → long

- **4-bit → 8-bit Example:**
  - Positive Case
    - Add 0’s?
    - 4-bit: 0010 = +2
    - 8-bit: 00000010 = +2

- Negative Case?
Polling Question [Int II - a]

- Which of the following 8-bit numbers has the same signed value as the 4-bit number \texttt{0b1100}?
  - Underlined digit = MSB
  - Vote at [http://pollev.com/rea](http://pollev.com/rea)

A. \texttt{0b 0000 1100}
B. \texttt{0b 1000 1100}
C. \texttt{0b 1111 1100}
D. \texttt{0b 1100 1100}
E. We’re lost...
Sign Extension

- **Task**: Given a $w$-bit signed integer $X$, convert it to a $w+k$-bit signed integer $X'$ with the same value.

- **Rule**: Add $k$ copies of sign bit.
  - Let $x_i$ be the $i$-th digit of $X$ in binary.
  - $X' = x_{w-1}, \ldots, x_{w-1}, x_{w-1}, x_{w-2}, \ldots, x_1, x_0$
Sign Extension Example

- Convert from smaller to larger integral data types
- C automatically performs sign extension
  - Java too

```
short int x = 12345;
int ix = (int) x;
short int y = -12345;
int iy = (int) y;
```

<table>
<thead>
<tr>
<th>Var</th>
<th>Decimal</th>
<th>Hex</th>
<th>Binary</th>
</tr>
</thead>
<tbody>
<tr>
<td>x</td>
<td>12345</td>
<td>30 39</td>
<td>00110000 00111001</td>
</tr>
<tr>
<td>ix</td>
<td>12345</td>
<td>00 00 30 39</td>
<td>00000000 00000000 00110000 00111001</td>
</tr>
<tr>
<td>y</td>
<td>-12345</td>
<td>CF C7</td>
<td>11001111 11000111</td>
</tr>
<tr>
<td>iy</td>
<td>-12345</td>
<td>FF FF CF C7</td>
<td>11111111 11111111 11001111 11000111</td>
</tr>
</tbody>
</table>
Summary

- Sign and unsigned variables in C
  - Bit pattern remains the same, just interpreted differently
  - Strange things can happen with our arithmetic when we convert/cast between sign and unsigned numbers
    - Type of variables affects behavior of operators (shifting, comparison)

- We can only represent so many numbers in \( w \) bits
  - When we exceed the limits, arithmetic overflow occurs
  - Sign extension tries to preserve value when expanding

- Shifting is a useful bitwise operator
  - Right shifting can be arithmetic (sign) or logical (0)
  - Can be used in multiplication with constant or bit masking
Practice Question

For the following expressions, find a value of `signed char x`, if there exists one, that makes the expression `TRUE`. Compare with your neighbor(s)!

- Assume we are using 8-bit arithmetic:
  - `x == (unsigned char) x`
  - `x >= 128U`
  - `x != (x>>2)<<2`
  - `x == -x`
    - Hint: there are two solutions
  - `(x < 128U) && (x > 0x3F)`
Some examples of using shift operators in combination with bitmasks, which you may find helpful for Lab 1.

- Extract the 2\textsuperscript{nd} most significant byte of an \texttt{int}
- Extract the sign bit of a signed \texttt{int}
- Conditionals as Boolean expressions
Using Shifts and Masks

- Extract the 2\textsuperscript{nd} most significant byte of an int:
  - First shift, then mask: \((x>>16) \& 0xFF\)

<table>
<thead>
<tr>
<th>(x)</th>
<th>00000001 00000010 00000011 00000100</th>
</tr>
</thead>
<tbody>
<tr>
<td>(x&gt;&gt;16)</td>
<td>00000000 00000000 00000011 00000100</td>
</tr>
<tr>
<td>0xFF</td>
<td>00000000 00000000 00000000 11111111</td>
</tr>
<tr>
<td>((x&gt;&gt;16) &amp; 0xFF)</td>
<td>00000000 00000000 00000000 00000100</td>
</tr>
</tbody>
</table>

- Or first mask, then shift: \((x \& 0xFF0000) >>16\)

<table>
<thead>
<tr>
<th>(x)</th>
<th>00000001 00000010 00000011 00000100</th>
</tr>
</thead>
<tbody>
<tr>
<td>0xFF0000</td>
<td>00000000 11111111 00000000 00000000</td>
</tr>
<tr>
<td>(x &amp; 0xFF0000)</td>
<td>00000000 00000010 00000000 00000000</td>
</tr>
<tr>
<td>((x&amp;0xFF0000) &gt;&gt;16)</td>
<td>00000000 00000000 00000000 00000100</td>
</tr>
</tbody>
</table>
Using Shifts and Masks

- Extract the *sign bit* of a signed `int`:
  - First shift, then mask: `(x >> 31) & 0x1`
    - Assuming arithmetic shift here, but this works in either case
    - Need mask to clear 1s possibly shifted in

<table>
<thead>
<tr>
<th>x</th>
<th>00000001 00000010 00000011 00000100</th>
</tr>
</thead>
<tbody>
<tr>
<td>x&gt;&gt;31</td>
<td>00000000 00000000 00000000 00000000</td>
</tr>
<tr>
<td>0x1</td>
<td>00000000 00000000 00000000 00000000</td>
</tr>
<tr>
<td>(x&gt;&gt;31) &amp; 0x1</td>
<td>00000000 00000000 00000000 00000000</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>x</th>
<th>10000001 00000010 00000011 00000100</th>
</tr>
</thead>
<tbody>
<tr>
<td>x&gt;&gt;31</td>
<td>11111111 11111111 11111111 11111111</td>
</tr>
<tr>
<td>0x1</td>
<td>00000000 00000000 00000000 00000000</td>
</tr>
<tr>
<td>(x&gt;&gt;31) &amp; 0x1</td>
<td>00000000 00000000 00000000 00000001</td>
</tr>
</tbody>
</table>
Using Shifts and Masks

- Conditionals as Boolean expressions
  - For int x, what does \((x \ll 31) \gg 31\) do?

<table>
<thead>
<tr>
<th>x=!!123</th>
<th>00000000 00000000 00000000 00000001</th>
</tr>
</thead>
<tbody>
<tr>
<td>x&lt;&lt;31</td>
<td>10000000 00000000 00000000 00000000</td>
</tr>
<tr>
<td>((x&lt;&lt;31)\gg31</td>
<td>11111111 11111111 11111111 11111111</td>
</tr>
<tr>
<td>!x</td>
<td>00000000 00000000 00000000 00000000</td>
</tr>
<tr>
<td>!x&lt;&lt;31</td>
<td>00000000 00000000 00000000 00000000</td>
</tr>
<tr>
<td>((!x&lt;&lt;31)\gg31</td>
<td>00000000 00000000 00000000 00000000</td>
</tr>
</tbody>
</table>

- Can use in place of conditional:
  - In C: `if(x) {a=y;} else {a=z;}` equivalent to `a=x?y:z;`
  - `a=((x<<31)>>31)&y) | ((!x<<31)>>31)&z);`