Integers II
CSE 351 Spring 2020

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http://xkcd.com/1953/
Administrivia

- hw4 due Friday – 11am
- hw5 due Monday – 11am
- Lab 1a due Monday (4/13)
  - Submit `pointer.c` and `lab1Areflect.txt` to Gradescope
- Lab 1b coming soon, due 4/20
  - Bit puzzles on number representation
  - Can start after today’s lecture, but floating point will be introduced next week
  - Section worksheet for tomorrow has helpful examples
  - Bonus slides at the end of today’s lecture have relevant examples
Extra Credit

- All labs starting with Lab 1b have extra credit portions
  - These are meant to be fun extensions to the labs

- Extra credit points *don't* affect your lab grades
  - From the course policies: “they will be accumulated over the course and will be used to bump up borderline grades at the end of the quarter.”
  - Make sure you finish the rest of the lab before attempting any extra credit
Integers

- Binary representation of integers
  - Unsigned and signed
- Shifting and arithmetic operations – useful for Lab 1a
- In C: Signed, Unsigned and Casting
- Consequences of finite width representations
  - Overflow, sign extension
Two’s Complement Arithmetic

- The same addition procedure works for both unsigned and two’s complement integers
  - **Simplifies hardware**: only one algorithm for addition
  - **Algorithm**: simple addition, discard the highest carry bit
    - Called modular addition: result is sum modulo $2^w$

### 4-bit Examples:

<table>
<thead>
<tr>
<th>HW</th>
<th>TC</th>
<th>HW</th>
<th>TC</th>
<th>HW</th>
<th>TC</th>
</tr>
</thead>
<tbody>
<tr>
<td>0100</td>
<td>+0011</td>
<td>1100</td>
<td>+0011</td>
<td>0100</td>
<td>+1101</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>=</td>
<td></td>
<td>=</td>
<td></td>
<td>=</td>
<td></td>
</tr>
</tbody>
</table>

= 7
= 7
= -1
Why Does Two’s Complement Work?

- For all representable positive integers \( x \), we want:

\[
\text{bit representation of } x + \text{bit representation of } -x \quad \text{(ignoring the carry-out bit)}
\]

- What are the 8-bit negative encodings for the following?

\[
\begin{align*}
00000001 & \quad 00000010 & \quad 11000011 \\
+ \, ???????? & + \, ???????? & + \, ???????? \\
00000000 & + \, 00000000 & + \, 00000000
\end{align*}
\]
Why Does Two’s Complement Work?

- For all representable positive integers $x$, we want:

\[
\text{bit representation of } x + \text{bit representation of } -x = \overline{0} \quad \text{(ignoring the carry-out bit)}
\]

- What are the 8-bit negative encodings for the following?

\[
\begin{align*}
00000001 + 11111111 & = 100000000 \\
00000010 + 11111110 & = 100000000 \\
11000011 + 00111101 & = 100000000
\end{align*}
\]

These are the bitwise complement plus 1!

\[-x \equiv \overline{x} + 1\]
Signed/ Unsigned Conversion Visualized

- Two’s Complement → Unsigned
  - Ordering Inversion
  - Negative → Big Positive
Values To Remember

- **Unsigned Values**
  - $U_{\text{Min}} = \text{0b}00...0 = 0$
  - $U_{\text{Max}} = \text{0b}11...1 = 2^w - 1$

- **Example: Values for $w = 64$**

<table>
<thead>
<tr>
<th></th>
<th>Decimal</th>
<th>Hex</th>
</tr>
</thead>
<tbody>
<tr>
<td>UMax</td>
<td>18,446,744,073,709,551,615</td>
<td>FF FF FF FF FF FF FF FF FF</td>
</tr>
<tr>
<td>Tmax</td>
<td>9,223,372,036,854,775,807</td>
<td>7F FF FF FF FF FF FF FF</td>
</tr>
<tr>
<td>Tmin</td>
<td>-9,223,372,036,854,775,808</td>
<td>80 00 00 00 00 00 00 00</td>
</tr>
<tr>
<td>-1</td>
<td>-1</td>
<td>FF FF FF FF FF FF FF FF</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>00 00 00 00 00 00 00 00</td>
</tr>
</tbody>
</table>
Integers

- Binary representation of integers
  - Unsigned and signed

- **Shifting and arithmetic operations** – useful for Lab 1a

- In C: Signed, Unsigned and Casting

- Consequences of finite width representations
  - Overflow, sign extension
Shift Operations

- **Left shift** \((x<<n)\) bit vector \(x\) by \(n\) positions
  - Throw away (drop) extra bits on left
  - Fill with 0s on right

- **Right shift** \((x>>n)\) bit-vector \(x\) by \(n\) positions
  - Throw away (drop) extra bits on right
  - Logical shift (for **unsigned** values)
    - Fill with 0s on left
  - Arithmetic shift (for **signed** values)
    - Replicate most significant bit on left
    - Maintains sign of \(x\)
Shift Operations

- **Left shift** \((x << n)\)
  - Fill with 0s on right

- **Right shift** \((x >> n)\)
  - Logical shift (for **unsigned** values)
    - Fill with 0s on left
  - Arithmetic shift (for **signed** values)
    - Replicate most significant bit on left

**Notes:**
- Shifts by \(n < 0\) or \(n \geq w\) \((w\) is bit width of \(x\)) are **undefined**
- **In C:** behavior of \(>>\) is determined by compiler
  - In gcc / C lang, depends on data type of \(x\) (signed/unsigned)
- **In Java:** logical shift is \(>>>\) and arithmetic shift is \(>>\)
Shifting Arithmetic?

- What are the following computing?
  - \(x \gg n\)
    - \(0b\ 0100 \gg 1 = 0b\ 0010\)
    - \(0b\ 0100 \gg 2 = 0b\ 0001\)
    - Divide by \(2^n\)
  - \(x \ll n\)
    - \(0b\ 0001 \ll 1 = 0b\ 0010\)
    - \(0b\ 0001 \ll 2 = 0b\ 0100\)
    - Multiply by \(2^n\)

- Shifting is faster than general multiply and divide operations
Left Shifting Arithmetic 8-bit Example

- No difference in left shift operation for unsigned and signed numbers (just manipulates bits)
  - Difference comes during interpretation: \( x \times 2^n \)?

<table>
<thead>
<tr>
<th>x = 25;</th>
<th>00011001</th>
<th>=</th>
<th>Signed 25</th>
<th>Unsigned 25</th>
</tr>
</thead>
<tbody>
<tr>
<td>L1=x&lt;&lt;2;</td>
<td>0001100100</td>
<td>=</td>
<td>Signed 100</td>
<td>Unsigned 100</td>
</tr>
<tr>
<td>L2=x&lt;&lt;3;</td>
<td>000110010000</td>
<td>=</td>
<td>Signed -56</td>
<td>Unsigned 200</td>
</tr>
<tr>
<td>L3=x&lt;&lt;4;</td>
<td>0001100100000</td>
<td>=</td>
<td>Signed -112</td>
<td>Unsigned 144</td>
</tr>
</tbody>
</table>
Right Shifting Arithmetic 8-bit Examples

- **Reminder:** C operator `>>` does *logical* shift on unsigned values and *arithmetic* shift on signed values

  - **Logical Shift:** $x / 2^n$?

    - $x_u = 240u; \ 11110000 = 240$
    - $R1_u = x_u >> 3; \ 00011110000 = 30$
    - $R2_u = x_u >> 5; \ 0000011110000 = 7$

  **rounding (down)**
Right Shifting Arithmetic 8-bit Examples

- **Reminder:** C operator `>>` does *logical* shift on unsigned values and *arithmetic* shift on signed values
  - **Arithmetic Shift:** $x / 2^n$?

\[
xs = -16; \quad 11110000 = -16 \\
R1s = xu >> 3; \quad 111111110000 = -2 \\
R2s = xu >> 5; \quad 1111111110000 = -1
\]

*rounding (down)*
Integers

- Binary representation of integers
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- Shifting and arithmetic operations – useful for Lab 1a
- In C: Signed, Unsigned and Casting
- Consequences of finite width representations
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In C: Signed vs. Unsigned

❖ Casting

❖ Bits are unchanged, just interpreted differently!
  • `int tx, ty;`
  • `unsigned int ux, uy;`

❖ *Explicit* casting
  • `tx = (int) ux;`
  • `uy = (unsigned int) ty;`

❖ *Implicit* casting can occur during assignments or function calls
  • `tx = ux;`
  • `uy = ty;`
Casting Surprises

- Integer literals (constants)
  - By default, integer constants are considered *signed* integers
    - Hex constants already have an explicit binary representation
  - Use “U” (or “u”) suffix to explicitly force *unsigned*
    - *Examples:* 0U, 4294967259u

- Expression Evaluation
  - When you mixed unsigned and signed in a single expression, then **signed values are implicitly cast to unsigned**
  - Including comparison operators <, >, ==, <=, >=
Casting Surprises

- 32-bit examples:
  - $T_{\text{Min}} = -2,147,483,648$, $T_{\text{Max}} = 2,147,483,647$

<table>
<thead>
<tr>
<th>Left Constant</th>
<th>Order</th>
<th>Right Constant</th>
<th>Interpretation</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td></td>
<td>0U</td>
<td>0</td>
</tr>
<tr>
<td>0000 0000 0000 0000 0000 0000 0000 0000</td>
<td></td>
<td>0000 0000 0000 0000 0000 0000 0000 0000</td>
<td>0</td>
</tr>
<tr>
<td>-1</td>
<td></td>
<td>0</td>
<td>0000 0000 0000 0000 0000 0000 0000 0000</td>
</tr>
<tr>
<td>1111 1111 1111 1111 1111 1111 1111 1111</td>
<td></td>
<td>0000 0000 0000 0000 0000 0000 0000 0000</td>
<td>0</td>
</tr>
<tr>
<td>-1</td>
<td></td>
<td>0U</td>
<td>0000 0000 0000 0000 0000 0000 0000 0000</td>
</tr>
<tr>
<td>1111 1111 1111 1111 1111 1111 1111 1111</td>
<td></td>
<td>0000 0000 0000 0000 0000 0000 0000 0000</td>
<td>0</td>
</tr>
<tr>
<td>2147483647</td>
<td></td>
<td>-2147483648</td>
<td>1000 0000 0000 0000 0000 0000 0000 0000</td>
</tr>
<tr>
<td>0111 1111 1111 1111 1111 1111 1111 1111</td>
<td></td>
<td>-2147483648</td>
<td>1000 0000 0000 0000 0000 0000 0000 0000</td>
</tr>
<tr>
<td>2147483647U</td>
<td></td>
<td>-2147483648</td>
<td>1000 0000 0000 0000 0000 0000 0000 0000</td>
</tr>
<tr>
<td>0111 1111 1111 1111 1111 1111 1111 1111</td>
<td></td>
<td>-2147483648</td>
<td>1000 0000 0000 0000 0000 0000 0000 0000</td>
</tr>
<tr>
<td>-1</td>
<td></td>
<td>-2</td>
<td>1111 1111 1111 1111 1111 1111 1111 1111</td>
</tr>
<tr>
<td>1111 1111 1111 1111 1111 1111 1111 1111</td>
<td></td>
<td>-2</td>
<td>1111 1111 1111 1111 1111 1111 1111 1111</td>
</tr>
<tr>
<td>(unsigned) -1</td>
<td></td>
<td>-2</td>
<td>1111 1111 1111 1111 1111 1111 1111 1111</td>
</tr>
<tr>
<td>1111 1111 1111 1111 1111 1111 1111 1111</td>
<td></td>
<td>-2</td>
<td>1111 1111 1111 1111 1111 1111 1111 1111</td>
</tr>
<tr>
<td>2147483647</td>
<td></td>
<td>2147483648U</td>
<td>1000 0000 0000 0000 0000 0000 0000 0000</td>
</tr>
<tr>
<td>0111 1111 1111 1111 1111 1111 1111 1111</td>
<td></td>
<td>2147483648U</td>
<td>1000 0000 0000 0000 0000 0000 0000 0000</td>
</tr>
<tr>
<td>2147483647</td>
<td></td>
<td>(int) 2147483648U</td>
<td>1000 0000 0000 0000 0000 0000 0000 0000</td>
</tr>
</tbody>
</table>
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- Binary representation of integers
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Arithmetic Overflow

- When a calculation produces a result that can’t be represented in the current encoding scheme
  - Integer range limited by fixed width
  - Can occur in both the positive and negative directions

- C and Java ignore overflow exceptions
  - You end up with a bad value in your program and no warning/indication... oops!

<table>
<thead>
<tr>
<th>Bits</th>
<th>Unsigned</th>
<th>Signed</th>
</tr>
</thead>
<tbody>
<tr>
<td>0000</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0001</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>0010</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>0011</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>0100</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>0101</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>0110</td>
<td>6</td>
<td>6</td>
</tr>
<tr>
<td>0111</td>
<td>7</td>
<td>7</td>
</tr>
<tr>
<td>1000</td>
<td>8</td>
<td>-8</td>
</tr>
<tr>
<td>1001</td>
<td>9</td>
<td>-7</td>
</tr>
<tr>
<td>1010</td>
<td>10</td>
<td>-6</td>
</tr>
<tr>
<td>1011</td>
<td>11</td>
<td>-5</td>
</tr>
<tr>
<td>1100</td>
<td>12</td>
<td>-4</td>
</tr>
<tr>
<td>1101</td>
<td>13</td>
<td>-3</td>
</tr>
<tr>
<td>1110</td>
<td>14</td>
<td>-2</td>
</tr>
<tr>
<td>1111</td>
<td>15</td>
<td>-1</td>
</tr>
</tbody>
</table>
Overflow: Unsigned

- **Addition:** drop carry bit ($-2^N$)

  \[
  \begin{array}{c}
  15 \\
  + \ 2 \\
  \hline
  17 \\
  \end{array}
  \quad
  \begin{array}{c}
  1111 \\
  + \ 0010 \\
  \hline
  10001 \\
  \end{array}
  \]

- **Subtraction:** borrow ($+2^N$)

  \[
  \begin{array}{c}
  1 \\
  - \ 2 \\
  \hline
  -1 \\
  \end{array}
  \quad
  \begin{array}{c}
  10001 \\
  - \ 0010 \\
  \hline
  1111 \quad \text{(Unsigned: } \pm 2^N \text{ because of modular arithmetic)}
  \end{array}
  \]
Overflow: Two’s Complement

- **Addition:** (+) + (+) = (−) result?

```
  6   0110
+ 3   + 0011
-----   -----
  9  1001
```

- **Subtraction:** (−) + (−) = (+)?

```
-7   1001
- 3   - 0011
-----   -----
-10  0110
```

For signed: overflow if operands have same sign and result’s sign is different
Sign Extension

- What happens if you convert a *signed* integral data type to a larger one?
  - *e.g.* char → short → int → long

- 4-bit → 8-bit Example:
  - Positive Case
    - 4-bit: 0010 = +2
    - 8-bit: 00000010 = +2
    - Add 0’s?
  - Negative Case?
Polling Question [Int II - a]

Which of the following 8-bit numbers has the same signed value as the 4-bit number 0b1100?

- Underlined digit = MSB
- Vote at [http://pollev.com/rea](http://pollev.com/rea)

A. 0b 0000 1100
B. 0b 1000 1100
C. 0b 1111 1100
D. 0b 1100 1100
E. We’re lost...
Sign Extension

- **Task:** Given a \(w\)-bit signed integer \(X\), convert it to \(w+k\)-bit signed integer \(X'\) *with the same value*

- **Rule:** Add \(k\) copies of sign bit
  - Let \(x_i\) be the \(i\)-th digit of \(X\) in binary
  - \(X' = x_{w-1}, \ldots, x_{w-1}, x_{w-1}, x_{w-2}, \ldots, x_1, x_0\)

\[\begin{array}{c}
\text{k copies of MSB} \\
\text{original X}
\end{array}\]

\[\begin{array}{c}
\text{k copies of MSB} \\
\text{original X}
\end{array}\]
Sign Extension Example

- Convert from smaller to larger integral data types
- C automatically performs sign extension
  - Java too

```java
short int x = 12345;
int ix = (int) x;
short int y = -12345;
int iy = (int) y;
```

<table>
<thead>
<tr>
<th>Var</th>
<th>Decimal</th>
<th>Hex</th>
<th>Binary</th>
</tr>
</thead>
<tbody>
<tr>
<td>x</td>
<td>12345</td>
<td>30 39</td>
<td>00110000 00111001</td>
</tr>
<tr>
<td>ix</td>
<td>12345</td>
<td>00 00 30 39</td>
<td>00000000 00000000 00110000 00111001</td>
</tr>
<tr>
<td>y</td>
<td>-12345</td>
<td>CF C7</td>
<td>11001111 11000111</td>
</tr>
<tr>
<td>iy</td>
<td>-12345</td>
<td>FF FF CF C7</td>
<td>11111111 11111111 11001111 11000111</td>
</tr>
</tbody>
</table>
Summary

- **Sign and unsigned variables in C**
  - Bit pattern remains the same, just *interpreted* differently
  - Strange things can happen with our arithmetic when we convert/cast between sign and unsigned numbers
    - Type of variables affects behavior of operators (shifting, comparison)

- **We can only represent so many numbers in** $w$ **bits**
  - When we exceed the limits, *arithmetic overflow* occurs
  - *Sign extension* tries to preserve value when expanding

- **Shifting is a useful bitwise operator**
  - Right shifting can be arithmetic (sign) or logical (0)
  - Can be used in multiplication with constant or bit masking
Practice Question

For the following expressions, find a value of `signed char x`, if there exists one, that makes the expression `TRUE`. Compare with your neighbor(s)!

- Assume we are using 8-bit arithmetic:
  - `x == (unsigned char) x`
  - `x >= 128U`
  - `x != (x>>2)<<2`
  - `x == -x`
    - Hint: there are two solutions
  - `(x < 128U) && (x > 0x3F)`
Some examples of using shift operators in combination with bitmasks, which you may find helpful for Lab 1.

- Extract the 2\textsuperscript{nd} most significant byte of an \texttt{int}
- Extract the sign bit of a signed \texttt{int}
- Conditionals as Boolean expressions
Using Shifts and Masks

- Extract the 2\textsuperscript{nd} most significant byte of an int:
  - First shift, then mask: $(x\gg 16) \& 0xFF$

<table>
<thead>
<tr>
<th>x</th>
<th>00000001 00000010 00000011 00000100</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x\gg 16$</td>
<td>00000000 00000000 00000001 00000010</td>
</tr>
<tr>
<td>0xFF</td>
<td>00000000 00000000 00000000 11111111</td>
</tr>
<tr>
<td>$(x\gg 16) &amp; 0xFF$</td>
<td>00000000 00000000 00000000 00000010</td>
</tr>
</tbody>
</table>

- Or first mask, then shift: $(x \& 0xFF0000) \gg 16$

<table>
<thead>
<tr>
<th>x</th>
<th>00000001 00000010 00000011 00000100</th>
</tr>
</thead>
<tbody>
<tr>
<td>0xFF0000</td>
<td>00000000 11111111 00000000 00000000</td>
</tr>
<tr>
<td>$x &amp; 0xFF0000$</td>
<td>00000000 00000010 00000000 00000000</td>
</tr>
<tr>
<td>$(x&amp;0xFF0000)\gg 16$</td>
<td>00000000 00000000 00000000 00000010</td>
</tr>
</tbody>
</table>
Using Shifts and Masks

- Extract the **sign bit** of a signed int:
  - First shift, then mask: \((x>>31) \& 0x1\)
    - Assuming arithmetic shift here, but this works in either case
    - Need mask to clear 1s possibly shifted in

<table>
<thead>
<tr>
<th></th>
<th>00000001 00000010 00000011 00000100</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>x</strong></td>
<td>10000001 00000010 00000011 00000100</td>
</tr>
<tr>
<td><strong>x&gt;&gt;31</strong></td>
<td>11111111 11111111 11111111 11111111</td>
</tr>
<tr>
<td><strong>0x1</strong></td>
<td>00000000 00000000 00000000 00000001</td>
</tr>
<tr>
<td>((x&gt;&gt;31) &amp; 0x1)</td>
<td>00000000 00000000 00000000 00000001</td>
</tr>
</tbody>
</table>
Using Shifts and Masks

- **Conditionals as Boolean expressions**
  - **For int** `x`, what does `(x<<31) >> 31` do?

<table>
<thead>
<tr>
<th>x=!!123</th>
<th>00000000 00000000 00000000 00000001</th>
</tr>
</thead>
<tbody>
<tr>
<td>x&lt;&lt;31</td>
<td>10000000 00000000 00000000 00000000</td>
</tr>
<tr>
<td>(x&lt;&lt;31)&gt;&gt;31</td>
<td>11111111 11111111 11111111 11111111</td>
</tr>
<tr>
<td>!x</td>
<td>00000000 00000000 00000000 00000000</td>
</tr>
<tr>
<td>!x&lt;&lt;31</td>
<td>00000000 00000000 00000000 00000000</td>
</tr>
<tr>
<td>(!x&lt;&lt;31)&gt;&gt;31</td>
<td>00000000 00000000 00000000 00000000</td>
</tr>
</tbody>
</table>

- Can use in place of conditional:
  - In C: `if(x) {a=y;} else {a=z;}` equivalent to `a=x?y:z;`
  - `a=((x<<31)>>31) & y) | ((!x<<31)>>31) & z);`