Integers II

CSE 351 Autumn 2020

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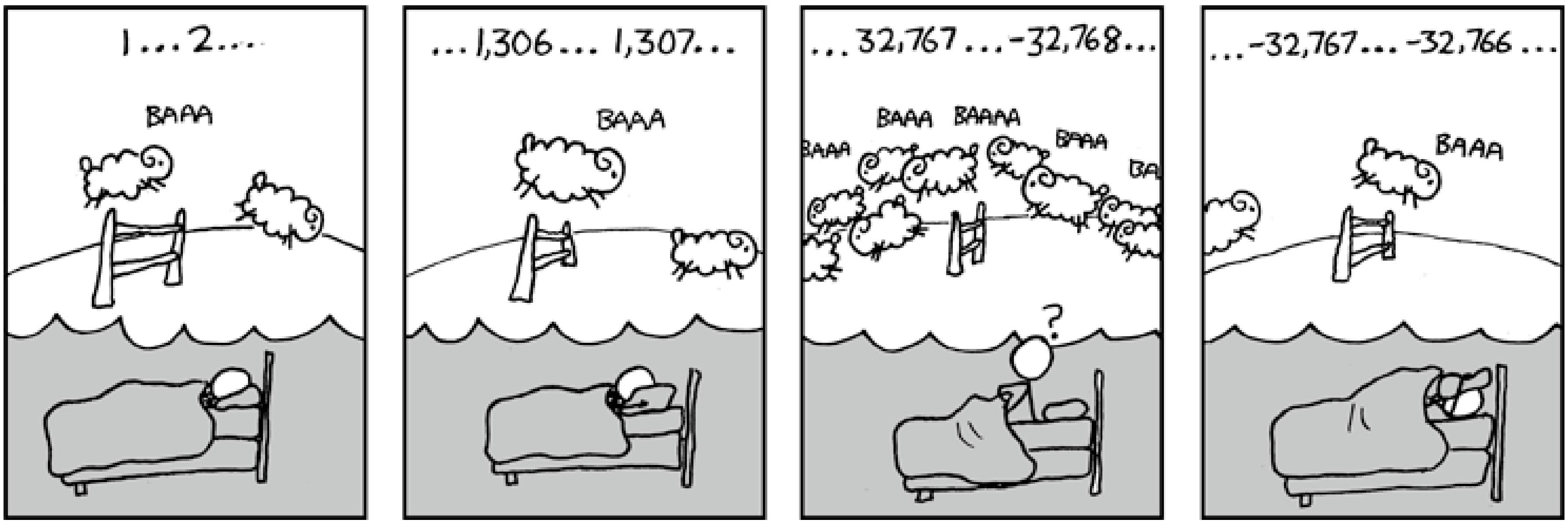


http://xkcd.com/571/

Administrivia

- hw4 due 10/12, hw5 due 10/14
- Lab 1a due Monday (10/12)
  - Submit `pointer.c` and `lab1Areflect.txt` to Gradescope
- Lab 1b released tomorrow, due 10/19
  - Bit manipulation on a custom number representation
  - Bonus slides at the end of today's lecture have relevant examples

Runnable Code Snippets on Ed

- Ed allows you to embed runnable code snippets (*e.g.*, readings, homework, discussion)
  - These are *editable* and *rerunnable*!
  - Hide compiler warnings, but will show compiler errors and runtime errors

- Suggested use
  - Good for experimental questions about basic behaviors in C
  - *NOT* entirely consistent with the CSE Linux environment, so should not be used for any lab-related work

Reading Review

- Terminology:
  - `UMin`, `UMax`, `TMin`, `TMax`
  - Type casting: implicit vs. explicit
  - Integer extension: zero extension vs. sign extension
  - Modular arithmetic and arithmetic overflow
  - Bit shifting: left shift, logical right shift, arithmetic right shift
- Questions from the Reading?

Review Questions

signed

most negative

represent $2^6 = 64$ numbers

- What is the value (and encoding) of `TMin` for a fictional 6-bit wide integer data type? $-2^{n-1} = -2^5 = \boxed{-32}$

  0b $\underset{-2^5}{1}\ \underset{2^4}{0}\ \underset{2^3}{0}\ \underset{2^2}{0}\ \underset{2^1}{0}\ \underset{2^0}{0}$

- For `unsigned char uc = 0xA1;`, what are the produced data for the cast `(short)uc`?

  signed, 2 bytes

  extension is based on original type: unsigned → zero extension $\boxed{\text{0x00A1}}$

- What is the result of the following expressions?
  - `(signed char)uc >> 2`
  - `(unsigned char)uc >> 3`

signed: 0b 1010 ~~0001~~ $\xrightarrow{\text{arithmetic}}$ 0b 1110 1000 = $\boxed{\text{0xE8}}$

unsigned: 0b 1010 ~~0001~~ $\xrightarrow{\text{logical}}$ 0b 0001 0100 = $\boxed{\text{0x14}}$

Why Does Two's Complement Work?

- For all representable positive integers x, we want:

additive inverse

$$
\begin{array}{r}
\textit{bit representation of } \; x \\
+\ \textit{bit representation of } -x \\
\hline
0
\end{array}
$$

(ignoring the carry-out bit)

  - What are the 8-bit negative encodings for the following?

$$
\begin{array}{r}
00000001 \\
11111111 \\
+\ ????????? \\
\hline
\cancel{1}\,00000000
\end{array}
\qquad
\begin{array}{r}
00000010 \\
11111110 \\
+\ ????????? \\
\hline
\cancel{1}\,00000000
\end{array}
\qquad
\begin{array}{r}
1111111 \\
11000011 \\
00111101 \\
+\ ????????? \\
\hline
\cancel{1}\,00000000
\end{array}
$$

Why Does Two's Complement Work?

- For all representable positive integers x, we want:

$$\begin{array}{r} \textit{bit representation of } x \\ +\ \textit{bit representation of } -x \\ \hline 0 \end{array}$$ (ignoring the carry-out bit)

0b1…1

$x + (\sim x) =$

$x + (\sim x) = -1$

$x + (\sim x + 1) = 0$

$\boxed{-x = \sim x + 1}$

- What are the 8-bit negative encodings for the following?

$$\begin{array}{r} 00000001 \\ +\ 11111111 \\ \hline \cancel{1}00000000 \end{array} \quad \begin{array}{r} 00000010 \\ +\ 11111110 \\ \hline \cancel{1}00000000 \end{array} \quad \begin{array}{r} 11000011 \\ +\ 00111101 \\ \hline \cancel{1}00000000 \end{array}$$

These are the bitwise complement plus 1!

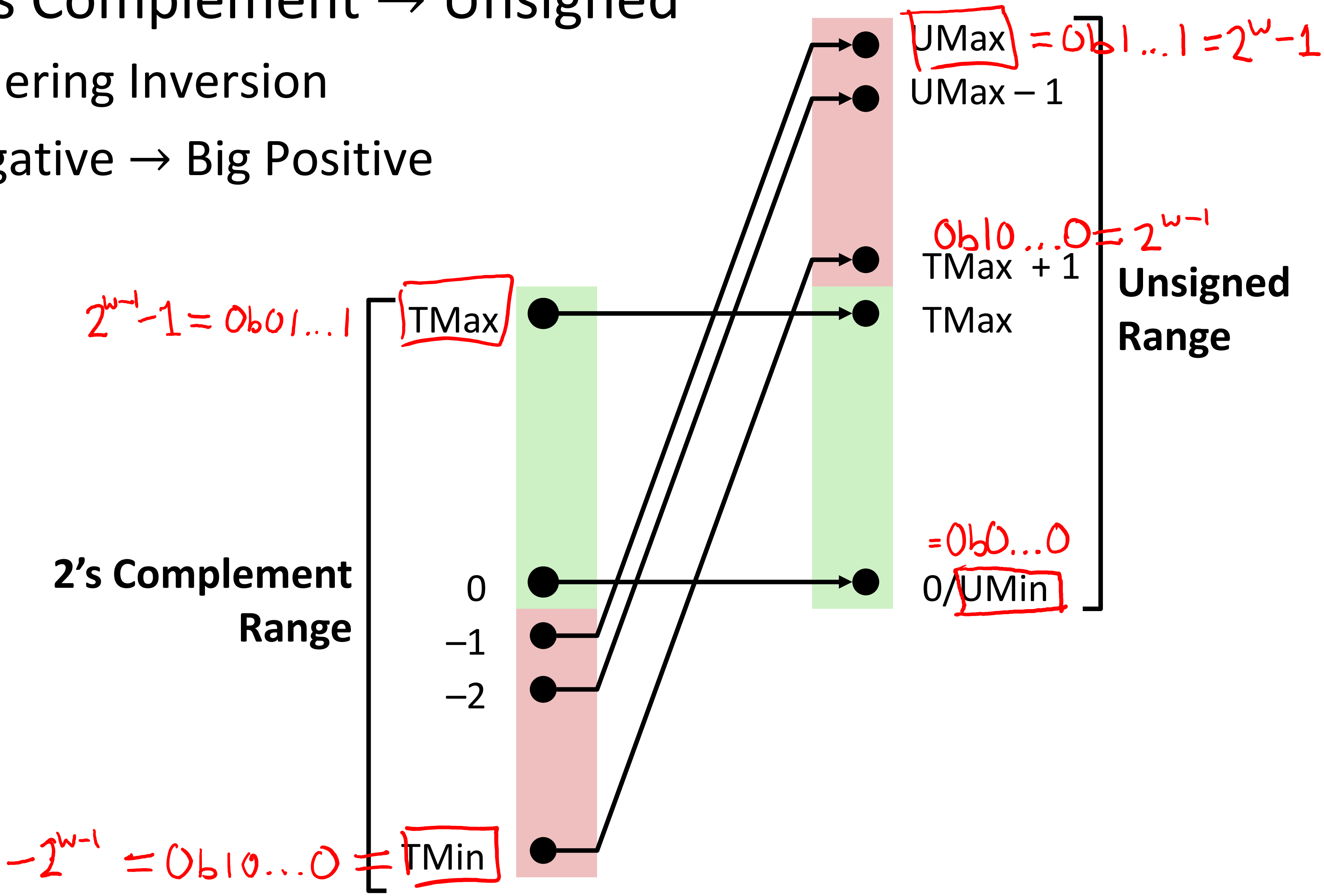
`-x == ~x + 1`

Integers

- **Binary representation of integers**
  - **Unsigned and signed**
  - **Casting in C**
- Consequences of finite width representations
  - Sign extension, overflow
- Shifting and arithmetic operations

Signed/Unsigned Conversion Visualized

- Two's Complement → Unsigned
  - Ordering Inversion
  - Negative → Big Positive



Values To Remember

- Unsigned Values
  - UMin = `0b00…0` = 0
  - UMax = `0b11…1` = $2^w - 1$
- Two's Complement Values
  - TMin = `0b10…0` = -2^{w-1}
  - TMax = `0b01…1` = $2^{w-1} - 1$
  - -1 = `0b11…1`
- **Example:** Values for $w = 64$

	Decimal	Hex
UMax	18,446,744,073,709,551,615	FF FF FF FF FF FF FF FF
TMax	9,223,372,036,854,775,807	7F FF FF FF FF FF FF FF
TMin	-9,223,372,036,854,775,808	80 00 00 00 00 00 00 00
-1	-1	FF FF FF FF FF FF FF FF
0	0	00 00 00 00 00 00 00 00

In C: Signed vs. Unsigned

- Casting
  - Bits are unchanged, just interpreted differently!
    - **int** tx, ty;
    - **unsigned int** ux, uy;
  - *Explicit* casting
    - tx = (**int**) ux;
    - uy = (**unsigned int**) ty;

(new_type) expression

  - *Implicit* casting can occur during assignments or function calls

cast to target variable/parameter type

    - tx = ux;
    - uy = ty;

(also implicitly occurs with printf format specifiers)

Casting Surprises

!!!

- Integer literals (constants)
  - By default, integer constants are considered *signed* integers
    - Hex constants already have an explicit binary representation
  - Use "U" (or "u") suffix to explicitly force *unsigned*
    - **Examples:** 0U, 4294967259u

- Expression Evaluation
  - When you mixed unsigned and signed in a single expression, then **signed values are implicitly cast to unsigned**
  - Including comparison operators <, >, ==, <=, >=

(unsigned "dominates")

Practice Question 1

- Assuming 8-bit data (*i.e.*, bit position 7 is the MSB), what will the following expression evaluate to?
  - UMin = 0, UMax = 255, TMin = -128, TMax = 127

signed

- `127 < (signed char) 128u`

$\underbrace{\texttt{(signed char) 128u}}_{\text{signed}}$ ← both sides are signed, so signed comparison

0b0111 1111 0b1000 0000

signed comparison:

0b0111 1111 0b1000 0000

127 < -128

$\boxed{\text{False}}$

unsigned comparison:

127 < 128

True

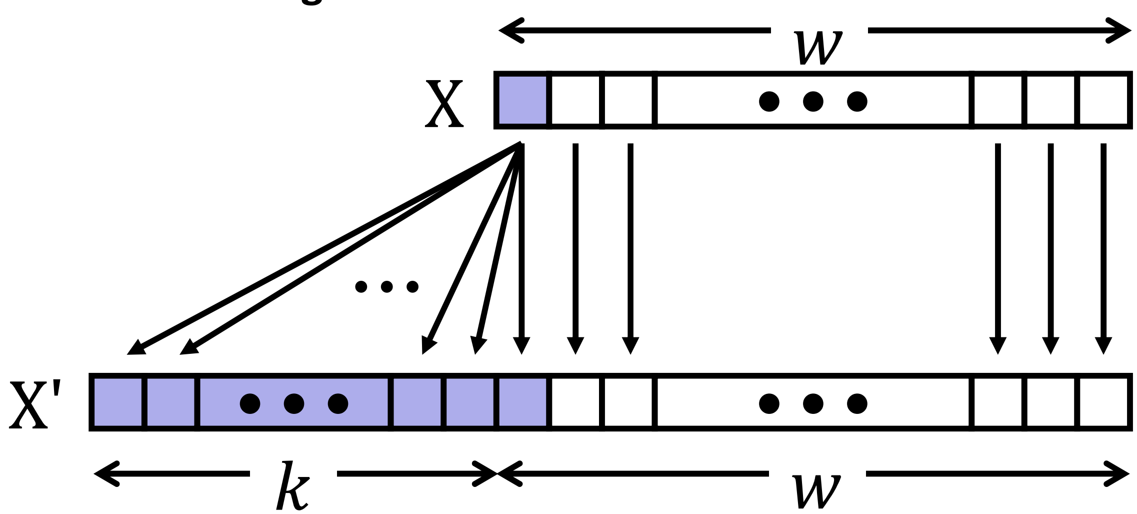
(e.g., if LHS was 127u)

Integers

- Binary representation of integers
  - Unsigned and signed
  - Casting in C
- **Consequences of finite width representations**
  - **Sign extension, overflow**
- Shifting and arithmetic operations

Sign Extension

- **Task:** Given a w-bit signed integer X, convert it to w+k-bit signed integer X' *with the same value*
- **Rule: Add k copies of sign bit**
  - Let x_i be the i-th digit of X in binary
  - $\mathrm{X}' = \underbrace{x_{w-1}, \dots, x_{w-1}}_{k \textbf{ copies of MSB}}, \underbrace{x_{w-1}, x_{w-2}, \dots, x_1, x_0}_{\textbf{original } \mathrm{X}}$



Two's Complement Arithmetic

- The same addition procedure works for both unsigned and two's complement integers
  - **Simplifies hardware:** only one algorithm for addition
  - **Algorithm:** simple addition, discard the highest carry bit
    - Called modular addition: result is sum *modulo* 2^w

Arithmetic Overflow

Bits	Unsigned	Signed
0000	0 UMin	0
0001	1	1
0010	2	2
0011	3	3
0100	4	4
0101	5	5
0110	6	6
0111	7	7 TMax
1000	8	-8 TMin
1001	9	-7
1010	10	-6
1011	11	-5
1100	12	-4
1101	13	-3
1110	14	-2
1111	15 UMax	-1

- When a calculation produces a result that can't be represented in the current encoding scheme
  - Integer range limited by fixed width UMin – UMax TMin – TMax
  - Can occur in both the positive and negative directions
- C and Java ignore overflow exceptions
  - You end up with a bad value in your program and no warning/indication… oops!

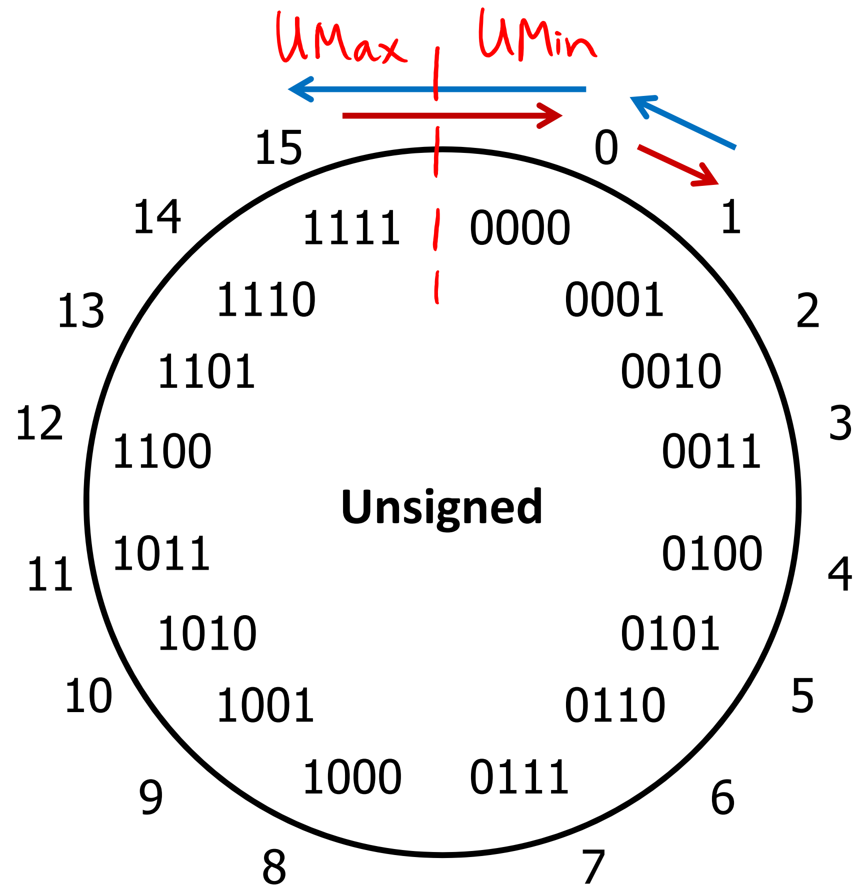
Overflow: Unsigned

- **Addition:** drop carry bit (-2^N)

$$\begin{array}{r} 15 \\ +\ \ 2 \\ \hline \cancel{17} \\ \color{red}{1} \end{array} \qquad \begin{array}{r} 1111 \\ +\ \ 0010 \\ \hline \cancel{1}0001 \end{array}$$

- **Subtraction:** borrow ($+2^N$)

$$\begin{array}{r} 1 \\ -\ \ 2 \\ \hline \cancel{-1} \\ \color{blue}{15} \end{array} \qquad \begin{array}{r} {\scriptstyle 1\ 1} \\ {\scriptstyle \cancel{2}\,\cancel{2}\,2} \\ \cancel{1}0001 \\ -\ \ 0010 \\ \hline 1111 \end{array}$$



$\pm 2^N$ because of modular arithmetic

$2^4 = 16$

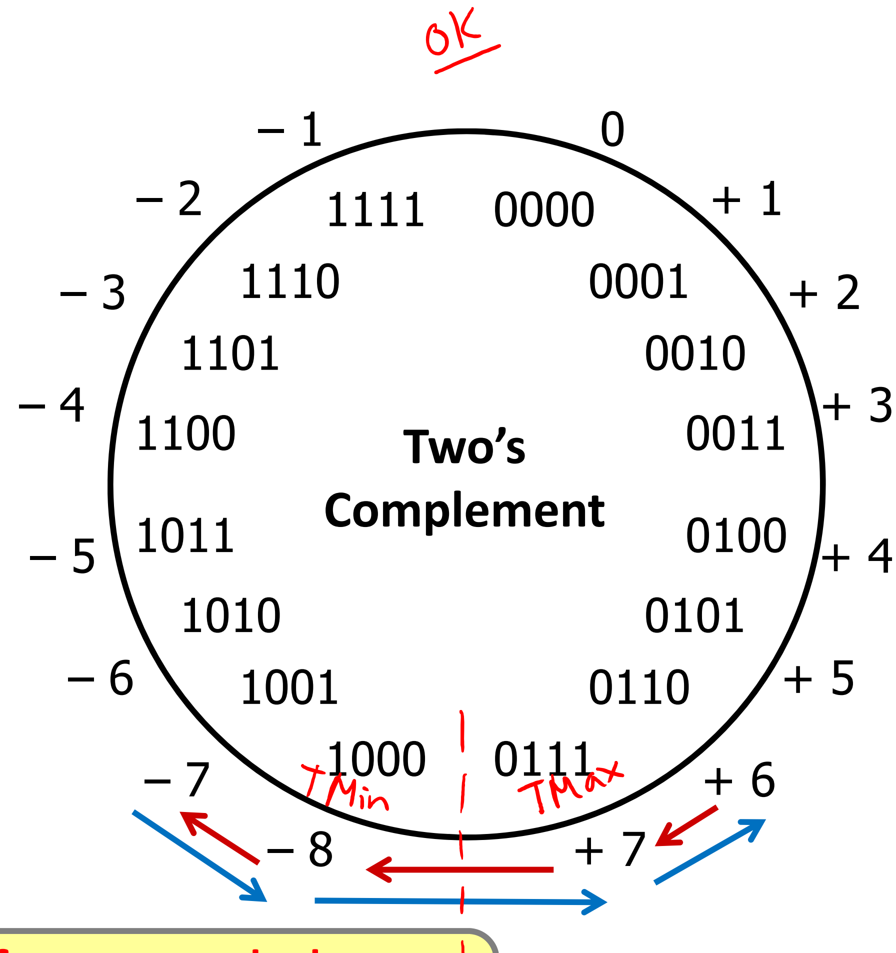
Overflow: Two's Complement

- **Addition:** (+) + (+) = (−) result?

$$\begin{array}{r} 6 \\ +\ 3 \\ \hline \cancel{9} \\ -7 \end{array} \qquad \begin{array}{r} 0110 \\ +\ 0011 \\ \hline 1001 \end{array}$$

- **Subtraction:** (−) + (−) = (+)?

$$\begin{array}{r} -7 \\ -\ 3 \\ \hline \cancel{-10} \\ 6 \end{array} \qquad \begin{array}{r} 1001 \\ -\ 0011 \\ \hline 0110 \end{array}$$



For signed: overflow if operands have same sign and result's sign is different

Practice Questions 2

- Assuming 8-bit integers:
  - `0x27` = 39 (signed) = 39 (unsigned)
  - `0xD9` = -39 (signed) = 217 (unsigned)
  - `0x7F` = 127 (signed) = 127 (unsigned)
  - `0x81` = -127 (signed) = 129 (unsigned)

- For the following additions, did signed and/or unsigned overflow occur?
  - `0x27 + 0x81`
  - `0x7F + 0xD9`

Integers

- Binary representation of integers
  - Unsigned and signed
  - Casting in C
- Consequences of finite width representations
  - Sign extension, overflow
- **Shifting and arithmetic operations**

Shift Operations

- Throw away (drop) extra bits that "fall off" the end
- Left shift (`x<<n`) bit vector `x` by `n` positions
  - Fill with `0`'s on right
- Right shift (`x>>n`) bit-vector `x` by `n` positions
  - Logical shift (for unsigned values)
    - Fill with `0`'s on left
  - Arithmetic shift (for signed values)
    - Replicate most significant bit on left (maintains sign of `x`)

8-bit example:

	x	0010 0010
	x<<3	0001 0000
logical:	x>>2	0000 1000
arithmetic:	x>>2	0000 1000

	x	1010 0010
	x<<3	0001 0000
logical:	x>>2	0010 1000
arithmetic:	x>>2	1110 1000

Shift Operations

digit $d_i \times 2^i$ changes power of 2 by n because it moved positions

- Arithmetic:
  - Left shift (`x<<n`) is equivalent to multiply by 2^n
  - Right shift (`x>>n`) is equivalent to divide by 2^n
  - Shifting is faster than general multiply and divide operations! (compiler will try to optimize for you)

- Notes:
  - Shifts by `n<0` or `n≥w` (`w` is bit width of `x`) are *undefined* behavior not guaranteed
  - **In C:** behavior of `>>` is determined by the compiler
    - In gcc / C lang, depends on data type of `x` (signed/unsigned) arithmetic/logical
  - **In Java:** logical shift is `>>>` and arithmetic shift is `>>`

Left Shifting Arithmetic 8-bit Example

- No difference in left shift operation for unsigned and signed numbers (just manipulates bits)
  - Difference comes during interpretation: `x*2^n`?

			Signed	Unsigned
`x = 25;`	`00011001`	=	25	25
`L1=x<<2;`	`~~00~~011001 00`	=	100	100
`L2=x<<3;`	`~~000~~11001 000`	=	-56	200
`L3=x<<4;`	`~~0001~~1001 0000`	=	-112	144

200
−256 → 2^8

signed overflow

400
−256 → 2^8

unsigned overflow

Right Shifting Arithmetic 8-bit Examples

- **Reminder:** C operator >> does *logical* shift on unsigned values and *arithmetic* shift on signed values
  - Logical Shift: `x/2`^{n}?

xu = 240u; 11110000 = 240 /8 = 30

R1u=xu>>3; 00011110~~000~~ = 30 /4 = 7.5

R2u=xu>>5; 00000111~~10000~~ = 7

rounding (down)

Right Shifting Arithmetic 8-bit Examples

- **Reminder:** C operator >> does *logical* shift on unsigned values and *arithmetic* shift on signed values
  - Arithmetic Shift: `x/2`^{n}?

`xs = -16;` 11110000 = -16

`R1s=xu>>3;` 11111110~~000~~ = -2 $/4 = -0.5$

`R2s=xu>>5;` 11111111~~10000~~ = -1

rounding (down)

Challenge Questions

UMin = 0 , UMax = 255
8-bits, so TMin = -128, TMax = 127

For the following expressions, find a value of `signed char x`, if there exists one, that makes the expression True.

- Assume we are using 8-bit arithmetic:

Expression	Example:	All solutions:
`x == (unsigned char) x` (unsigned)	x = 0	works for all x
`x >= 128U` (unsigned) 0b1000 0000	x = -1	any x < 0
`x != (x>>2)<<2`	x = 3	any x where lowest two bits are not 0b00
`x == -x` • Hint: there are two solutions	x = 0	① x = 0b0...0 = 0 ② x = 0b10...0 = -128
`(x < 128U) && (x > 0x3F)`		any x where upper two bits are exactly 0b01

Summary

- Sign and unsigned variables in C
  - Bit pattern remains the same, just *interpreted* differently
  - Strange things can happen with our arithmetic when we convert/cast between sign and unsigned numbers
    - Type of variables affects behavior of operators (shifting, comparison)
- We can only represent so many numbers in *w* bits
  - When we exceed the limits, *arithmetic overflow* occurs
  - *Sign extension* tries to preserve value when expanding
- Shifting is a useful bitwise operator
  - Right shifting can be arithmetic (sign) or logical (0)
  - Can be used in multiplication with constant or bit masking

BONUS SLIDES

Some examples of using shift operators in combination with bitmasks, which you may find helpful for Lab 1b.

- Extract the 2nd most significant byte of an `int`
- Extract the sign bit of a signed `int`
- Conditionals as Boolean expressions

Using Shifts and Masks

- Extract the 2nd most significant *byte* of an `int`:
  - First shift, then mask: `(x>>16) & 0xFF`

x	00000001 00000010 00000011 00000100
x>>16	00000000 00000000 00000001 00000010
0xFF	00000000 00000000 00000000 11111111
(x>>16) & 0xFF	00000000 00000000 00000000 00000010

  - Or first mask, then shift: `(x & 0xFF0000)>>16`

x	00000001 00000010 00000011 00000100
0xFF0000	00000000 11111111 00000000 00000000
x & 0xFF0000	00000000 00000010 00000000 00000000
(x&0xFF0000)>>16	00000000 00000000 00000000 00000010

Using Shifts and Masks

- Extract the *sign bit* of a signed `int`:
  - First shift, then mask: `(x>>31) & 0x1`
    - Assuming arithmetic shift here, but this works in either case
    - Need mask to clear `1`s possibly shifted in

x	00000001 00000010 00000011 00000100
x>>31	00000000 00000000 00000000 00000000
0x1	00000000 00000000 00000000 00000001
(x>>31) & 0x1	00000000 00000000 00000000 00000000

x	10000001 00000010 00000011 00000100
x>>31	11111111 11111111 11111111 11111111
0x1	00000000 00000000 00000000 00000001
(x>>31) & 0x1	00000000 00000000 00000000 00000001

Using Shifts and Masks

- Conditionals as Boolean expressions
  - For **`int`** `x`, what does `(x<<31)>>31` do?

`x=!!123`	00000000 00000000 00000000 00000001
`x<<31`	10000000 00000000 00000000 00000000
`(x<<31)>>31`	11111111 11111111 11111111 11111111
`!x`	00000000 00000000 00000000 00000000
`!x<<31`	00000000 00000000 00000000 00000000
`(!x<<31)>>31`	00000000 00000000 00000000 00000000

  - Can use in place of conditional:
    - In C: `if(x) {a=y;} else {a=z;}` equivalent to `a=x?y:z;`
    - `a=(((x<<31)>>31)&y) | (((!x<<31)>>31)&z);`