Floating Point I

CSE 351 Spring 2019

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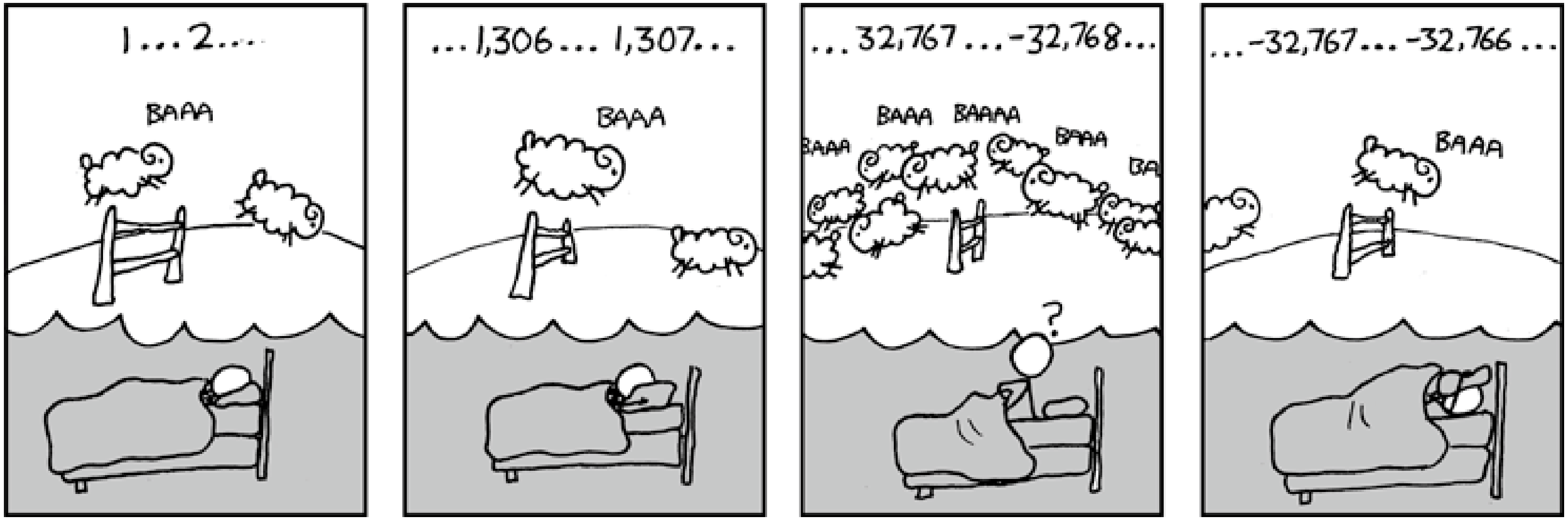


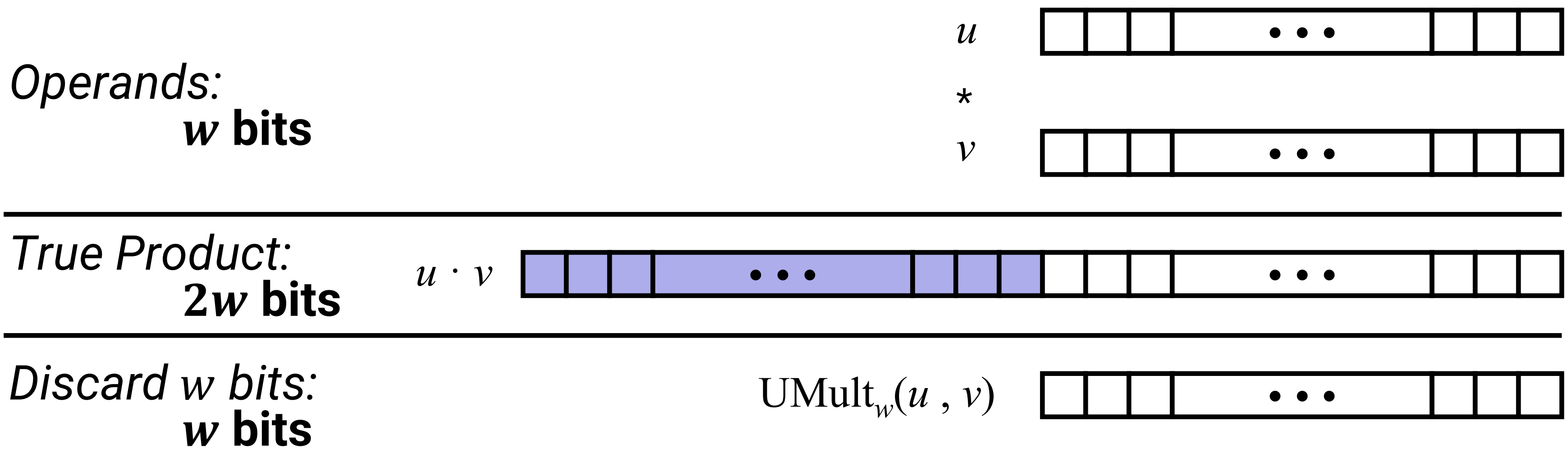
http://xkcd.com/571/

Administrivia

- Lab 1a due Monday 4/15 at 11:59 pm
  - Submit `pointer.c` and `lab1Areflect.txt`
- Lab 1b due Monday (4/22)
  - Submit `bits.c` and `lab1Breflect.txt`

- Homework 2 coming soon, due Wednesday (4/24)
  - On Integers, Floating Point, and x86-64

Unsigned Multiplication in C



- Standard Multiplication Function
  - Ignores high order *w* bits
- Implements Modular Arithmetic
  - $\text{UMult}_w(u\ ,\ v) = u \cdot v \mod 2^w$

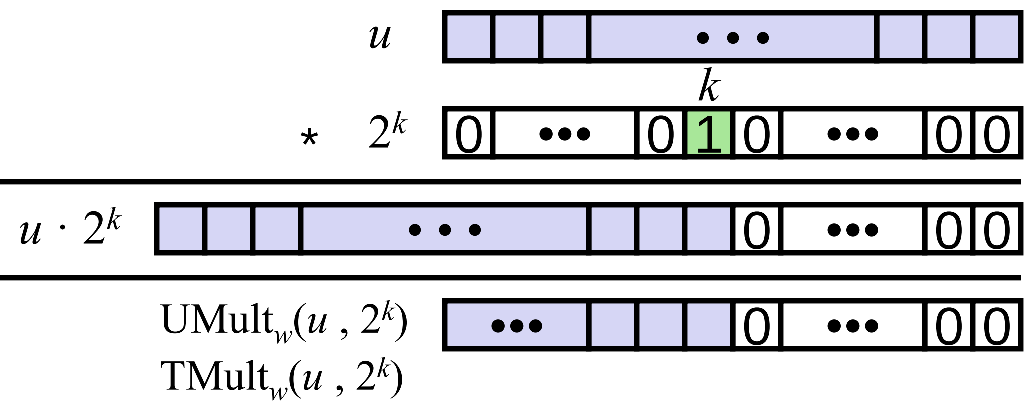
Multiplication with shift and add

- Operation `u<<k` gives `u*`2^k
  - Both signed and unsigned

Operands: ***w* bits**

True Product: ***w + k* bits**

Discard k bits: ***w* bits**



- <u>Examples:</u>
  - `u<<3           ==  u * 8`
  - `u<<5 - u<<3    ==  u * 24`
  - Most machines shift and add faster than multiply
    - ***Compiler generates this code automatically***

Number Representation Revisited

- We know how to represent:
  - Signed and Unsigned Integers
  - Characters (ASCII)
  - Addresses
- How do we encode the following:
  - Real numbers (*e.g.* 3.14159)
  - Very large numbers (*e.g.* 6.02×10^{23})
  - Very small numbers (*e.g.* 6.626×10^{-34})
  - Special numbers (*e.g.* ∞, NaN)

Floating Point

Floating Point Topics

- **Fractional binary numbers**
- IEEE floating-point standard
- Floating-point operations and rounding
- Floating-point in C

- There are many more details that we won’t cover
  - It’s a 58-page standard…

Floating Point Summary

- Floats also suffer from the fixed number of bits available to represent them
  - Can get overflow/underflow, just like `int`s
  - "Gaps" produced in representable numbers means we can lose precision, unlike `int`s
    - Some "simple fractions" have no exact representation (*e.g.* 0.2)
    - "Every operation gets a slightly wrong result"
- Floating point arithmetic not associative or distributive
  - *Mathematically* equivalent ways of writing an expression may compute different results
- Never test floating point values for equality!
- Careful when converting between `int`s and `float`s!

Representation of Fractions

- "Binary Point," like decimal point, signifies boundary between integer and fractional parts:

Example 6-bit representation: xx.yyyy

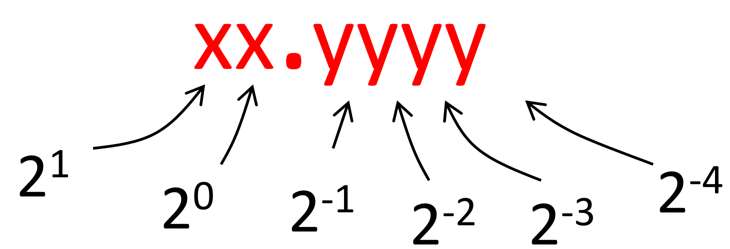
2^{1} 2^{0} 2^{-1} 2^{-2} 2^{-3} 2^{-4}

- Example: $10.1010_2 = 1\times2^{1} + 1\times2^{-1} + 1\times2^{-3} = 2.625_{10}$

Representation of Fractions

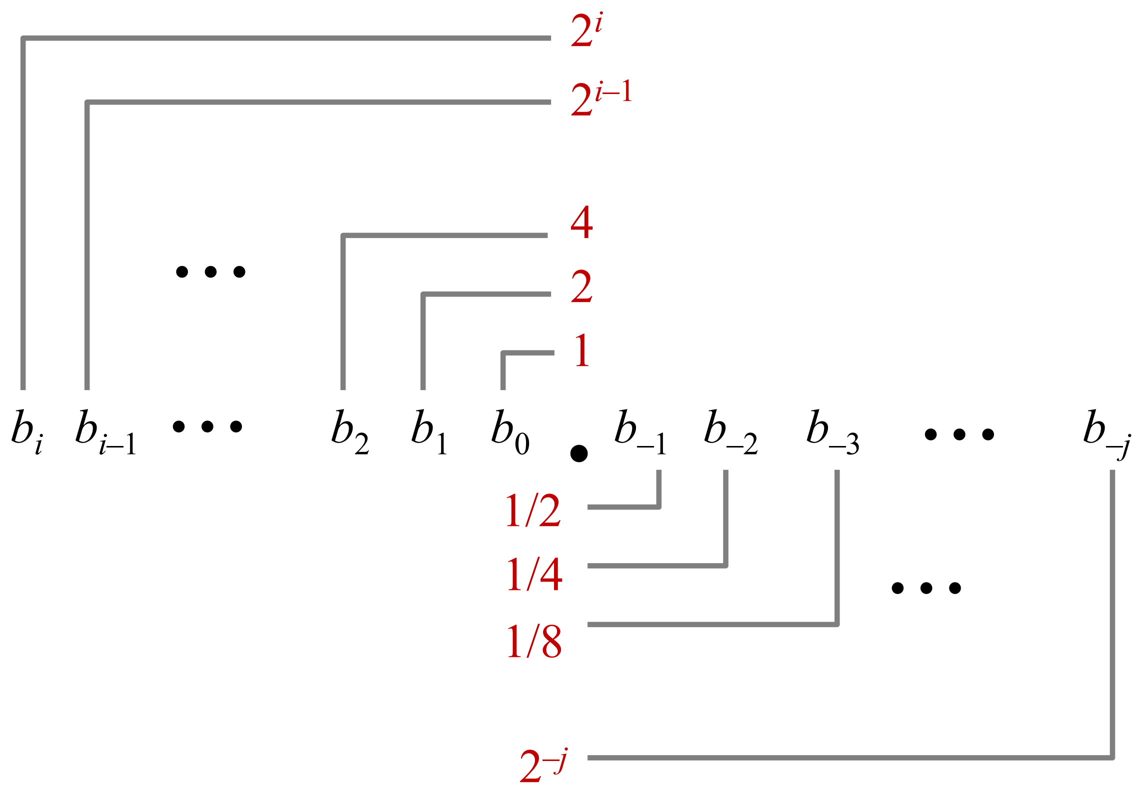
- "Binary Point," like decimal point, signifies boundary between integer and fractional parts:

Example 6-bit representation:



- In this 6-bit representation:
  - What is the encoding and value of the smallest (most negative) number?
  - What is the encoding and value of the largest (most positive) number?
  - What is the smallest number greater than 2 that we can represent?

Fractional Binary Numbers



- **Representation**
  - Bits to right of "binary point" represent fractional powers of 2
  - Represents rational number: $\sum_{k=-j}^{i} b_k \cdot 2^k$

Fractional Binary Numbers

- Value Representation
  - 5 and 3/4 `101.11`$_2$
  - 2 and 7/8 `10.111`$_2$
  - 47/64 `0.101111`$_2$
- Observations
  - Shift left = multiply by power of 2
  - Shift right = divide by power of 2
  - Numbers of the form **`0.111111`**…$_2$ are just below 1.0
    - $1/2 + 1/4 + 1/8 + \ldots + 1/2^i + \ldots \rightarrow 1.0$
    - Use notation $1.0 - \varepsilon$

Limits of Representation

- Limitations:
  - Even given an arbitrary number of bits, can only **exactly** represent numbers of the form $x * 2^y$ (y can be negative)
  - Other rational numbers have repeating bit representations

Value:	**Binary Representation:**
• 1/3 = $0.333333\ldots_{10}$ =	$0.01010101[01]\ldots_2$
• 1/5 =	$0.001100110011[0011]\ldots_2$
• 1/10 =	$0.0001100110011[0011]\ldots_2$

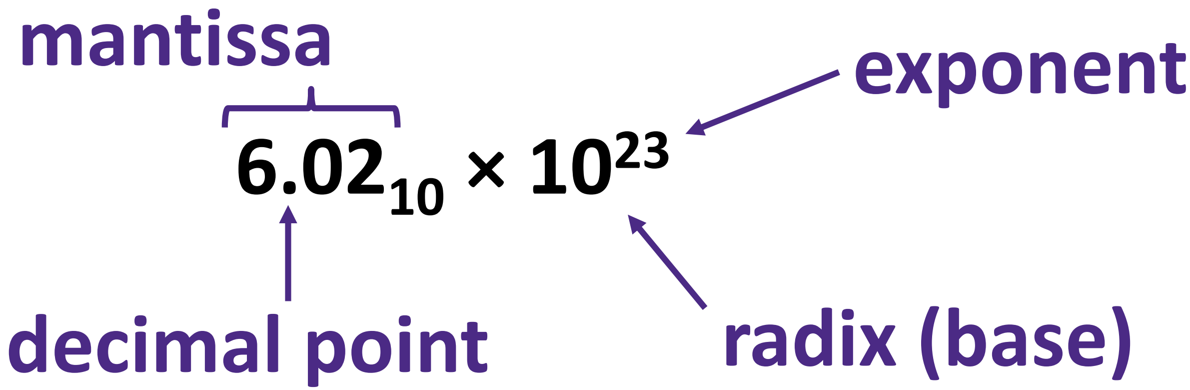
Fixed Point Representation

- Implied binary point. Two example schemes:
  - #1: the binary point is between bits 2 and 3
    - $b_7\ b_6\ b_5\ b_4\ b_3$ [.] $b_2\ b_1\ b_0$
  - #2: the binary point is between bits 4 and 5
    - $b_7\ b_6\ b_5$ [.] $b_4\ b_3\ b_2\ b_1\ b_0$
- Wherever we put the binary point, with fixed point representations there is a trade off between the amount of range and precision we have
- Fixed point = fixed *range* and fixed *precision*
  - range: difference between largest and smallest numbers possible
  - precision: smallest possible difference between any two numbers
- Hard to pick how much you need of each!

Floating Point Representation

- Analogous to scientific notation
  - In Decimal:
    - Not 12000000, but 1.2×10^7 In C: 1.2e7
    - Not 0.0000012, but 1.2×10^{-6} In C: 1.2e-6
  - In Binary:
    - Not 11000.000, but 1.1×2^4
    - Not 0.000101, but 1.01×2^{-4}
- We have to divvy up the bits we have (e.g., 32) among:
  - the sign (1 bit)
  - the mantissa (significand)
  - the exponent

Scientific Notation (Decimal)



- *Normalized form*: exactly one digit (non-zero) to left of decimal point
- Alternatives to representing 1/1,000,000,000
  - Normalized: 1.0×10^{-9}
  - Not normalized: $0.1 \times 10^{-8}, 10.0 \times 10^{-10}$

Scientific Notation (Binary)

mantissa

exponent

$$1.01_2 \times 2^{-1}$$

binary point

radix (base)

- Computer arithmetic that supports this called floating point due to the "floating" of the binary point
  - Declare such variable in C as `float` (or `double`)

Scientific Notation Translation

- Convert from scientific notation to binary point
  - Perform the multiplication by shifting the decimal until the exponent disappears
    - Example: $1.011_2 \times 2^4 = 10110_2 = 22_{10}$
    - Example: $1.011_2 \times 2^{-2} = 0.01011_2 = 0.34375_{10}$
- Convert from binary point to *normalized* scientific notation
  - Distribute out exponents until binary point is to the right of a single digit
    - Example: $1101.001_2 = 1.101001_2 \times 2^3$
- **Practice:** Convert 11.375_{10} to binary scientific notation

Floating Point Topics

- Fractional binary numbers
- **IEEE floating-point standard**
- Floating-point operations and rounding
- Floating-point in C

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  - It's a 58-page standard…

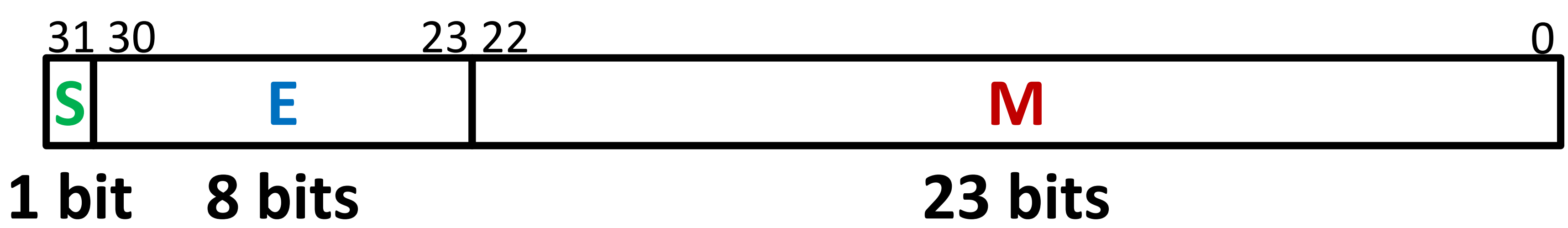
IEEE Floating Point

- IEEE 754
  - Established in 1985 as uniform standard for floating point arithmetic
  - Main idea: make numerically sensitive programs portable
  - Specifies two things: representation and result of floating operations
  - Now supported by all major CPUs

- Driven by numerical concerns
  - **Scientists**/numerical analysts want them to be as **real** as possible
  - **Engineers** want them to be **easy to implement** and **fast**
  - In the end:
    - Scientists mostly won out
    - Nice standards for rounding, overflow, underflow, but...
    - Hard to make fast in hardware
    - **Float operations can be an order of magnitude slower than integer ops**

Floating Point Encoding

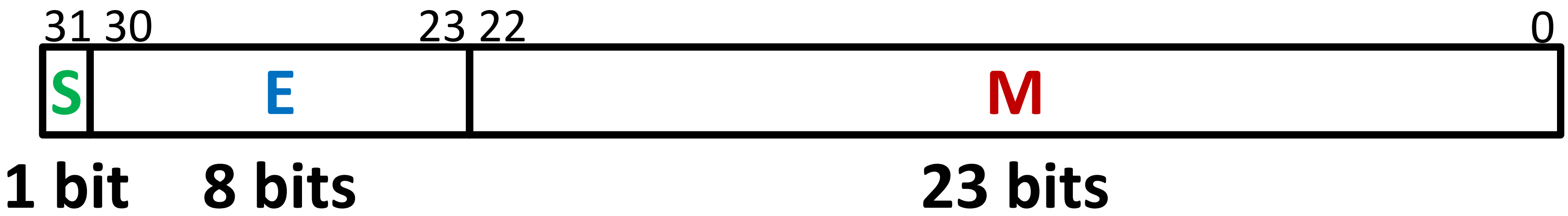
- Use normalized, base 2 scientific notation:
  - Value: $\pm 1 \times \text{Mantissa} \times 2^{\text{Exponent}}$
  - Bit Fields: $(-1)^{S} \times 1.M \times 2^{(E-\text{bias})}$
- Representation Scheme:
  - Sign bit (0 is positive, 1 is negative)
  - Mantissa (a.k.a. significand) is the fractional part of the number in normalized form and encoded in bit vector **M**
  - Exponent weights the value by a (possibly negative) power of 2 and encoded in the bit vector **E**



The Exponent Field

- Use biased notation
  - Read exponent as unsigned, but with *bias* of $2^{w-1}-1$ = 127
  - Representable exponents roughly ½ positive and ½ negative
  - Exponent 0 (Exp = 0) is represented as E = 0b 0111 1111
- Why biased?
  - Makes floating point arithmetic easier
  - Makes somewhat compatible with two's complement
- **Practice:** To encode in biased notation, add the bias then encode in unsigned:
  - Exp = 1 → → E = 0b
  - Exp = 127 → → E = 0b
  - Exp = -63 → → E = 0b

The Mantissa (Fraction) Field



$$(-1)^{S} \times (1\ .\ M) \times 2^{(E-\text{bias})}$$

- Note the implicit 1 in front of the M bit vector
  - Example: 0b 0011 1111 1100 0000 0000 0000 0000 0000
    is read as $1.1_2 = 1.5_{10}$, *not* $0.1_2 = 0.5_{10}$
  - Gives us an extra bit of *precision*
- Mantissa "limits"
  - Low values near M = 0b0…0 are close to 2^{Exp}
  - High values near M = 0b1…1 are close to 2^{Exp+1}

Peer Instruction Question

- What is the correct value encoded by the following floating point number?
  - 0b 0 10000000 11000000000000000000000
  - Vote at http://pollev.com/rea

A. + 0.75

B. + 1.5

C. + 2.75

D. + 3.5

E. We're lost…

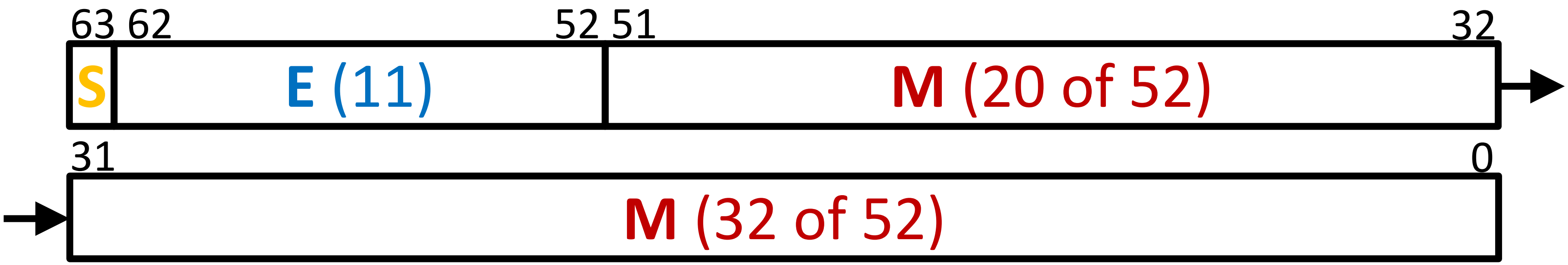
Precision and Accuracy

- Precision is a count of the number of bits in a computer word used to represent a value
  - Capacity for accuracy
- Accuracy is a measure of the difference between the *actual value of a number* and its computer representation

  - *High precision permits high accuracy but doesn't guarantee it. It is possible to have high precision but low accuracy.*
  - **Example:** `float pi = 3.14;`
    - `pi` will be represented using all 24 bits of the mantissa (highly precise), but is only an approximation (not accurate)

Need Greater Precision?

- Double Precision (vs. Single Precision) in 64 bits

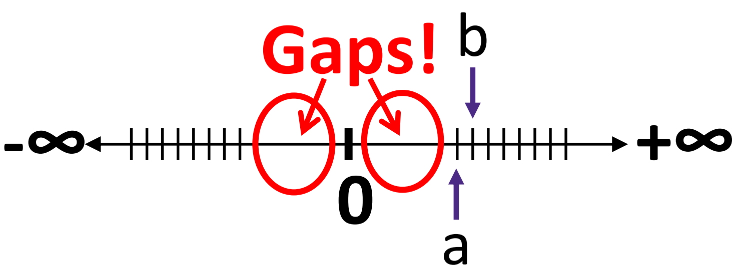


  - C variable declared as `double`
  - Exponent bias is now $2^{10}-1 = 1023$
  - **Advantages:** greater precision (larger mantissa), greater range (larger exponent)
  - **Disadvantages:** more bits used, slower to manipulate

Representing Very Small Numbers

- But wait… what happened to zero?
  - Using standard encoding 0x00000000 =
  - *Special case:* E and M all zeros = 0
    - Two zeros! But at least 0x00000000 = 0 like integers

- New numbers closest to 0:
  - $a = 1.0…0_2 \times 2^{-126} = 2^{-126}$
  - $b = 1.0…01_2 \times 2^{-126} = 2^{-126} + 2^{-149}$
  - Normalization and implicit 1 are to blame
  - *Special case:* E = 0, M ≠ 0 are denormalized numbers



Denorm Numbers

This is extra (non-testable) material

- Denormalized numbers
  - No leading 1
  - Uses implicit exponent of –126 even though E = 0x00

- Denormalized numbers close the gap between zero and the smallest normalized number
  - Smallest norm: $\pm\ 1.0\ldots0_{two} \times 2^{-126} = \pm\ 2^{-126}$
  - Smallest denorm: $\pm\ 0.0\ldots01_{two} \times 2^{-126} = \pm\ 2^{-149}$
    - There is still a gap between zero and the smallest denormalized number

So much closer to 0

Other Special Cases

- E = 0xFF, M = 0: ± ∞
  - *e.g.* division by 0
  - Still work in comparisons!
- E = 0xFF, M ≠ 0: Not a Number (NaN)
  - *e.g.* square root of negative number, 0/0, ∞–∞
  - NaN propagates through computations
  - Value of M can be useful in debugging
- New largest value (besides ∞)?
  - E = 0xFF has now been taken!
  - E = 0xFE has largest: $1.1\ldots1_2 \times 2^{127} = 2^{128} - 2^{104}$

Floating Point Encoding Summary

E	M	Meaning
0x00	0	± 0
0x00	non-zero	± denorm num
0x01 – 0xFE	anything	± norm num
0xFF	0	± ∞
0xFF	non-zero	NaN

Summary

- Floating point approximates real numbers:

31	30 … 23	22 … 0
S	E (8)	M (23)

  - Handles large numbers, small numbers, special numbers
  - Exponent in biased notation (bias = $2^{w-1}–1$)
    - Outside of representable exponents is *overflow* and *underflow*
  - Mantissa approximates fractional portion of binary point
    - Implicit leading 1 (normalized) except in special cases
    - Exceeding length causes *rounding*

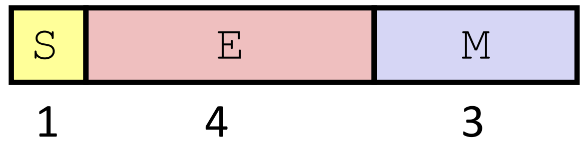
E	M	Meaning
0x00	0	± 0
0x00	non-zero	± denorm num
0x01 – 0xFE	anything	± norm num
0xFF	0	± ∞
0xFF	non-zero	NaN

BONUS SLIDES

An example that applies the IEEE Floating Point concepts to a smaller (8-bit) representation scheme.

These slides expand on material covered today, so while you don't need to read these, the information is "fair game."

Tiny Floating Point Example



- 8-bit Floating Point Representation
  - The sign bit is in the most significant bit (MSB)
  - The next four bits are the exponent, with a bias of $2^{4-1}-1 = 7$
  - The last three bits are the mantissa
- Same general form as IEEE Format
  - Normalized binary scientific point notation
  - Similar special cases for 0, denormalized numbers, NaN, ∞

Dynamic Range (Positive Only)

	S	E	M	Exp	Value	
Denormalized numbers	0	0000	000	-6	0	
	0	0000	001	-6	1/8*1/64 = 1/512	closest to zero
	0	0000	010	-6	2/8*1/64 = 2/512	
	…					
	0	0000	110	-6	6/8*1/64 = 6/512	
	0	0000	111	-6	7/8*1/64 = 7/512	largest denorm
Normalized numbers	0	0001	000	-6	8/8*1/64 = 8/512	smallest norm
	0	0001	001	-6	9/8*1/64 = 9/512	
	…					
	0	0110	110	-1	14/8*1/2 = 14/16	
	0	0110	111	-1	15/8*1/2 = 15/16	closest to 1 below
	0	0111	000	0	8/8*1 = 1	
	0	0111	001	0	9/8*1 = 9/8	closest to 1 above
	0	0111	010	0	10/8*1 = 10/8	
	…					
	0	1110	110	7	14/8*128 = 224	
	0	1110	111	7	15/8*128 = 240	largest norm
	0	1111	000	n/a	inf	

Special Properties of Encoding

- Floating point zero (0^+) exactly the same bits as integer zero
  - All bits = 0
- Can (Almost) Use Unsigned Integer Comparison
  - Must first compare sign bits
  - Must consider $0^- = 0^+ = 0$
  - NaNs problematic
    - Will be greater than any other values
    - What should comparison yield?
  - Otherwise OK
    - Denorm vs. normalized
    - Normalized vs. infinity