Integers II
CSE 351 Autumn 2019

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http://xkcd.com/1953/
Administrivia

- hw4 due 10/7, hw5 due 10/9
- Lab 1a due Monday (10/7)
  - Submit `pointer.c` and `lab1Areflect.txt` to Gradescope
- Lab 1b released today, due 10/14
  - Bit puzzles on number representation
  - Can start after today’s lecture, but floating point will be introduced next week
  - Section worksheet from yesterday has helpful examples, too
  - Bonus slides at the end of today’s lecture have relevant examples
Extra Credit

- All labs starting with Lab 1b have extra credit portions
  - These are meant to be fun extensions to the labs

- Extra credit points *don't* affect your lab grades
  - From the course policies: “they will be accumulated over the course and will be used to bump up borderline grades at the end of the quarter.”
  - Make sure you finish the rest of the lab before attempting any extra credit
Integers

- Binary representation of integers
  - Unsigned and signed
  - Casting in C
- Consequences of finite width representations
  - Overflow, sign extension
- Shifting and arithmetic operations
Two’s Complement Arithmetic

- The same addition procedure works for both unsigned and two’s complement integers
  - **Simplifies hardware**: only one algorithm for addition
  - **Algorithm**: simple addition, **discard the highest carry bit**
    - Called modular addition: result is sum \( \text{modulo } 2^w \)

- **4-bit Examples:**

<table>
<thead>
<tr>
<th>HW</th>
<th>TC</th>
<th>HW</th>
<th>TC</th>
<th>HW</th>
<th>TC</th>
</tr>
</thead>
<tbody>
<tr>
<td>0100</td>
<td>+0011</td>
<td>1100</td>
<td>+0011</td>
<td>0100</td>
<td>+1101</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>=</td>
<td></td>
<td>=</td>
<td></td>
<td>=</td>
<td></td>
</tr>
</tbody>
</table>

  - Example 1: \(0100 + 0011 = 0111\) (in decimal, \(4 + 3 = 7\))
  - Example 2: \(1100 + 0011 = 1111\) (in decimal, \(-4 + 3 = -1\) or \(10001\))
Why Does Two’s Complement Work?

- For all representable positive integers \( x \), we want:

  \[
  \begin{align*}
  \text{bit representation of } x & \\
  + \text{bit representation of } -x & \\
  \hline
  0 & \text{(ignoring the carry-out bit)}
  \end{align*}
  \]

- What are the 8-bit negative encodings for the following?

  \[
  \begin{align*}
  00000001 & + \ ? \ ? \ ? \ ? \ ? \ ? \ ? \\
  00000010 & + \ ? \ ? \ ? \ ? \ ? \ ? \ ? \\
  11000011 & + \ ? \ ? \ ? \ ? \ ? \ ? \ ?
  \end{align*}
  \]

  \[
  \begin{align*}
  00000000 & \\
  00000000 & \\
  00000000 &
  \end{align*}
  \]
Why Does Two’s Complement Work?

- For all representable positive integers \( x \), we want:

\[
\begin{align*}
\text{bit representation of } x & \\
+ \text{bit representation of } -x & \\
\end{align*}
\]

0 (ignoring the carry-out bit)

- What are the 8-bit negative encodings for the following?

\[
\begin{align*}
00000001 + 11111111 & = 100000000 \\
00000010 + 11111110 & = 100000000 \\
11000011 + 00111101 & = 100000000
\end{align*}
\]

These are the bitwise complement plus 1!

\[-x == \sim x + 1\]
Signed/Unsigned Conversion Visualized

- Two’s Complement → Unsigned
  - Ordering Inversion
  - Negative → Big Positive
Values To Remember

- **Unsigned Values**
  - $U_{\text{Min}} = 0b00...0$
    - $= 0$
  - $U_{\text{Max}} = 0b11...1$
    - $= 2^w - 1$

- **Example:** Values for $w = 64$

<table>
<thead>
<tr>
<th></th>
<th>Decimal</th>
<th>Hex</th>
</tr>
</thead>
<tbody>
<tr>
<td>UMax</td>
<td>18,446,744,073,709,551,615</td>
<td>FF FF FF FF FF FF FF FF FF</td>
</tr>
<tr>
<td>Tmax</td>
<td>9,223,372,036,854,775,807</td>
<td>7F FF FF FF FF FF FF FF FF</td>
</tr>
<tr>
<td>Tmin</td>
<td>-9,223,372,036,854,775,808</td>
<td>80 00 00 00 00 00 00 00 00</td>
</tr>
<tr>
<td>-1</td>
<td>-1</td>
<td>FF FF FF FF FF FF FF FF FF</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>00 00 00 00 00 00 00 00 00</td>
</tr>
</tbody>
</table>

- **Two’s Complement Values**
  - $T_{\text{Min}} = 0b10...0$
    - $= -2^{w-1}$
  - $T_{\text{Max}} = 0b01...1$
    - $= 2^{w-1} - 1$
  - $-1 = 0b11...1$
In C: Signed vs. Unsigned

- Casting
  - Bits are unchanged, just interpreted differently!
    - `int` `tx, ty;`
    - `unsigned int` `ux, uy;`
  - *Explicit* casting
    - `tx = (int) ux;`
    - `uy = (unsigned int) ty;`
  - *Implicit* casting can occur during assignments or function calls
    - `tx = ux;`
    - `uy = ty;`
Casting Surprises

- Integer literals (constants)
  - By default, integer constants are considered *signed* integers
    - Hex constants already have an explicit binary representation
  - Use “U” (or “u”) suffix to explicitly force *unsigned*
    - Examples: 0U, 4294967259u

- Expression Evaluation
  - When you mixed unsigned and signed in a single expression, then *signed values are implicitly cast to unsigned*
  - Including comparison operators <, >, ==, <=, >=
Casting Surprises

- **32-bit examples:**
  - TMin = -2,147,483,648, TMax = 2,147,483,647

<table>
<thead>
<tr>
<th>Left Constant</th>
<th>Order</th>
<th>Right Constant</th>
<th>Interpretation</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td></td>
<td>0U</td>
<td>0000 0000 0000 0000 0000 0000 0000 0000</td>
</tr>
<tr>
<td>-1</td>
<td></td>
<td>0</td>
<td>0000 0000 0000 0000 0000 0000 0000 0000</td>
</tr>
<tr>
<td>-1</td>
<td></td>
<td>0U</td>
<td>0000 0000 0000 0000 0000 0000 0000 0000</td>
</tr>
<tr>
<td>2147483647</td>
<td></td>
<td>-2147483648</td>
<td>1000 0000 0000 0000 0000 0000 0000 0000</td>
</tr>
<tr>
<td>2147483647U</td>
<td></td>
<td>-2147483648</td>
<td>1000 0000 0000 0000 0000 0000 0000 0000</td>
</tr>
<tr>
<td>-1</td>
<td></td>
<td>-2</td>
<td>1111 1111 1111 1111 1111 1111 1111 1111</td>
</tr>
<tr>
<td>(unsigned) -1</td>
<td></td>
<td>-2</td>
<td>1111 1111 1111 1111 1111 1111 1111 1110</td>
</tr>
<tr>
<td>2147483647</td>
<td></td>
<td>2147483648U</td>
<td>1000 0000 0000 0000 0000 0000 0000 0000</td>
</tr>
<tr>
<td>2147483647</td>
<td></td>
<td>(int) 2147483648U</td>
<td>1000 0000 0000 0000 0000 0000 0000 0000</td>
</tr>
</tbody>
</table>
Integers

- Binary representation of integers
  - Unsigned and signed
  - Casting in C

- Consequences of finite width representations
  - Overflow, sign extension

- Shifting and arithmetic operations
Arithmetic Overflow

- When a calculation produces a result that can’t be represented in the current encoding scheme
  - Integer range limited by fixed width
  - Can occur in both the positive and negative directions

- C and Java ignore overflow exceptions
  - You end up with a bad value in your program and no warning/indication... oops!

<table>
<thead>
<tr>
<th>Bits</th>
<th>Unsigned</th>
<th>Signed</th>
</tr>
</thead>
<tbody>
<tr>
<td>0000</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0001</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>0010</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>0011</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>0100</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>0101</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>0110</td>
<td>6</td>
<td>6</td>
</tr>
<tr>
<td>0111</td>
<td>7</td>
<td>7</td>
</tr>
<tr>
<td>1000</td>
<td>8</td>
<td>-8</td>
</tr>
<tr>
<td>1001</td>
<td>9</td>
<td>-7</td>
</tr>
<tr>
<td>1010</td>
<td>10</td>
<td>-6</td>
</tr>
<tr>
<td>1011</td>
<td>11</td>
<td>-5</td>
</tr>
<tr>
<td>1100</td>
<td>12</td>
<td>-4</td>
</tr>
<tr>
<td>1101</td>
<td>13</td>
<td>-3</td>
</tr>
<tr>
<td>1110</td>
<td>14</td>
<td>-2</td>
</tr>
<tr>
<td>1111</td>
<td>15</td>
<td>-1</td>
</tr>
</tbody>
</table>
Overflow: Unsigned

- **Addition**: drop carry bit \(-2^N\)
  
  \[
  \begin{array}{c}
  15 \\
  + 2 \\
  \hline
  17 \\
  \end{array}
  \quad \begin{array}{c}
  1111 \\
  + \ 0010 \\
  \hline
  10001 \\
  \end{array}
  \]

  \[
  \begin{array}{c}
  1
  \end{array}
  \]

- **Subtraction**: borrow \(+2^N\)
  
  \[
  \begin{array}{c}
  1 \\
  - 2 \\
  \hline
  -1 \\
  \end{array}
  \quad \begin{array}{c}
  10001 \\
  - \ 0010 \\
  \hline
  1111 \\
  \end{array}
  \]

\(\pm 2^N\) because of modular arithmetic
Overflow: Two’s Complement

- **Addition:** \((+) + (+) = (-)\) result?

\[
\begin{array}{c}
6 \\
+ 3 \\
\hline
9 \\
\hline
-7
\end{array}
\]

- **Subtraction:** \((-) + (-) = (+)?

\[
\begin{array}{c}
-7 \\
- 3 \\
\hline
-10
\end{array}
\]

\[
\begin{array}{c}
1001 \\
- 0011 \\
\hline
0110
\end{array}
\]

For signed: overflow if operands have same sign and result’s sign is different
Sign Extension

- What happens if you convert a *signed* integral data type to a larger one?
  - *e.g.* char → short → int → long

- 4-bit → 8-bit Example:
  - Positive Case
    - 4-bit: 0010 = +2
    - 8-bit: 00000010 = +2
  - Add 0’s?

- Negative Case?
Polling Question

- Which of the following 8-bit numbers has the same signed value as the 4-bit number 0b1100?
  - Underlined digit = MSB
  - Vote at http://PollEv.com/justinh

A. 0b 0000 1100
B. 0b 1000 1100
C. 0b 1111 1100
D. 0b 1100 1100
E. We’re lost...
Sign Extension

- **Task:** Given a $w$-bit signed integer $X$, convert it to a $w+k$-bit signed integer $X'$ with the same value.

- **Rule:** Add $k$ copies of the sign bit.
  - Let $x_i$ be the $i$-th digit of $X$ in binary.
  - $X' = x_{w-1}, \ldots, x_{w-1}, x_{w-1}, x_{w-2}, \ldots, x_1, x_0$

- Diagram:
  - $X$ is represented by $w$ bits.
  - $X'$ is $w+k$ bits with $k$ copies of the MSB and $w$ bits of the original $X$.
  - The sign bit is extended to the left.
Sign Extension Example

- Convert from smaller to larger integral data types
- C automatically performs sign extension
  - Java too

```
short int x = 12345;
int ix = (int) x;
short int y = -12345;
int iy = (int) y;
```

<table>
<thead>
<tr>
<th>Var</th>
<th>Decimal</th>
<th>Hex</th>
<th>Binary</th>
</tr>
</thead>
<tbody>
<tr>
<td>x</td>
<td>12345</td>
<td>30 39</td>
<td>00110000 00111001</td>
</tr>
<tr>
<td>ix</td>
<td>12345</td>
<td>00 00 30 39</td>
<td>00000000 00000000 00110000 00111001</td>
</tr>
<tr>
<td>y</td>
<td>-12345</td>
<td>CF C7</td>
<td>11001111 11000111</td>
</tr>
<tr>
<td>iy</td>
<td>-12345</td>
<td>FF FF CF C7</td>
<td>11111111 11111111 11001111 11000111</td>
</tr>
</tbody>
</table>
Integers

- Binary representation of integers
  - Unsigned and signed
  - Casting in C
- Consequences of finite width representations
  - Overflow, sign extension
- **Shifting and arithmetic operations**
Shift Operations

- **Left shift** \((x<<n)\) bit vector \(x\) by \(n\) positions
  - Throw away (drop) extra bits on left
  - Fill with 0s on right

- **Right shift** \((x>>n)\) bit-vector \(x\) by \(n\) positions
  - Throw away (drop) extra bits on right
  - Logical shift (for **unsigned** values)
    - Fill with 0s on left
  - Arithmetic shift (for **signed** values)
    - Replicate most significant bit on left
    - Maintains sign of \(x\)
Shift Operations

- **Left shift** \((x << n)\)
  - Fill with 0s on right

- **Right shift** \((x >> n)\)
  - Logical shift (for **unsigned** values)
    - Fill with 0s on left
  - Arithmetic shift (for **signed** values)
    - Replicate most significant bit on left

**Notes:**

- Shifts by \(n < 0\) or \(n \geq w\) (bit width of \(x\)) are **undefined**
- **In C:** behavior of \(\gg\) is determined by compiler
  - In gcc / C lang, depends on data type of \(x\) (signed/unsigned)
- **In Java:** logical shift is \(\gg\gg\) and arithmetic shift is \(\gg\)
Shifting Arithmetic?

- What are the following computing?
  - $x >> n$
    - $0b\ 0100 >> 1 = 0b\ 0010$
    - $0b\ 0100 >> 2 = 0b\ 0001$
  - **Divide** by $2^n$
  - $x << n$
    - $0b\ 0001 << 1 = 0b\ 0010$
    - $0b\ 0001 << 2 = 0b\ 0100$
    - **Multiply** by $2^n$

- Shifting is faster than general multiply and divide operations
Left Shifting Arithmetic 8-bit Example

- No difference in left shift operation for unsigned and signed numbers (just manipulates bits)
  - Difference comes during interpretation: \( x \times 2^n \)?

<table>
<thead>
<tr>
<th>Signed</th>
<th>Unsigned</th>
</tr>
</thead>
<tbody>
<tr>
<td>25</td>
<td>25</td>
</tr>
<tr>
<td>100</td>
<td>100</td>
</tr>
<tr>
<td>-56</td>
<td>200</td>
</tr>
<tr>
<td>-112</td>
<td>144</td>
</tr>
</tbody>
</table>

\( x = 25; \)  \( x = 25; \)
\( L1 = x << 2; \)  \( L1 = x << 2; \)
\( L2 = x << 3; \)  \( L2 = x << 3; \)
\( L3 = x << 4; \)  \( L3 = x << 4; \)
Right Shifting Arithmetic 8-bit Examples

- **Reminder**: C operator `>>` does *logical* shift on *unsigned* values and *arithmetic* shift on *signed* values
  - **Logical Shift**: $x / 2^n$?

  $x_u = 240_u; \ 11110000 \quad = \quad 240$

  $R1_u = x_u >> 3; \ 00011110000 \quad = \quad 30$

  $R2_u = x_u >> 5; \ 0000011110000 \quad = \quad 7$

  *rounding (down)*
Right Shifting Arithmetic 8-bit Examples

- **Reminder:** C operator `>>` does *logical* shift on unsigned values and *arithmetic* shift on signed values
  - **Arithmetic Shift:** $x / 2^n$

\[
x_{s} = -16; \quad 11110000 \quad = \quad -16
\]

\[
R_{1s}=x_{u}>>3; \quad 11111110000 \quad = \quad -2
\]

\[
R_{2s}=x_{u}>>5; \quad 1111111110000 \quad = \quad -1
\]
Practice Question

For the following expressions, find a value of **signed char** `x`, if there exists one, that makes the expression **TRUE**. Compare with your neighbor(s)!

- Assume we are using 8-bit arithmetic:
  - `x == (unsigned char) x`
  - `x >= 128U`
  - `x != (x>>2)<<2`
  - `x == -x`
    - Hint: there are two solutions
  - `(x < 128U) && (x > 0x3F)`
Summary

- Sign and unsigned variables in C
  - Bit pattern remains the same, just interpreted differently
  - Strange things can happen with our arithmetic when we convert/cast between sign and unsigned numbers
    - Type of variables affects behavior of operators (shifting, comparison)

- We can only represent so many numbers in \( w \) bits
  - When we exceed the limits, \textit{arithmetic overflow} occurs
  - \textit{Sign extension} tries to preserve value when expanding

- Shifting is a useful bitwise operator
  - Right shifting can be arithmetic (sign) or logical (0)
  - Can be used in multiplication with constant or bit masking
Some examples of using shift operators in combination with bitmasks, which you may find helpful for Lab 1.

- Extract the 2\textsuperscript{nd} most significant byte of an \texttt{int}
- Extract the sign bit of a signed \texttt{int}
- Conditionals as Boolean expressions
Using Shifts and Masks

- Extract the 2\textsuperscript{nd} most significant \textit{byte} of an \texttt{int}:
  - First shift, then mask: \((x\gg16) \& 0xFF\)

<table>
<thead>
<tr>
<th>\textbf{x}</th>
<th>00000001 00000010 00000011 00000100</th>
</tr>
</thead>
<tbody>
<tr>
<td>\textbf{x\gg16}</td>
<td>00000000 00000000 00000011 00000100</td>
</tr>
<tr>
<td>\textbf{0xFF}</td>
<td>00000000 00000000 00000000 11111111</td>
</tr>
<tr>
<td>( (x\gg16) &amp; 0xFF)</td>
<td>00000000 00000000 00000000 00000100</td>
</tr>
</tbody>
</table>

- Or first mask, then shift: \((x \& 0xFF0000) \gg 16\)

<table>
<thead>
<tr>
<th>\textbf{x}</th>
<th>00000001 00000010 00000011 00000100</th>
</tr>
</thead>
<tbody>
<tr>
<td>\textbf{0xFF0000}</td>
<td>00000000 11111111 00000000 00000000</td>
</tr>
<tr>
<td>\textbf{x &amp; 0xFF0000}</td>
<td>00000000 00000010 00000000 00000000</td>
</tr>
<tr>
<td>( (x&amp;0xFF0000) \gg 16)</td>
<td>00000000 00000000 00000000 00000100</td>
</tr>
</tbody>
</table>
Using Shifts and Masks

- Extract the **sign bit** of a signed `int`:
  - First shift, then mask: `(x>>31) & 0x1`
    - Assuming arithmetic shift here, but this works in either case
    - Need mask to clear 1s possibly shifted in

<table>
<thead>
<tr>
<th>x</th>
<th>00000001 00000010 00000011 00000100</th>
</tr>
</thead>
<tbody>
<tr>
<td>x&gt;&gt;31</td>
<td>00000000 00000000 00000000 00000000</td>
</tr>
<tr>
<td>0x1</td>
<td>00000000 00000000 00000000 00000001</td>
</tr>
<tr>
<td>(x&gt;&gt;31) &amp; 0x1</td>
<td>00000000 00000000 00000000 00000000</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>x</th>
<th>10000001 00000010 00000011 00000100</th>
</tr>
</thead>
<tbody>
<tr>
<td>x&gt;&gt;31</td>
<td>11111111 11111111 11111111 11111111</td>
</tr>
<tr>
<td>0x1</td>
<td>00000000 00000000 00000000 00000001</td>
</tr>
<tr>
<td>(x&gt;&gt;31) &amp; 0x1</td>
<td>00000000 00000000 00000000 00000001</td>
</tr>
</tbody>
</table>
Using Shifts and Masks

- Conditionals as Boolean expressions
  - For `int x`, what does `(x<<31)>>31` do?

<table>
<thead>
<tr>
<th>x=!!123</th>
<th>00000000 00000000 00000000 00000001</th>
</tr>
</thead>
<tbody>
<tr>
<td>x&lt;&lt;31</td>
<td>10000000 00000000 00000000 00000000</td>
</tr>
<tr>
<td>(x&lt;&lt;31)&gt;&gt;31</td>
<td>11111111 11111111 11111111 11111111</td>
</tr>
<tr>
<td>!x</td>
<td>00000000 00000000 00000000 00000000 0</td>
</tr>
<tr>
<td>!x&lt;&lt;31</td>
<td>00000000 00000000 00000000 00000000 0</td>
</tr>
<tr>
<td>(!x&lt;&lt;31)&gt;&gt;31</td>
<td>00000000 00000000 00000000 00000000 0</td>
</tr>
</tbody>
</table>

- Can use in place of conditional:
  - In C: `if(x) {a=y;} else {a=z;}` equivalent to `a=x?y:z;`
  - `a=((x<<31)>>31)&y) | (((!x<<31)>>31)&z);`