Integers II

CSE 351 Autumn 2019

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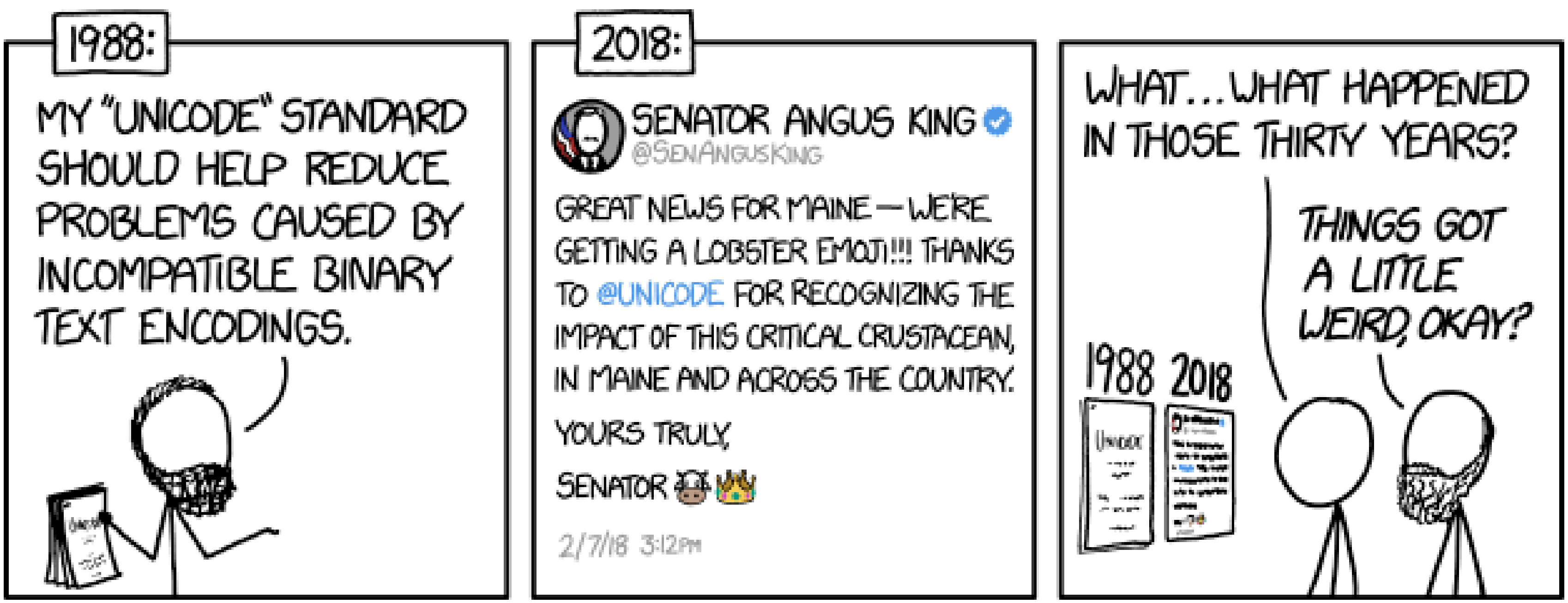


http://xkcd.com/1953/

Administrivia

- hw4 due 10/7, hw5 due 10/9
- Lab 1a due Monday (10/7)
  - Submit `pointer.c` and `lab1Areflect.txt` to Gradescope
- Lab 1b released today, due 10/14
  - Bit puzzles on number representation
  - Can start after today's lecture, but floating point will be introduced next week
  - Section worksheet from yesterday has helpful examples, too
  - Bonus slides at the end of today's lecture have relevant examples

Extra Credit

- All labs starting with Lab 1b have extra credit portions
  - These are meant to be fun extensions to the labs
- Extra credit points *don't* affect your lab grades
  - From the course policies: "they will be accumulated over the course and will be used to bump up borderline grades at the end of the quarter."
  - Make sure you finish the rest of the lab before attempting any extra credit

Integers

- **Binary representation of integers**
  - **Unsigned and signed**
  - **Casting in C**
- Consequences of finite width representations
  - Overflow, sign extension
- Shifting and arithmetic operations

Two's Complement Arithmetic

MSB has negative weight: 0b $\underset{-8}{\_}\ \underset{+4}{\_}\ \underset{+2}{\_}\ \underset{+1}{\_}$

- The same addition procedure works for both unsigned and two's complement integers
  - **Simplifies hardware:** only one algorithm for addition
  - **Algorithm:** simple addition, discard the highest carry bit
    - Called modular addition: result is sum *modulo* 2^w

- **4-bit Examples:**

HW	TC
0100	+4
+0011	+3
= 0111	+7

HW	TC
1100	-4
+0011	+3
= 1111	-1

HW	TC
0100	+4
+1101	-3
= ~~1~~0001	+1

Why Does Two's Complement Work?

- For all representable positive integers x, we want:

additive inverse

$$\begin{array}{r} \textit{bit representation of } x \\ + \textit{ bit representation of } -x \\ \hline 0 \end{array}$$

(ignoring the carry-out bit)

  - What are the 8-bit negative encodings for the following?

$$\begin{array}{r} 00000001 \\ 11111111 \\ + \ ???????? \\ \hline \cancel{1}\ 00000000 \end{array} \qquad \begin{array}{r} 00000010 \\ 11111110 \\ + \ ???????? \\ \hline \cancel{1}\ 00000000 \end{array} \qquad \begin{array}{r} 1111111 \\ 11000011 \\ 00111101 \\ + \ ???????? \\ \hline \cancel{1}\ 00000000 \end{array}$$

Why Does Two's Complement Work?

- For all representable positive integers x, we want:

0b1...1

$x + (\sim x) =$

$$\begin{array}{r} \textit{bit representation of } \; x \\ + \; \textit{bit representation of } -x \\ \hline 0 \end{array}$$

(ignoring the carry-out bit)

$x + (\sim x) = -1$

$x + (\sim x + 1) = 0$

$\boxed{-x = \sim x + 1}$

  - What are the 8-bit negative encodings for the following?

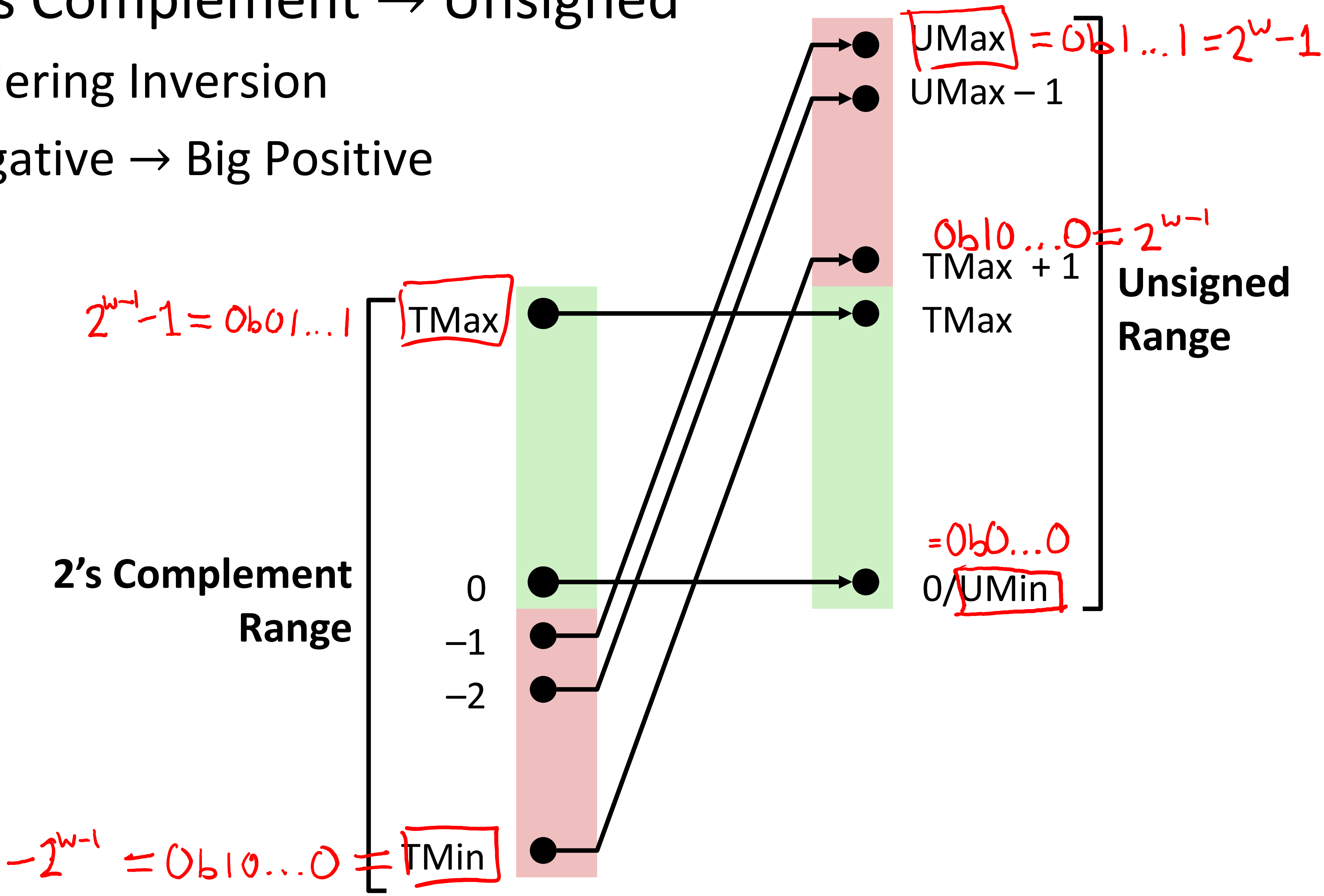
$$\begin{array}{r} 00000001 \\ +\; 11111111 \\ \hline \cancel{1}00000000 \end{array} \qquad \begin{array}{r} 00000010 \\ +\; 11111110 \\ \hline \cancel{1}00000000 \end{array} \qquad \begin{array}{r} 11000011 \\ +\; 00111101 \\ \hline \cancel{1}00000000 \end{array}$$

These are the bitwise complement plus 1!

`-x == ~x + 1`

Signed/Unsigned Conversion Visualized

- Two's Complement → Unsigned
  - Ordering Inversion
  - Negative → Big Positive



Values To Remember

- Unsigned Values
  - UMin = `0b00…0` = 0
  - UMax = `0b11…1` = $2^w - 1$
- Two's Complement Values
  - TMin = `0b10…0` = -2^{w-1}
  - TMax = `0b01…1` = $2^{w-1} - 1$
  - -1 = `0b11…1`
- **Example:** Values for $w = 64$

	Decimal	Hex
UMax	18,446,744,073,709,551,615	FF FF FF FF FF FF FF FF
TMax	9,223,372,036,854,775,807	7F FF FF FF FF FF FF FF
TMin	-9,223,372,036,854,775,808	80 00 00 00 00 00 00 00
-1	-1	FF FF FF FF FF FF FF FF
0	0	00 00 00 00 00 00 00 00

In C: Signed vs. Unsigned

- Casting
  - Bits are unchanged, just interpreted differently!
    - **int** tx, ty;
    - **unsigned int** ux, uy;
  - *Explicit* casting
    - tx = (**int**) ux;
    - uy = (**unsigned int**) ty;

    (new_type) expression
  - *Implicit* casting can occur during assignments or function calls cast to target variable/parameter type
    - tx = ux;
    - uy = ty;

    (also implicitly occurs with printf format specifiers)

Casting Surprises

!!!

- Integer literals (constants)
  - By default, integer constants are considered *signed* integers
    - Hex constants already have an explicit binary representation
  - Use "U" (or "u") suffix to explicitly force *unsigned*
    - **Examples:** 0U, 4294967259u

- Expression Evaluation
  - When you mixed unsigned and signed in a single expression, then **signed values are implicitly cast to <u>unsigned</u>**
  - Including comparison operators <, >, ==, <=, >=

(unsigned "dominates")

Casting Surprises

!!!

- 32-bit examples:
  - TMin = -2,147,483,648, TMax = 2,147,483,647

Left Constant	Order	Right Constant	Interpretation
0 0000 0000 0000 0000 0000 0000 0000 0000	==	**0U** 0000 0000 0000 0000 0000 0000 0000 0000	unsigned
-1 1111 1111 1111 1111 1111 1111 1111 1111	<	**0** 0000 0000 0000 0000 0000 0000 0000 0000	signed
-1 1111 1111 1111 1111 1111 1111 1111 1111	>	**0U** 0000 0000 0000 0000 0000 0000 0000 0000	unsigned
2147483647 0111 1111 1111 1111 1111 1111 1111 1111	>	**-2147483648** 1000 0000 0000 0000 0000 0000 0000 0000	signed
2147483647U 0111 1111 1111 1111 1111 1111 1111 1111	<	**-2147483648** 1000 0000 0000 0000 0000 0000 0000 0000	unsigned
-1 1111 1111 1111 1111 1111 1111 1111 1111	>	**-2** 1111 1111 1111 1111 1111 1111 1111 1110	signed
(unsigned) -1 1111 1111 1111 1111 1111 1111 1111 1111	>	**-2** 1111 1111 1111 1111 1111 1111 1111 1110	unsigned
2147483647 0111 1111 1111 1111 1111 1111 1111 1111	<	**2147483648U** 1000 0000 0000 0000 0000 0000 0000 0000	unsigned
2147483647 0111 1111 1111 1111 1111 1111 1111 1111	>	**(int) 2147483648U** 1000 0000 0000 0000 0000 0000 0000 0000	signed

Integers

- Binary representation of integers
  - Unsigned and signed
  - Casting in C
- **Consequences of finite width representations**
  - **Overflow, sign extension**
- Shifting and arithmetic operations

Arithmetic Overflow

Bits	Unsigned	Signed
0000	0 UMin	0
0001	1	1
0010	2	2
0011	3	3
0100	4	4
0101	5	5
0110	6	6
0111	7	7 TMax
1000	8	-8 TMin
1001	9	-7
1010	10	-6
1011	11	-5
1100	12	-4
1101	13	-3
1110	14	-2
1111	15 UMax	-1

- When a calculation produces a result that can't be represented in the current encoding scheme
  - Integer range limited by fixed width UMin – UMax TMin – TMax
  - Can occur in both the positive and negative directions
- C and Java ignore overflow exceptions
  - You end up with a bad value in your program and no warning/indication… oops!

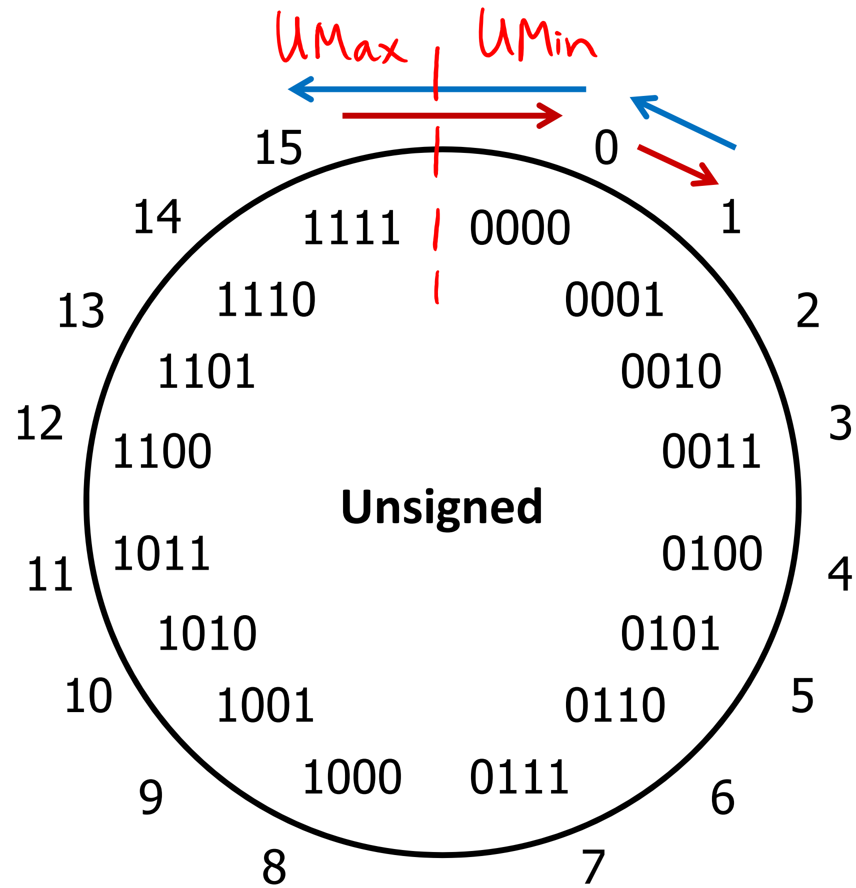
Overflow: Unsigned

- **Addition:** drop carry bit (-2^N)

$$\begin{array}{r} 15 \\ +\ \ 2 \\ \hline \cancel{17} \\ 1 \end{array} \qquad \begin{array}{r} 1111 \\ +\ \ 0010 \\ \hline \cancel{1}0001 \end{array}$$

- **Subtraction:** borrow ($+2^N$)

$$\begin{array}{r} 1 \\ -\ \ 2 \\ \hline \cancel{-1} \\ 15 \end{array} \qquad \begin{array}{r} \cancel{1}0001 \\ -\ \ 0010 \\ \hline 1111 \end{array}$$



$\pm 2^N$ because of modular arithmetic

$2^4 = 16$

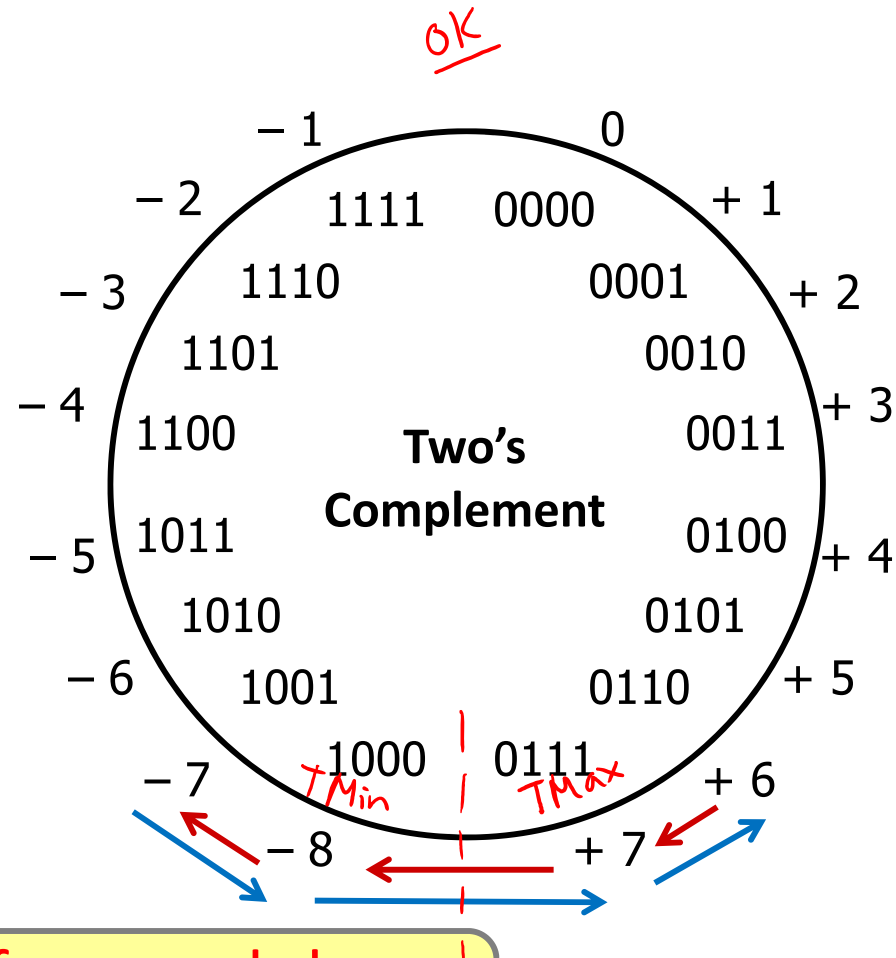
Overflow: Two's Complement

- **Addition:** (+) + (+) = (–) result?

$$\begin{array}{r} 6 \\ +\;\;3 \\ \hline \cancel{9} \\ -7 \end{array} \qquad \begin{array}{r} 0110 \\ +\;\;0011 \\ \hline 1001 \end{array}$$

- **Subtraction:** (–) + (–) = (+)?

$$\begin{array}{r} -7 \\ -\;\;3 \\ \hline \cancel{-10} \\ 6 \end{array} \qquad \begin{array}{r} 1001 \\ -\;\;0011 \\ \hline 0110 \end{array}$$



For signed: overflow if operands have same sign and result's sign is different

Sign Extension

- What happens if you convert a *signed* integral data type to a larger one?
  - *e.g.* `char` → `short` → `int` → `long`
    1 byte 2 bytes 4 bytes 8 bytes
- **4-bit → 8-bit Example:**
  - Positive Case
    - ✓ Add 0's?
  - Negative Case?

4-bit:	`0010`	=	+2
8-bit:	**`00000010`**	=	**+2**

Polling Question

- Which of the following 8-bit numbers has the same *signed* value as the 4-bit number **0b1100**?
  - Underlined digit = MSB
  - Vote at http://PollEv.com/justinh

−8 4 2 1

−8+4= −4

−x = 0 0 1 1
+1
0 1 0 0 = 4 ⇒ x = −4

A. ~~0b 0000 1100~~ (add zeros) positive!

B. ~~0b 1000 1100~~ (add leading 1) much too negative: $-2^7+2^3+2^2 = -116$

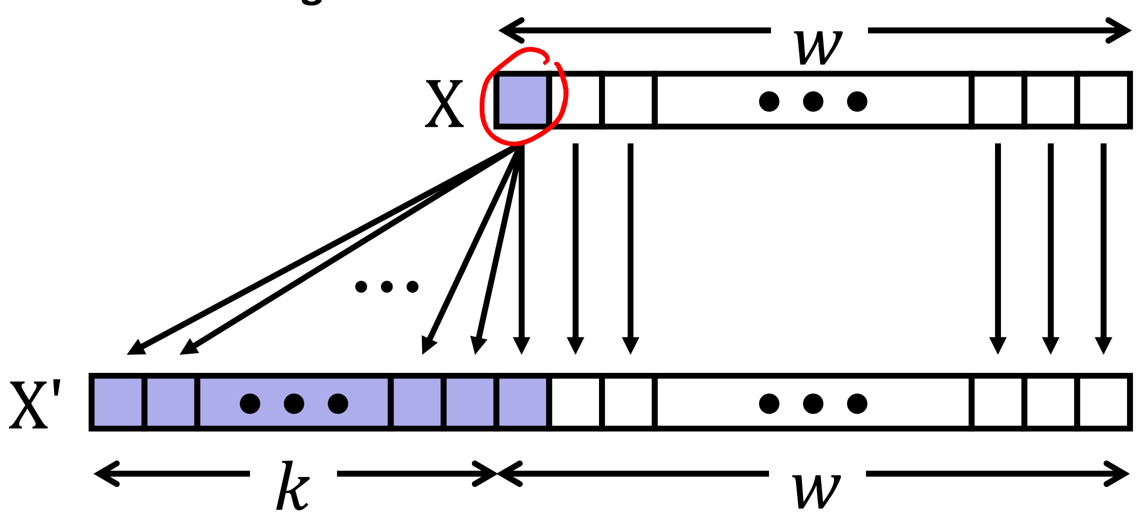
C. $\boxed{\text{0b 1111 1100 (add ones)}}$ correct! $-y$ = 0b 0000 0011 +1 = 4, y = −4

D. ~~0b 1100 1100~~ (duplicate) $-2^7+2^6+2^3+2^2 = -52$

E. We're lost…

Sign Extension

- **Task:** Given a w-bit signed integer X, convert it to w+k-bit signed integer X′ *with the same value*
- **Rule: Add k copies of sign bit**
  - Let x_i be the i-th digit of X in binary
  - $\mathrm{X}' = \underbrace{x_{w-1}, \ldots, x_{w-1}}_{k \text{ copies of MSB}}, \underbrace{x_{w-1}, x_{w-2}, \ldots, x_1, x_0}_{\text{original X}}$



Sign Extension Example

- Convert from smaller to larger integral data types
- C automatically performs sign extension
  - Java too

```
short int x =   12345;
int      ix = (int) x;
short int y =  -12345;
int      iy = (int) y;
```

0b 0011

Var	Decimal	Hex	Binary
x	12345	30 39	00110000 00111001
ix	12345	00 00 30 39	00000000 00000000 00110000 00111001
y	-12345	CF C7	11001111 11000111
iy	-12345	FF FF CF C7	11111111 11111111 11001111 11000111

0b 1100

Integers

- Binary representation of integers
  - Unsigned and signed
  - Casting in C
- Consequences of finite width representations
  - Overflow, sign extension
- **Shifting and arithmetic operations**

Shift Operations

- Left shift (`x<<n`) bit vector `x` by `n` positions
  - Throw away (drop) extra bits on left
  - Fill with `0`s on right
- Right shift (`x>>n`) bit-vector `x` by `n` positions
  - Throw away (drop) extra bits on right
  - Logical shift (for unsigned values)
    - Fill with `0`s on left
  - Arithmetic shift (for signed values)
    - Replicate most significant bit on left
    - Maintains sign of `x`

Shift Operations

8-bit example:

	x	0010 0010
	x<<3	0001 0000
logical:	x>>2	0000 1000
arithmetic:	x>>2	0000 1000

	x	1010 0010
	x<<3	0001 0000
logical:	x>>2	0010 1000
arithmetic:	x>>2	1110 1000

- Left shift (x<<n)
  - Fill with 0s on right
- Right shift (x>>n)
  - Logical shift (for unsigned values)
    - Fill with 0s on left
  - Arithmetic shift (for signed values)
    - Replicate most significant bit on left
- Notes:
  - Shifts by n<0 or n≥w (w is bit width of x) are *undefined* behavior not guaranteed
  - **In C:** behavior of >> is determined by compiler arithmetic/logical
    - In gcc / C lang, depends on data type of x (signed/unsigned)
  - **In Java:** logical shift is >>> and arithmetic shift is >>

Shifting Arithmetic?

- What are the following computing?
  - `x>>n`
    - $\overset{4}{\texttt{0b 0100}}\ \texttt{>> 1 =}\ \overset{2}{\texttt{0b 0010}}$
    - $\overset{4}{\texttt{0b 0100}}\ \texttt{>> 2 =}\ \overset{1}{\texttt{0b 0001}}$
    - Divide by 2^n
  - `x<<n`
    - $\overset{1}{\texttt{0b 0001}}\ \texttt{<< 1 =}\ \overset{2}{\texttt{0b 0010}}$
    - $\overset{1}{\texttt{0b 0001}}\ \texttt{<< 2 =}\ \overset{4}{\texttt{0b 0100}}$
    - Multiply by 2^n
- Shifting is faster than general multiply and divide operations

Left Shifting Arithmetic 8-bit Example

- No difference in left shift operation for unsigned and signed numbers (just manipulates bits)
  - Difference comes during interpretation: `x*2`^{n}?

			Signed	Unsigned
`x = 25;`	00011001	=	25	25
`L1=x<<2;`	~~00~~01100100	=	100	100
`L2=x<<3;`	~~000~~11001000	=	-56	200
`L3=x<<4;`	~~0001~~10010000	=	-112	144

200
−256 → 2^8

signed overflow

400
−256 → 2^8

unsigned overflow

Right Shifting Arithmetic 8-bit Examples

- **Reminder:** C operator >> does *logical* shift on unsigned values and *arithmetic* shift on signed values
  - Logical Shift: $x/2^n$?

`xu = 240u;`	`11110000`	=	240 /8 = 30
`R1u=xu>>3;`	`00011110`~~`000`~~	=	30 /4 = 7.5
`R2u=xu>>5;`	`00000111`~~`10000`~~	=	7

rounding (down)

Right Shifting Arithmetic 8-bit Examples

- **Reminder:** C operator >> does *logical* shift on unsigned values and *arithmetic* shift on signed values
  - Arithmetic Shift: `x/2`^{n}?

`xs = -16;` 11110000 = -16

`R1s=xu>>3;` 11111110~~000~~ = -2 /4 = -0.5

`R2s=xu>>5;` 11111111~~10000~~ = -1

rounding (down)

Practice Question

UMin = 0 , UMax = 255
8-bits, so TMin = -128, TMax = 127

For the following expressions, find a value of **`signed char`** `x`, if there exists one, that makes the expression `TRUE`. Compare with your neighbor(s)!

- Assume we are using 8-bit arithmetic:

Expression	Example:	All solutions:
`x == (unsigned char) x` (unsigned)	x = 0	works for all x
`x >= 128U` (unsigned; 0b1000 0000)	x = -1	any x < 0
`x != (x>>2)<<2`	x = 3	any x where lowest two bits are not 0b00
`x == -x` • Hint: there are two solutions	x = 0	① x = 0b0...0 = 0 ② x = 0b10...0 = -128
`(x < 128U) && (x > 0x3F)`		any x where upper two bits are exactly 0b01

Summary

- Sign and unsigned variables in C
  - Bit pattern remains the same, just *interpreted* differently
  - Strange things can happen with our arithmetic when we convert/cast between sign and unsigned numbers
    - Type of variables affects behavior of operators (shifting, comparison)
- We can only represent so many numbers in *w* bits
  - When we exceed the limits, *arithmetic overflow* occurs
  - *Sign extension* tries to preserve value when expanding
- Shifting is a useful bitwise operator
  - Right shifting can be arithmetic (sign) or logical (0)
  - Can be used in multiplication with constant or bit masking

BONUS SLIDES

Some examples of using shift operators in combination with bitmasks, which you may find helpful for Lab 1.

- Extract the 2nd most significant byte of an `int`
- Extract the sign bit of a signed `int`
- Conditionals as Boolean expressions

Using Shifts and Masks

- Extract the 2nd most significant *byte* of an `int`:
  - First shift, then mask: `(x>>16) & 0xFF`

`x`	00000001 00000010 00000011 00000100
`x>>16`	00000000 00000000 00000001 00000010
`0xFF`	00000000 00000000 00000000 11111111
`(x>>16) & 0xFF`	00000000 00000000 00000000 00000010

  - Or first mask, then shift: `(x & 0xFF0000)>>16`

`x`	00000001 00000010 00000011 00000100
`0xFF0000`	00000000 11111111 00000000 00000000
`x & 0xFF0000`	00000000 00000010 00000000 00000000
`(x&0xFF0000)>>16`	00000000 00000000 00000000 00000010

Using Shifts and Masks

- Extract the *sign bit* of a signed `int`:
  - First shift, then mask: `(x>>31) & 0x1`
    - Assuming arithmetic shift here, but this works in either case
    - Need mask to clear `1`s possibly shifted in

x	00000001 00000010 00000011 00000100
x>>31	00000000 00000000 00000000 00000000
0x1	00000000 00000000 00000000 00000001
(x>>31) & 0x1	00000000 00000000 00000000 00000000

x	10000001 00000010 00000011 00000100
x>>31	11111111 11111111 11111111 11111111
0x1	00000000 00000000 00000000 00000001
(x>>31) & 0x1	00000000 00000000 00000000 00000001

Using Shifts and Masks

- Conditionals as Boolean expressions
  - For **`int`** `x`, what does `(x<<31)>>31` do?

`x=!!123`	`00000000 00000000 00000000 00000001`
`x<<31`	`10000000 00000000 00000000 00000000`
`(x<<31)>>31`	`11111111 11111111 11111111 11111111`
`!x`	`00000000 00000000 00000000 00000000`
`!x<<31`	`00000000 00000000 00000000 00000000`
`(!x<<31)>>31`	`00000000 00000000 00000000 00000000`

  - Can use in place of conditional:
    - In C: `if(x) {a=y;} else {a=z;}` equivalent to `a=x?y:z;`
    - `a=(((x<<31)>>31)&y) | (((!x<<31)>>31)&z);`