Floating Point

CSE 351 Winter 2018

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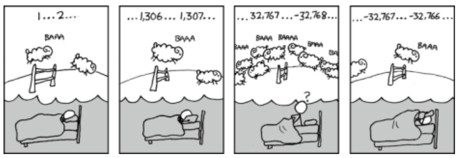
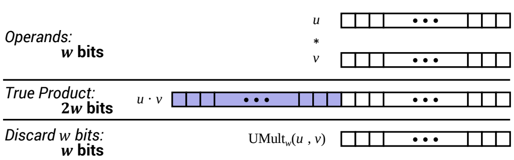


http://xkcd.com/571/

Administrivia

- Lab 1 due Friday (1/19)
  - Submit `bits.c` and `pointer.c`
- Homework 2 out since 1/15, due 1/24
  - On Integers, Floating Point, and x86-64

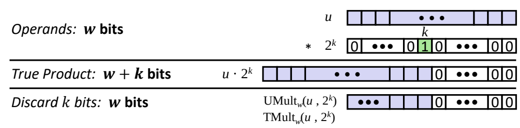
Unsigned Multiplication in C



- Standard Multiplication Function
  - Ignores high order w bits
- Implements Modular Arithmetic
  - $\text{UMult}_w(u, v) = u \cdot v \mod 2^w$

Multiplication with shift and add

- Operation `u<<k` gives `u*2`^{k}
  - Both signed and unsigned



- Examples:
  - `u<<3          ==  u * 8`
  - `u<<5 - u<<3   ==  u * 24`
  - Most machines shift and add faster than multiply
    - ***Compiler generates this code automatically***

Number Representation Revisited

- What can we represent in one word?
  - Signed and Unsigned Integers
  - Characters (ASCII)
  - Addresses
- How do we encode the following:
  - Real numbers (*e.g.* 3.14159)
  - Very large numbers (*e.g.* 6.02×10^{23})
  - Very small numbers (*e.g.* 6.626×10^{-34})
  - Special numbers (*e.g.* ∞, NaN)

Floating Point

Floating Point Topics

- **Fractional binary numbers**
- IEEE floating-point standard
- Floating-point operations and rounding
- Floating-point in C

- There are many more details that we won't cover
  - It's a 58-page standard…

Representation of Fractions

- "Binary Point," like decimal point, signifies boundary between integer and fractional parts:

  Example 6-bit representation: xx.yyyy

  2^1 2^0 2^{-1} 2^{-2} 2^{-3} 2^{-4}

- Example: $10.1010_2 = 1{\times}2^1 + 1{\times}2^{-1} + 1{\times}2^{-3} = 2.625_{10}$

- Binary point numbers that match the 6-bit format above range from 0 (00.0000_2) to 3.9375 (11.1111_2)

Scientific Notation (Decimal)

mantissa → $6.02_{10} \times 10^{23}$ ← exponent

decimal point   radix (base)

- *Normalized form*: exactly one digit (non-zero) to left of decimal point
- Alternatives to representing 1/1,000,000,000
  - Normalized: $1.0{\times}10^{-9}$
  - Not normalized: $0.1{\times}10^{-8}$, $10.0{\times}10^{-10}$

Scientific Notation (Binary)

mantissa → $1.01_2 \times 2^{-1}$ ← exponent

binary point   radix (base)

- Computer arithmetic that supports this called floating point due to the "floating" of the binary point
  - Declare such variable in C as `float` (or `double`)

Scientific Notation Translation

- Convert from scientific notation to binary point to decimal
  - Perform the multiplication by shifting the decimal until the exponent disappears
    - Example: $1.011_2{\times}2^4 = 10110_2 = 22_{10}$
    - Example: $1.011_2{\times}2^{-2} = 0.01011_2 = 0.34375_{10}$
- Convert from binary point to *normalized* scientific notation
  - Distribute out exponents until binary point is to the right of a single digit
    - Example: $1101.001_2 = 1.101001_2{\times}2^3$

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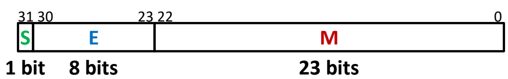
IEEE Floating Point

- IEEE 754
  - Established in 1985 as uniform standard for floating point arithmetic
  - Main idea: make numerically sensitive programs portable
  - Specifies two things: representation and result of floating operations
  - Now supported by all major CPUs
- Driven by numerical concerns
  - **Scientists**/numerical analysts want them to be as **real** as possible
  - **Engineers** want them to be **easy to implement** and **fast**
  - In the end:
    - Scientists mostly won out
    - Nice standards for rounding, overflow, underflow, but…
    - Hard to make fast in hardware
    - **Float operations can be an order of magnitude slower than integer ops**

Floating Point Encoding

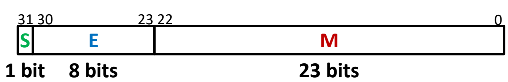
- Use normalized, base 2 scientific notation:
  - Value: $\pm 1 \times$ Mantissa $\times 2^{\text{Exponent}}$
  - Bit Fields: $(-1)^S \times 1.M \times 2^{(E-\text{bias})}$
- Representation Scheme:
  - Sign bit (0 is positive, 1 is negative)
  - Mantissa (a.k.a. significand) is the fractional part of the number in normalized form and encoded in bit vector **M**
  - Exponent weights the value by a (possibly negative) power of 2 and encoded in the bit vector **E**



The Exponent Field

- Use biased notation
  - Read exponent as unsigned, but with *bias* of $2^{w-1}-1 = 127$
  - Representable exponents roughly ½ positive and ½ negative
  - Exponent 0 (Exp = 0) is represented as E = 0b 0111 1111
- Why biased?
  - Makes floating point arithmetic easier
  - Makes somewhat compatible with two's complement
- **Practice:** To encode in biased notation, add the bias then encode in unsigned:
  - Exp = 1 → → E = 0b
  - Exp = 127 → → E = 0b
  - Exp = -63 → → E = 0b

The Mantissa (Fraction) Field



$$(-1)^S \times (1\,.\,M) \times 2^{(E-\text{bias})}$$

- Note the implicit 1 in front of the M bit vector
  - Example: 0b 0011 1111 1100 0000 0000 0000 0000 0000 is read as $1.1_2 = 1.5_{10}$, *not* $0.1_2 = 0.5_{10}$
  - Gives us an extra bit of *precision*
- Mantissa "limits"
  - Low values near M = 0b0…0 are close to 2^{Exp}
  - High values near M = 0b1…1 are close to $2^{\text{Exp}+1}$

Peer Instruction Question

- What is the correct value encoded by the following floating point number?
  - 0b 0 10000000 11000000000000000000000

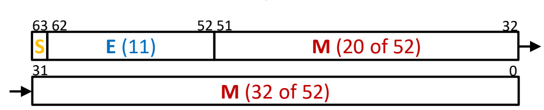
A. + 0.75
B. + 1.5
C. + 2.75
D. + 3.5
E. We're lost…

Precision and Accuracy

- Precision is a count of the number of bits in a computer word used to represent a value
  - Capacity for accuracy
- Accuracy is a measure of the difference between the *actual value of a number* and its computer representation
  - *High precision permits high accuracy but doesn't guarantee it. It is possible to have high precision but low accuracy.*
  - **Example:** `float pi = 3.14;`
    - `pi` will be represented using all 24 bits of the mantissa (highly precise), but is only an approximation (not accurate)

Need Greater Precision?

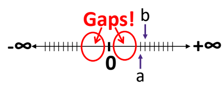
- Double Precision (vs. Single Precision) in 64 bits



  - C variable declared as `double`
  - Exponent bias is now $2^{10}-1 = 1023$
  - **Advantages:** greater precision (larger mantissa), greater range (larger exponent)
  - **Disadvantages:** more bits used, slower to manipulate

Representing Very Small Numbers

- But wait… what happened to zero?
  - Using standard encoding 0x00000000 =
  - *Special case:* E and M all zeros = 0
    - Two zeros! But at least 0x00000000 = 0 like integers
- New numbers closest to 0:
  - a = $1.0\ldots0_2 \times 2^{-126} = 2^{-126}$
  - b = $1.0\ldots01_2 \times 2^{-126} = 2^{-126} + 2^{-149}$
  - Normalization and implicit 1 are to blame
  - *Special case:* E = 0, M ≠ 0 are denormalized numbers



Other Special Cases

- E = 0xFF, M = 0: ± ∞
  - *e.g.* division by 0
  - Still work in comparisons!
- E = 0xFF, M ≠ 0: Not a Number (NaN)
  - *e.g.* square root of negative number, 0/0, ∞–∞
  - NaN propagates through computations
  - Value of M can be useful in debugging
- New largest value (besides ∞)?
  - E = 0xFF has now been taken!
  - E = 0xFE has largest: $1.1\ldots1_2 \times 2^{127} = 2^{128} - 2^{104}$

Summary

- Floating point approximates real numbers:

31	30 … 23	22 … 0
S	E (8)	M (23)

  - Handles large numbers, small numbers, special numbers
  - Exponent in biased notation (bias = $2^{w-1}-1$)
    - Outside of representable exponents is *overflow* and *underflow*
  - Mantissa approximates fractional portion of binary point
    - Implicit leading 1 (normalized) except in special cases
    - Exceeding length causes *rounding*

Exponent	Mantissa	Meaning
0x00	0	± 0
0x00	non-zero	± denorm num
0x01 – 0xFE	anything	± norm num
0xFF	0	± ∞
0xFF	non-zero	NaN

Floating point topics

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- **Floating-point operations and rounding**
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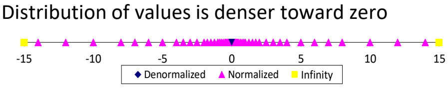
Distribution of Values

- What ranges are NOT representable?
  - Between largest norm and infinity Overflow
  - Between zero and smallest denorm Underflow
  - Between norm numbers? Rounding
- Given a FP number, what's the bit pattern of the next largest representable number?
  - What is this "step" when Exp = 0?
  - What is this "step" when Exp = 100?
- Distribution of values is denser toward zero



Floating Point Operations: Basic Idea

Value = $(-1)^S \times \text{Mantissa} \times 2^{\text{Exponent}}$

S	E	M

- `x +f y = Round(x + y)`
- `x *f y = Round(x * y)`
- Basic idea for floating point operations:
  - First, compute the exact result
  - Then *round* the result to make it fit into desired precision:
    - Possibly over/underflow if exponent outside of range
    - Possibly drop least-significant bits of mantissa to fit into M bit vector

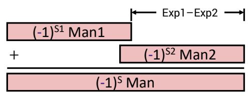
Floating Point Addition

Line up the binary points!

- $(-1)^{S1}\times Man1\times 2^{Exp1} + (-1)^{S2}\times Man2\times 2^{Exp2}$
  - Assume Exp1 > Exp2

```
  1.010*2^2       1.0100*2^2
+ 1.000*2^-1    + 0.0010*2^2
    ???           1.0110*2^2
```

- Exact Result: $(-1)^{S}\times Man\times 2^{Exp}$
  - Sign S, mantissa Man:
    - Result of signed align & add
  - Exponent E: E1



- Adjustments:
  - If Man ≥ 2, shift Man right, increment Exp
  - If Man < 1, shift Man left k positions, decrement Exp by k
  - Over/underflow if Exp out of range
  - Round Man to fit mantissa precision

Floating Point Multiplication

- $(-1)^{S1}\times Man1\times 2^{Exp1} \times (-1)^{S2}\times Man2\times 2^{Exp2}$
- Exact Result: $(-1)^{S}\times M\times 2^{E}$
  - Sign S: S1 ^ S2
  - Mantissa Man: Man1 × Man2
  - Exponent Exp: Exp1 + Exp2
- Adjustments:
  - If Man ≥ 2, shift Man right, increment Exp
  - Over/underflow if Exp out of range
  - Round Man to fit mantissa precision

Mathematical Properties of FP Operations

- Exponent overflow yields +∞ or -∞
- Floats with value +∞, -∞, and NaN can be used in operations
  - Result usually still +∞, -∞, or NaN; but not always intuitive
- Floating point operations do not work like real math, due to rounding
  - Not associative: `(3.14+1e100)-1e100 != 3.14+(1e100-1e100)`
    `0` `3.14`
  - Not distributive: `100*(0.1+0.2) != 100*0.1+100*0.2`
    `30.000000000000003553` `30`
  - Not cumulative
    - Repeatedly adding a very small number to a large one may do nothing

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Floating Point in C

!!!

- C offers two (well, 3) levels of precision

`float`	`1.0f`	single precision (32-bit)
`double`	`1.0`	double precision (64-bit)
`long double`	`1.0L`	*("double double" or quadruple)* precision (64-128 bits)

- `#include <math.h>` to get `INFINITY` and `NAN` constants
- Equality (==) comparisons between floating point numbers are tricky, and often return unexpected results, so just avoid them!

Floating Point Conversions in C

!!!

- Casting between `int`, `float`, and `double` **changes** the bit representation
  - `int` → `float`
    - May be rounded (not enough bits in mantissa: 23)
    - Overflow impossible
  - `int` or `float` → `double`
    - Exact conversion (all 32-bit `int`s representable)
  - `long` → `double`
    - Depends on word size (32-bit is exact, 64-bit may be rounded)
  - `double` or `float` → `int`
    - Truncates fractional part (rounded toward zero)
    - "Not defined" when out of range or NaN: generally sets to `Tmin` (even if the value is a very big positive)

Number Representation Really Matters

- **1991:** Patriot missile targeting error
  - clock skew due to conversion from integer to floating point
- **1996:** Ariane 5 rocket exploded ($1 billion)
  - overflow converting 64-bit floating point to 16-bit integer
- **2000:** Y2K problem
  - limited (decimal) representation: overflow, wrap-around
- **2038:** Unix epoch rollover
  - Unix epoch = seconds since 12am, January 1, 1970
  - signed 32-bit integer representation rolls over to TMin in 2038
- **Other related bugs:**
  - 1982: Vancouver Stock Exchange 10% error in less than 2 years
  - 1994: Intel Pentium FDIV (floating point division) HW bug ($475 million)
  - 1997: USS Yorktown "smart" warship stranded: divide by zero
  - 1998: Mars Climate Orbiter crashed: unit mismatch ($193 million)

Floating Point Summary

- Floats also suffer from the fixed number of bits available to represent them
  - Can get overflow/underflow
  - "Gaps" produced in representable numbers means we can lose precision, unlike `int`s
    - Some "simple fractions" have no exact representation (*e.g.* 0.2)
    - "Every operation gets a slightly wrong result"
- Floating point arithmetic not associative or distributive
  - Mathematically equivalent ways of writing an expression may compute different results
- Never test floating point values for equality!
- Careful when converting between `int`s and `float`s!

Floating Point Summary

- Converting between integral and floating point data types *does* change the bits
  - Floating point rounding is a HUGE issue!
    - Limited mantissa bits cause inaccurate representations
    - Floating point arithmetic is NOT associative or distributive

Denorm Numbers

This is extra (non-testable) material

- Denormalized numbers
  - No leading 1
  - Uses implicit exponent of –126 even though E = 0x00
- Denormalized numbers close the gap between zero and the smallest normalized number
  - Smallest norm: $\pm\ 1.0\ldots0_{two}\times 2^{-126} = \pm\ 2^{-126}$
  - Smallest denorm: $\pm\ 0.0\ldots01_{two}\times 2^{-126} = \pm\ 2^{-149}$
    - There is still a gap between zero and the smallest denormalized number

So much closer to 0

Floating Point and the Programmer

```
#include <stdio.h>

int main(int argc, char* argv[]) {
  float f1 = 1.0;
  float f2 = 0.0;
  int i;
  for (i = 0; i < 10; i++)
    f2 += 1.0/10.0;

  printf("0x%08x  0x%08x\n", *(int*)&f1, *(int*)&f2);
  printf("f1 = %10.9f\n", f1);
  printf("f2 = %10.9f\n\n", f2);

  f1 = 1E30;
  f2 = 1E-30;
  float f3 = f1 + f2;
  printf("f1 == f3? %s\n", f1 == f3 ? "yes" : "no" );

  return 0;
}
```

```
$ ./a.out
0x3f800000  0x3f800001
f1 = 1.000000000
f2 = 1.000000119

f1 == f3? yes
```


BONUS SLIDES

An example that applies the IEEE Floating Point concepts to a smaller (8-bit) representation scheme.

Tiny Floating Point Example

s	exp	man
1	4	3

- 8-bit Floating Point Representation
  - The sign bit is in the most significant bit (MSB)
  - The next four bits are the exponent, with a bias of $2^{4-1}–1 = 7$
  - The last three bits are the mantissa
- Same general form as IEEE Format
  - Normalized binary scientific point notation
  - Similar special cases for 0, denormalized numbers, NaN, ∞

Dynamic Range (Positive Only)

	S E M	Exp	Value	
Denormalized numbers	0 0000 000	-6	0	
	0 0000 001	-6	1/8*1/64 = 1/512	closest to zero
	0 0000 010	-6	2/8*1/64 = 2/512	
	…			
	0 0000 110	-6	6/8*1/64 = 6/512	
	0 0000 111	-6	7/8*1/64 = 7/512	largest denorm
Normalized numbers	0 0001 000	-6	8/8*1/64 = 8/512	smallest norm
	0 0001 001	-6	9/8*1/64 = 9/512	
	…			
	0 0110 110	-1	14/8*1/2 = 14/16	
	0 0110 111	-1	15/8*1/2 = 15/16	closest to 1 below
	0 0111 000	0	8/8*1 = 1	
	0 0111 001	0	9/8*1 = 9/8	closest to 1 above
	0 0111 010	0	10/8*1 = 10/8	
	…			
	0 1110 110	7	14/8*128 = 224	
	0 1110 111	7	15/8*128 = 240	largest norm
	0 1111 000	n/a	inf	

Special Properties of Encoding

- Floating point zero (0^+) exactly the same bits as integer zero
  - All bits = 0
- Can (Almost) Use Unsigned Integer Comparison
  - Must first compare sign bits
  - Must consider $0^- = 0^+ = 0$
  - NaNs problematic
    - Will be greater than any other values
    - What should comparison yield?
  - Otherwise OK
    - Denorm vs. normalized
    - Normalized vs. infinity