Integers II

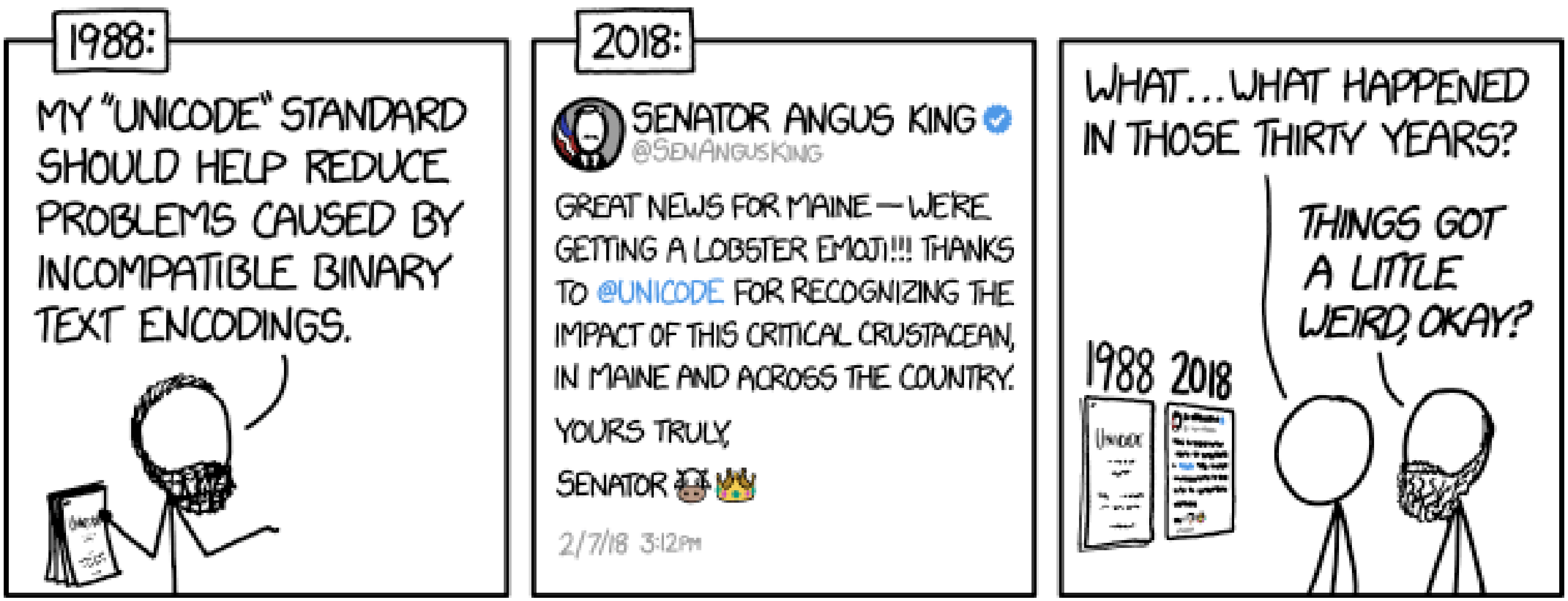
CSE 351 Autumn 2018

Instructor:

Justin Hsia

Teaching Assistants:

Akshat Aggarwal, An Wang, Andrew Hu, Brian Dai, Britt Henderson, James Shin, Kevin Bi, Kory Watson, Riley Germundson, Sophie Tian, Teagan Horkan



http://xkcd.com/1953/

Administrivia

- Lab 1a due Monday (10/8)
  - Submit `pointer.c` and `lab1Areflect.txt` to Canvas
- Lab 1b released today, due 10/12
  - Bit puzzles on number representation
  - Can start after today’s lecture, but floating point will be introduced next week
  - Section worksheet from yesterday has helpful examples, too
  - Bonus slides at the end of today’s lecture have relevant examples

Extra Credit

- All labs starting with Lab 1b have extra credit portions
  - These are meant to be fun extensions to the labs
- Extra credit points *don't* affect your lab grades
  - From the course policies: "they will be accumulated over the course and will be used to bump up borderline grades at the end of the quarter."
  - Make sure you finish the rest of the lab before attempting any extra credit

Integers

- **Binary representation of integers**
  - **Unsigned and signed**
  - **Casting in C**
- Consequences of finite width representations
  - Overflow, sign extension
- Shifting and arithmetic operations

Two's Complement Arithmetic

- The same addition procedure works for both unsigned and two's complement integers
  - **Simplifies hardware:** only one algorithm for addition
  - **Algorithm:** simple addition, discard the highest carry bit
    - Called modular addition: result is sum *modulo* 2^w

- **4-bit Examples:**

4 +3	0100 +0011	-4 +3	1100 +0011	4 -3	0100 +1101
=7	0111	=-1	1111	=1	~~1~~0001

Why Does Two's Complement Work?

- For all representable positive integers x, we want:

additive inverse

$$\left\{\begin{array}{r} \textit{bit representation of} \;\; x \\ + \; \textit{bit representation of} \; -x \\ \hline 0 \end{array}\right.$$

(ignoring the carry-out bit)

  - What are the 8-bit negative encodings for the following?

$$\begin{array}{rr} & 00000001 \\ & 11111111 \\ + & ???????? \\ \hline \cancel{1} & 00000000 \end{array} \qquad \begin{array}{rr} & 00000010 \\ & 11111110 \\ + & ???????? \\ \hline \cancel{1} & 00000000 \end{array} \qquad \begin{array}{rr} & 1\,1\,11\,111 \\ & 11000011 \\ & 00111101 \\ + & ???????? \\ \hline \cancel{1} & 00000000 \end{array}$$

Why Does Two's Complement Work?

- For all representable positive integers x, we want:

$$\begin{array}{r} \textit{bit representation of } x \\ +\ \textit{bit representation of } -x \\ \hline 0 \end{array}$$ (ignoring the carry-out bit)

0b1…1

$x + (\sim x) =$

$x + (\sim x) = -1$

$x + (\sim x + 1) = 0$

$\boxed{-x = \sim x + 1}$

  - What are the 8-bit negative encodings for the following?

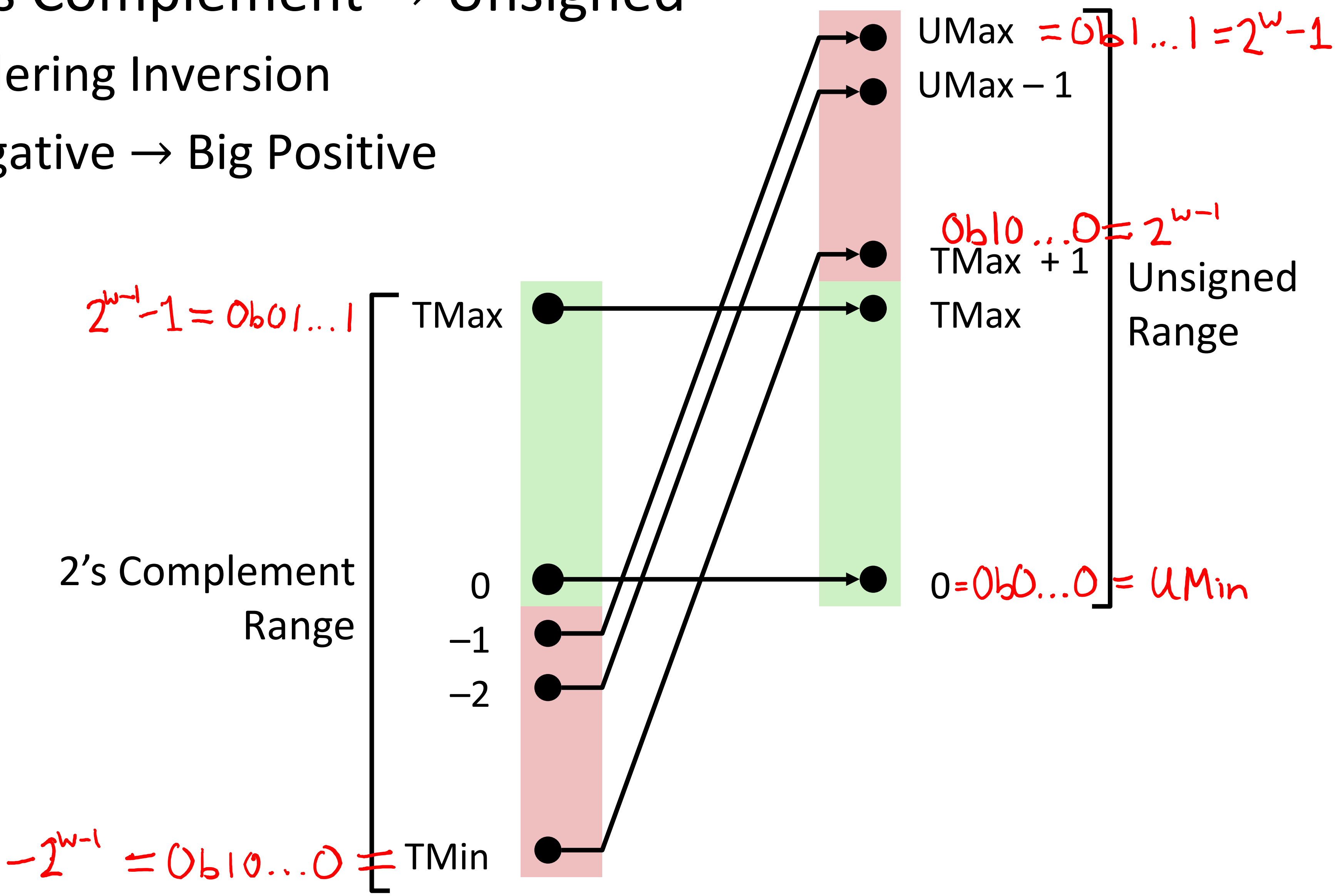
$$\begin{array}{r} 00000001 \\ +\ 11111111 \\ \hline \cancel{1}00000000 \end{array} \qquad \begin{array}{r} 00000010 \\ +\ 11111110 \\ \hline \cancel{1}00000000 \end{array} \qquad \begin{array}{r} 11000011 \\ +\ 00111101 \\ \hline \cancel{1}00000000 \end{array}$$

These are the bitwise complement plus 1!

`-x == ~x + 1`

Signed/Unsigned Conversion Visualized

- Two's Complement → Unsigned
  - Ordering Inversion
  - Negative → Big Positive



Values To Remember

- Unsigned Values
  - UMin = `0b00…0`
    = 0
  - UMax = `0b11…1`
    = $2^w - 1$

- Two's Complement Values
  - TMin = `0b10…0`
    = -2^{w-1}
  - TMax = `0b01…1`
    = $2^{w-1} - 1$
  - -1 = `0b11…1`

- **Example:** Values for $w = 64$

	Decimal	Hex
UMax	18,446,744,073,709,551,615	`FF FF FF FF FF FF FF FF`
TMax	9,223,372,036,854,775,807	`7F FF FF FF FF FF FF FF`
TMin	-9,223,372,036,854,775,808	`80 00 00 00 00 00 00 00`
-1	-1	`FF FF FF FF FF FF FF FF`
0	0	`00 00 00 00 00 00 00 00`

In C: Signed vs. Unsigned

- Casting
  - Bits are unchanged, just interpreted differently!
    - **int** tx, ty;
    - **unsigned int** ux, uy;
  - *Explicit* casting
    - tx = (**int**) ux;
    - uy = (**unsigned int**) ty;
  - *Implicit* casting can occur during assignments or function calls cast to target variable/parameter type
    - tx = ux;
    - uy = ty; (also implicitly occurs with printf format specifiers)

Casting Surprises

!!!

- Integer literals (constants)
  - By default, integer constants are considered *signed* integers
    - Hex constants already have an explicit binary representation
  - Use "U" (or "u") suffix to explicitly force *unsigned*
    - **Examples:** 0U, 4294967259u
- Expression Evaluation
  - When you mixed unsigned and signed in a single expression, then **signed values are implicitly cast to unsigned** (unsigned "dominates")
  - Including comparison operators <, >, ==, <=, >=

Casting Surprises

!!!

- 32-bit examples:
  - TMin = -2,147,483,648, TMax = 2,147,483,647

Left Constant	Order	Right Constant	Interpretation
0 0000 0000 0000 0000 0000 0000 0000 0000	==	**0U** 0000 0000 0000 0000 0000 0000 0000 0000	unsigned
-1 1111 1111 1111 1111 1111 1111 1111 1111	<	**0** 0000 0000 0000 0000 0000 0000 0000 0000	signed
-1 1111 1111 1111 1111 1111 1111 1111 1111	>	**0U** 0000 0000 0000 0000 0000 0000 0000 0000	unsigned
2147483647 0111 1111 1111 1111 1111 1111 1111 1111	>	**-2147483648** 1000 0000 0000 0000 0000 0000 0000 0000	signed
2147483647U 0111 1111 1111 1111 1111 1111 1111 1111	<	**-2147483648** 1000 0000 0000 0000 0000 0000 0000 0000	unsigned
-1 1111 1111 1111 1111 1111 1111 1111 1111	>	**-2** 1111 1111 1111 1111 1111 1111 1111 1110	signed
(unsigned) -1 1111 1111 1111 1111 1111 1111 1111 1111	>	**-2** 1111 1111 1111 1111 1111 1111 1111 1110	unsigned
2147483647 0111 1111 1111 1111 1111 1111 1111 1111	<	**2147483648U** 1000 0000 0000 0000 0000 0000 0000 0000	unsigned
2147483647 0111 1111 1111 1111 1111 1111 1111 1111	>	**(int) 2147483648U** 1000 0000 0000 0000 0000 0000 0000 0000	signed

Integers

- Binary representation of integers
  - Unsigned and signed
  - Casting in C
- **Consequences of finite width representations**
  - **Overflow, sign extension**
- Shifting and arithmetic operations

Arithmetic Overflow

Bits	Unsigned	Signed
0000	0 UMin	0
0001	1	1
0010	2	2
0011	3	3
0100	4	4
0101	5	5
0110	6	6
0111	7	7 TMax
1000	8	-8 TMin
1001	9	-7
1010	10	-6
1011	11	-5
1100	12	-4
1101	13	-3
1110	14	-2
1111	15 UMax	-1

- When a calculation produces a result that can't be represented in the current encoding scheme
  - Integer range limited by fixed width UMin – UMax TMin – TMax
  - Can occur in both the positive and negative directions
- C and Java ignore overflow exceptions
  - You end up with a bad value in your program and no warning/indication… oops!

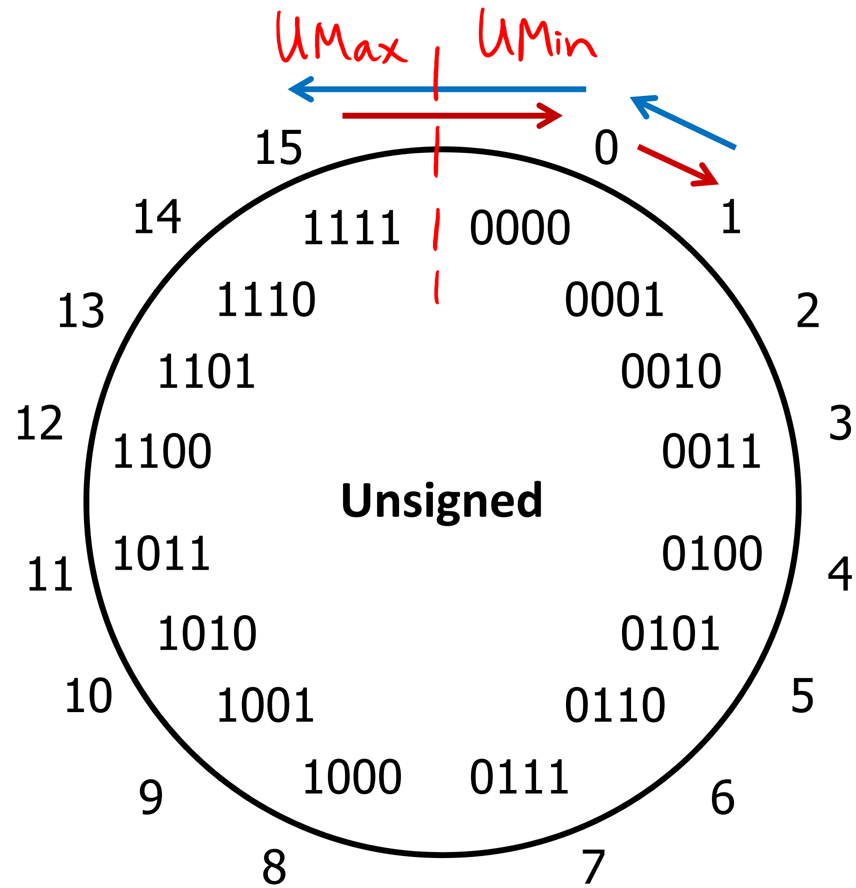
Overflow: Unsigned

- **Addition:** drop carry bit (-2^N)

$$\begin{array}{r} 15 \\ +\ \ 2 \\ \hline \cancel{17} \\ 1 \end{array} \qquad \begin{array}{r} 1111 \\ +\ \ 0010 \\ \hline \cancel{1}0001 \end{array}$$

- **Subtraction:** borrow ($+2^N$)

$$\begin{array}{r} 1 \\ -\ \ 2 \\ \hline \cancel{-1} \\ 15 \end{array} \qquad \begin{array}{r} {\scriptstyle 1\ 1} \\ {\scriptstyle \cancel{2}\ \cancel{2}\ 2} \\ \cancel{1}0001 \\ -\ \ 0010 \\ \hline 1111 \end{array}$$



$\pm 2^N$ because of modular arithmetic

$2^4 = 16$

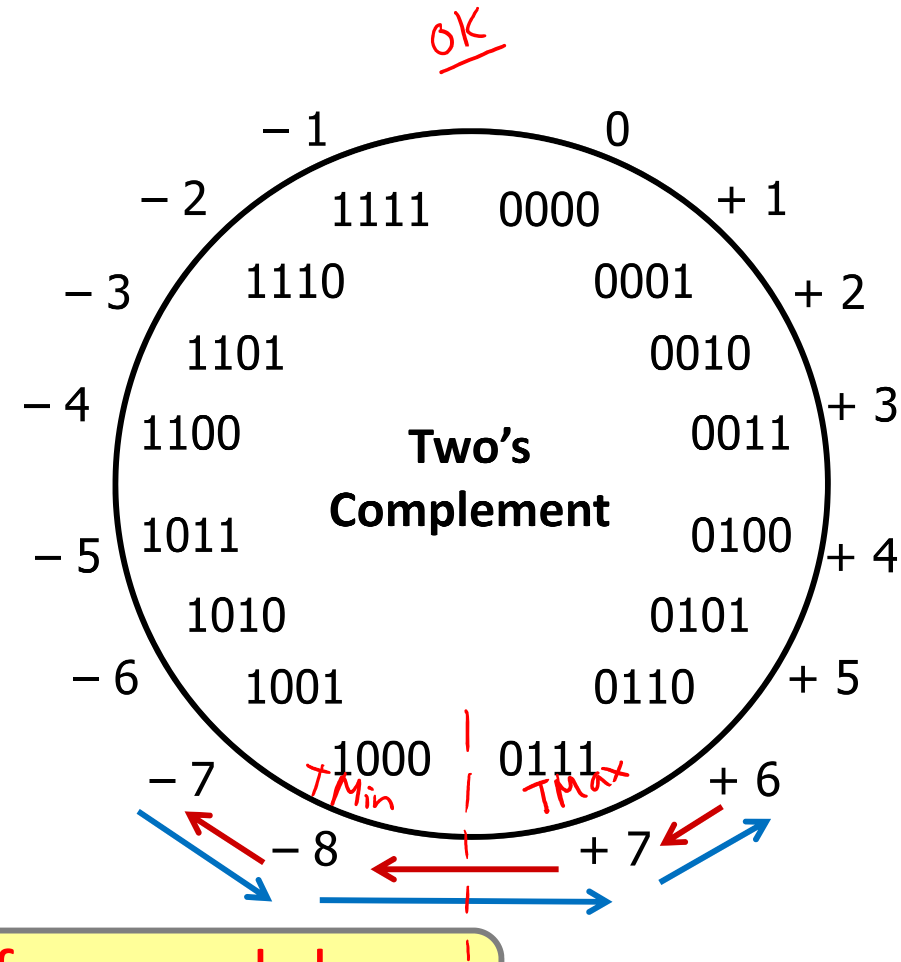
Overflow: Two's Complement

- **Addition:** (+) + (+) = (–) result?

6	0110
+ 3	+ 0011
~~9~~	1001
–7	

- **Subtraction:** (–) + (–) = (+)?

–7	1001
– 3	– 0011
~~–10~~	0110
6	



For signed: overflow if operands have same sign and result's sign is different

Sign Extension

- What happens if you convert a *signed* integral data type to a larger one?
  - *e.g.* `char` → `short` → `int` → `long`

    1 byte, 2 bytes, 4 bytes, 8 bytes

- **4-bit → 8-bit Example:**
  - Positive Case
    - ✓ Add 0's?

4-bit: 0010 = +2

8-bit: 00000010 = +2

  - Negative Case?

Peer Instruction Question

- Which of the following 8-bit numbers has the same *signed* value as the 4-bit number **0b<u>1</u>100**?
  - Underlined digit = MSB
  - Vote at http://PollEv.com/justinh

$-8\ 4\ 2\ 1$ $-8+4=-4$

$-x = 0011$, $+1$, $0100 = 4 \Rightarrow x=-4$

A. ~~0b 0000 1100~~ (add zeros) positive!

B. ~~0b 1000 1100~~ (add leading 1) much too negative: $-2^7+2^3+2^2 = -116$

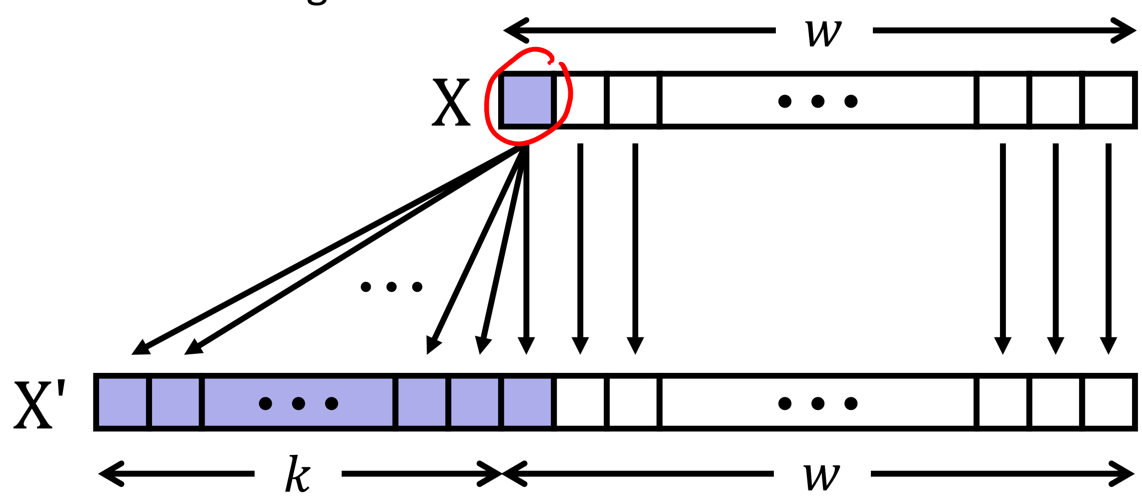
C. $\boxed{\text{0b 1111 1100 (add ones)}}$ correct! $-y = $ 0b 0000 0011 $+1 = 4$, $y=-4$

D. ~~0b 1100 1100~~ (duplicate) $-2^7+2^6+2^3+2^2 = -52$

E. We're lost…

Sign Extension

- **Task:** Given a w-bit signed integer X, convert it to w+k-bit signed integer X' *with the same value*
- **Rule: Add k copies of sign bit**
  - Let x_i be the i-th digit of X in binary
  - $\mathrm{X}' = \underbrace{x_{w-1}, \ldots, x_{w-1}}_{k \text{ copies of MSB}}, \underbrace{x_{w-1}, x_{w-2}, \ldots, x_1, x_0}_{\text{original } \mathrm{X}}$



Sign Extension Example

- Convert from smaller to larger integral data types
- C automatically performs sign extension
  - Java too

```
short int x =   12345;
int      ix = (int) x;
short int y =  -12345;
int      iy = (int) y;
```

0b 0011

Var	Decimal	Hex	Binary
x	12345	30 39	00110000 00111001
ix	12345	00 00 30 39	00000000 00000000 00110000 00111001
y	-12345	CF C7	11001111 11000111
iy	-12345	FF FF CF C7	11111111 11111111 11001111 11000111

0b 1100

Integers

- Binary representation of integers
  - Unsigned and signed
  - Casting in C
- Consequences of finite width representations
  - Overflow, sign extension
- **Shifting and arithmetic operations**

Shift Operations

- Left shift (`x<<n`) bit vector `x` by `n` positions
  - Throw away (drop) extra bits on left
  - Fill with `0`s on right
- Right shift (`x>>n`) bit-vector `x` by `n` positions
  - Throw away (drop) extra bits on right
  - Logical shift (for unsigned values)
    - Fill with `0`s on left
  - Arithmetic shift (for signed values)
    - Replicate most significant bit on left
    - Maintains sign of `x`

Shift Operations

8-bit example:

	x	0010 0010
	x<<3	0001 0000
logical:	x>>2	0000 1000
arithmetic:	x>>2	0000 1000

	x	1010 0010
	x<<3	0001 0000
logical:	x>>2	0010 1000
arithmetic:	x>>2	1110 1000

- Left shift (`x<<n`)
  - Fill with `0`s on right
- Right shift (`x>>n`)
  - Logical shift (for unsigned values)
    - Fill with 0s on left
  - Arithmetic shift (for signed values)
    - Replicate most significant bit on left
- Notes:
  - Shifts by `n<0` or `n≥w` (bit width of x) are *undefined*: behavior not guaranteed
  - **In C:** behavior of >> is determined by compiler
    - In gcc / C lang, depends on data type of `x` (signed/unsigned) arithmetic/logical
  - **In Java:** logical shift is >>> and arithmetic shift is >>

Shifting Arithmetic?

- What are the following computing?
  - `x>>n`
    - $\overset{4}{\texttt{0b 0100}} \texttt{ >> 1 = } \overset{2}{\texttt{0b 0010}}$
    - $\overset{4}{\texttt{0b 0100}} \texttt{ >> 2 = } \overset{1}{\texttt{0b 0001}}$
    - Divide by 2^n
  - `x<<n`
    - $\overset{1}{\texttt{0b 0001}} \texttt{ << 1 = } \overset{2}{\texttt{0b 0010}}$
    - $\overset{1}{\texttt{0b 0001}} \texttt{ << 2 = } \overset{4}{\texttt{0b 0100}}$
    - Multiply by 2^n
- Shifting is faster than general multiply and divide operations

Left Shifting Arithmetic 8-bit Example

- No difference in left shift operation for unsigned and signed numbers (just manipulates bits)
  - Difference comes during interpretation: `x*2`^{n}?

			Signed	Unsigned
`x = 25;`	`00011001`	=	25	25
`L1=x<<2;`	~~`00`~~`01100100`	=	100	100
`L2=x<<3;`	~~`000`~~`11001000`	=	-56	200
`L3=x<<4;`	~~`0001`~~`10010000`	=	-112	144

200
−256 → 2^8

400
−256 → 2^8

signed overflow

unsigned overflow

Right Shifting Arithmetic 8-bit Examples

- **Reminder:** C operator >> does *logical* shift on unsigned values and *arithmetic* shift on signed values
  - Logical Shift: $x/2^n$?

```
xu  =  240u;  11110000      =  240
R1u=xu>>3;    00011110000   =   30
R2u=xu>>5;    0000011110000 =    7
```

$240/8 = 30$

$30/4 = 7.5$

rounding (down)

Right Shifting Arithmetic 8-bit Examples

- **Reminder:** C operator >> does *logical* shift on unsigned values and *arithmetic* shift on signed values
  - Arithmetic Shift: $x/2^n$?

```
xs = -16;      11110000         = -16
R1s=xu>>3;     11111110~~000~~      =  -2
R2s=xu>>5;     1111111~~10000~~   =  -1
```

$/4 = -0.5$

rounding (down)

Peer Instruction Question

UMin = 0 , UMax = 255
8-bits, so TMin = -128, TMax = 127

For the following expressions, find a value of **`signed char`** `x`, if there exists one, that makes the expression `TRUE`. Compare with your neighbor(s)!

- Assume we are using 8-bit arithmetic:

Expression	Example:	All solutions:
`x == (unsigned char) x` (unsigned)	x = 0	works for all x
`x >= 128U` (unsigned) 0b1000 0000	x = -1	any x < 0
`x != (x>>2)<<2`	x = 3	any x where lowest two bits are not 0b00
`x == -x` — Hint: there are two solutions	x = 0	① x = 0b0...0 = 0 ② x = 0b10...0 = -128
`(x < 128U) && (x > 0x3F)`	x = 64	any x where upper two bits are exactly 0b01

Summary

- Sign and unsigned variables in C
  - Bit pattern remains the same, just *interpreted* differently
  - Strange things can happen with our arithmetic when we convert/cast between sign and unsigned numbers
    - Type of variables affects behavior of operators (shifting, comparison)
- We can only represent so many numbers in *w* bits
  - When we exceed the limits, *arithmetic overflow* occurs
  - *Sign extension* tries to preserve value when expanding
- Shifting is a useful bitwise operator
  - Right shifting can be arithmetic (sign) or logical (0)
  - Can be used in multiplication with constant or bit masking

BONUS SLIDES

Some examples of using shift operators in combination with bitmasks, which you may find helpful for Lab 1.

- Extract the 2nd most significant byte of an `int`
- Extract the sign bit of a signed `int`
- Conditionals as Boolean expressions

Using Shifts and Masks

- Extract the 2nd most significant *byte* of an `int`:
  - First shift, then mask: `(x>>16) & 0xFF`

x	00000001 00000010 00000011 00000100
x>>16	00000000 00000000 00000001 00000010
0xFF	00000000 00000000 00000000 11111111
(x>>16) & 0xFF	00000000 00000000 00000000 00000010

  - Or first mask, then shift: `(x & 0xFF0000)>>16`

x	00000001 00000010 00000011 00000100
0xFF0000	00000000 11111111 00000000 00000000
x & 0xFF0000	00000000 00000010 00000000 00000000
(x&0xFF0000)>>16	00000000 00000000 00000000 00000010

Using Shifts and Masks

- Extract the *sign bit* of a signed `int`:
  - First shift, then mask: `(x>>31) & 0x1`
    - Assuming arithmetic shift here, but this works in either case
    - Need mask to clear `1`s possibly shifted in

x	00000001 00000010 00000011 00000100
x>>31	00000000 00000000 00000000 00000000
0x1	00000000 00000000 00000000 00000001
(x>>31) & 0x1	00000000 00000000 00000000 00000000

x	10000001 00000010 00000011 00000100
x>>31	11111111 11111111 11111111 11111111
0x1	00000000 00000000 00000000 00000001
(x>>31) & 0x1	00000000 00000000 00000000 00000001

Using Shifts and Masks

- Conditionals as Boolean expressions
  - For **`int`** `x`, what does `(x<<31)>>31` do?

`x=!!123`	`00000000 00000000 00000000 00000001`
`x<<31`	`10000000 00000000 00000000 00000000`
`(x<<31)>>31`	`11111111 11111111 11111111 11111111`
`!x`	`00000000 00000000 00000000 00000000`
`!x<<31`	`00000000 00000000 00000000 00000000`
`(!x<<31)>>31`	`00000000 00000000 00000000 00000000`

  - Can use in place of conditional:
    - In C: `if(x) {a=y;} else {a=z;}` equivalent to `a=x?y:z;`
    - `a=(((x<<31)>>31)&y) | (((!x<<31)>>31)&z);`