Floating Point

CSE 351 Autumn 2017

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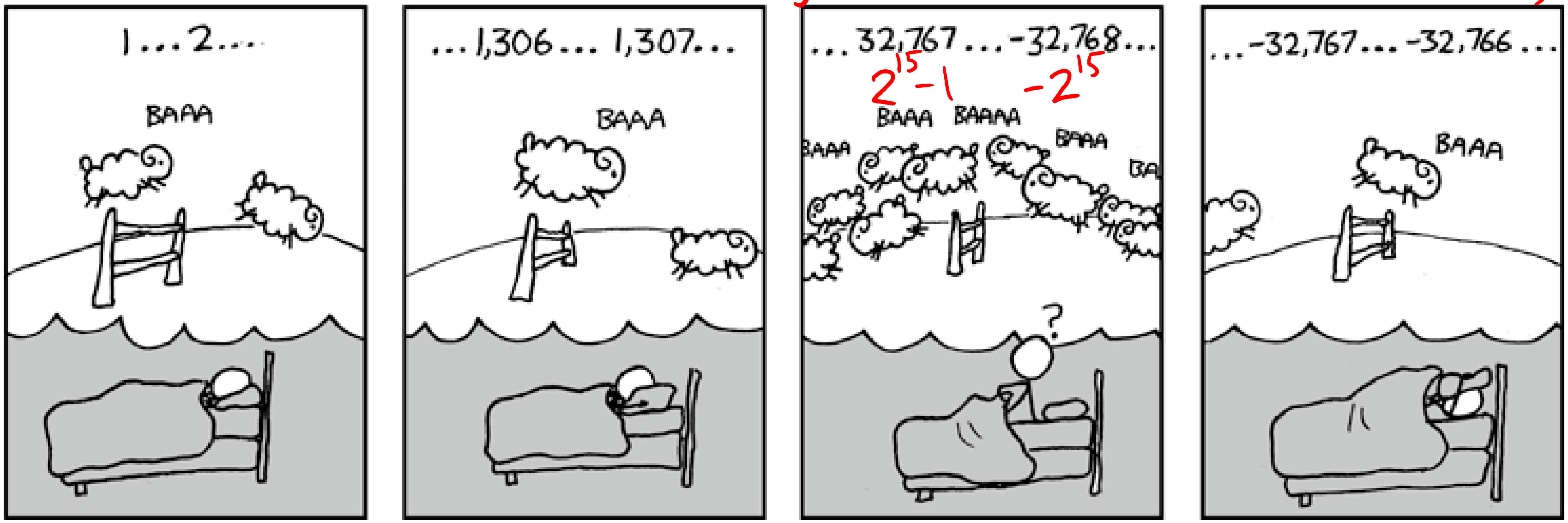
signed overflow in 16 bits → short (in C)



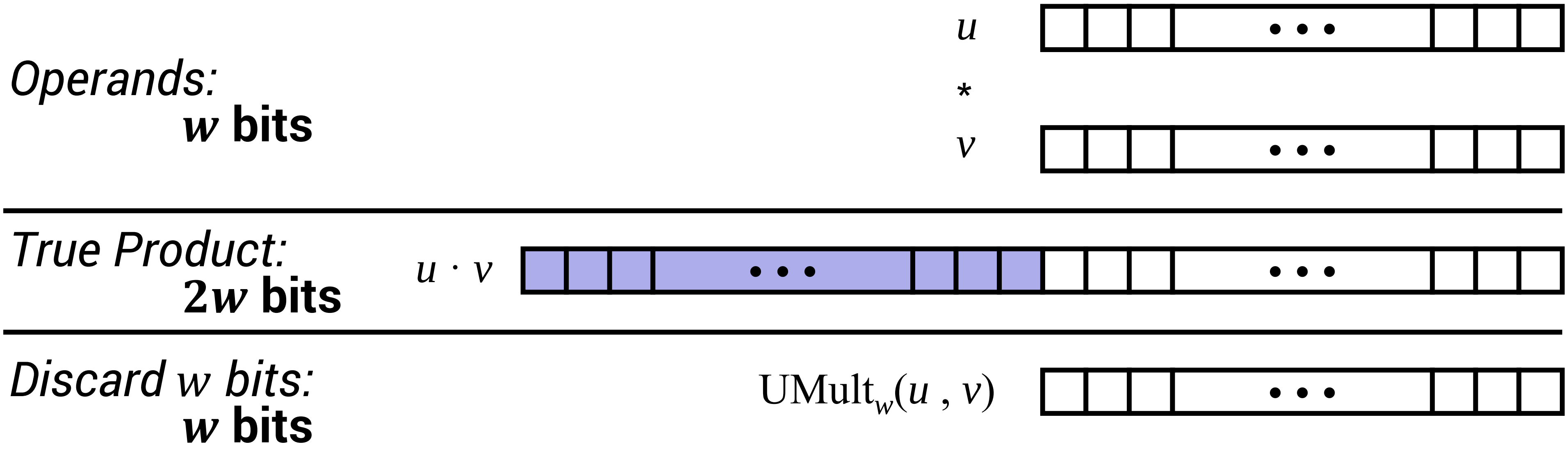
$2^{15}-1$ -2^{15}

http://xkcd.com/571/

Administrivia

- Lab 1 Prelim due tonight at 11:59pm
  - Only submit `bits.c`
- Lab 1 due Friday (10/13)
  - Submit `bits.c`, `pointer.c`, `lab1reflect.txt`
- Homework 2 released tomorrow, due 10/20
  - On Integers, Floating Point, and x86-64

Unsigned Multiplication in C



Operands: w **bits**

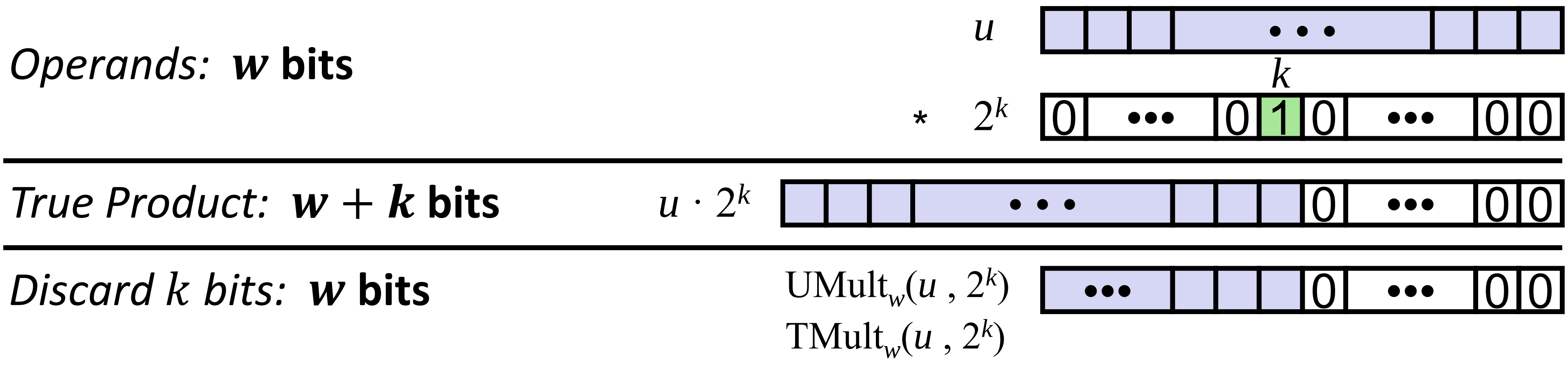
True Product: $2w$ **bits**

Discard w bits: w **bits**

- Standard Multiplication Function
  - Ignores high order w bits
- Implements Modular Arithmetic
  - $\mathrm{UMult}_w(u , v) = u \cdot v \mod 2^w$

Multiplication with shift and add

- Operation `u<<k` gives `u*2`^{k}
  - Both signed and unsigned



Operands: w **bits**

True Product: $w + k$ **bits**

Discard k *bits:* w **bits**

$\mathrm{UMult}_w(u\ ,\ 2^k)$

$\mathrm{TMult}_w(u\ ,\ 2^k)$

- Examples:
  - `u<<3 == u * 8`
  - `u<<5 - u<<3 == u * 24` → 32-8 ↳ 16+8

    u<<4 + u<<3
  - Most machines shift and add faster than multiply
    - ***Compiler generates this code automatically***

Number Representation Revisited

- What can we represent in one word?
  - Signed and Unsigned Integers
  - Characters (ASCII)
  - Addresses
- How do we encode the following:
  - Real numbers (*e.g.* 3.14159) π
  - Very large numbers (*e.g.* 6.02×10^{23}) Avogadro's
  - Very small numbers (*e.g.* 6.626×10^{-34}) Planck's
  - Special numbers (*e.g.* ∞, NaN)

Floating Point

Floating Point Topics

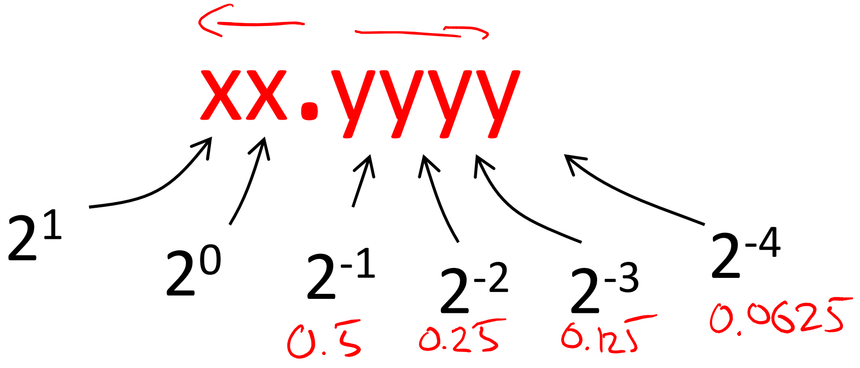
- **Fractional binary numbers**
- IEEE floating-point standard
- Floating-point operations and rounding
- Floating-point in C

- There are many more details that we won't cover
  - It's a 58-page standard…

Representation of Fractions

- "Binary Point," like decimal point, signifies boundary between integer and fractional parts:

Example 6-bit representation:



xx.yyyy

2^1 2^0 2^{-1} 2^{-2} 2^{-3} 2^{-4}

0.5 0.25 0.125 0.0625

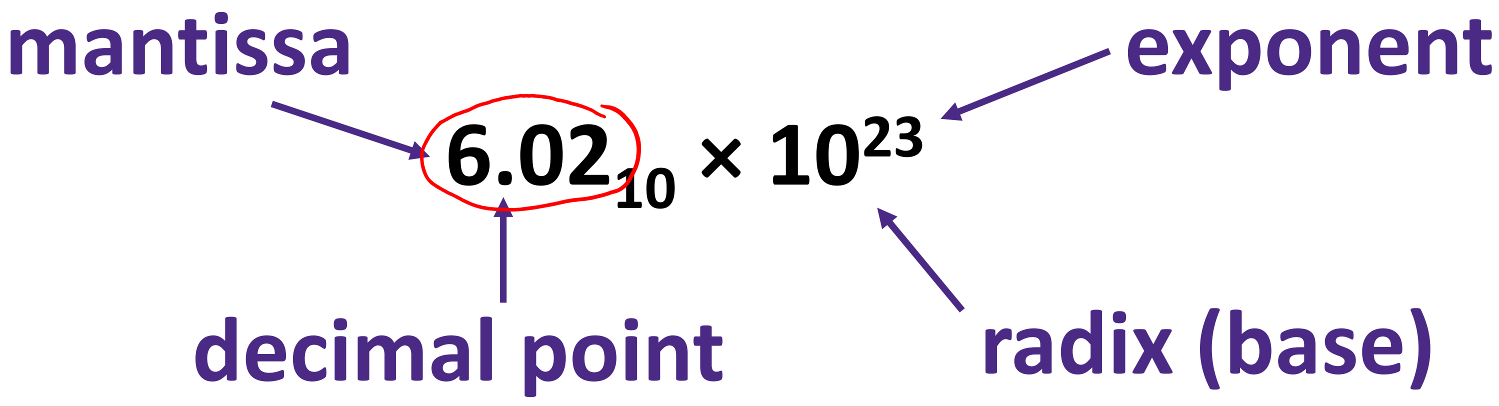
- Example: $10.1010_2 = 1\times2^1 + 1\times2^{-1} + 1\times2^{-3} = 2.625_{10}$

- Binary point numbers that match the 6-bit format above range from 0 (00.0000_2) to 3.9375 (11.1111_2) 2^{-4}

$= 4 - 2^{-4}$

$$\begin{array}{r} 11.1111 \\ +1 \\ \hline 100.0000 \end{array}$$

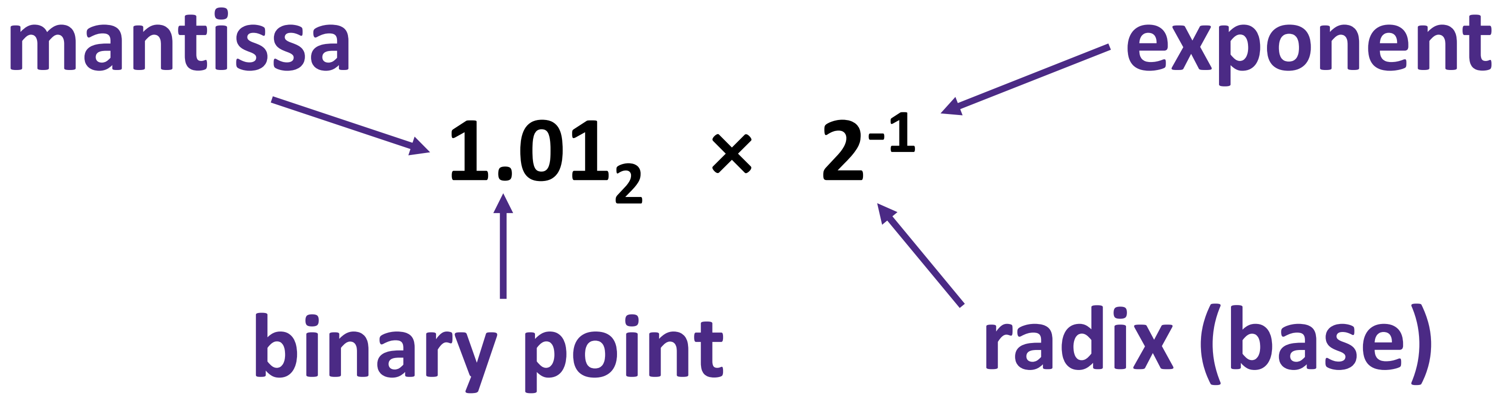
Scientific Notation (Decimal)



- *Normalized form*: exactly one digit (non-zero) to left of decimal point

- Alternatives to representing 1/1,000,000,000
    - Normalized: 1.0×10^{-9} $(1.0\times10^{-1})\times10^{-8}$ $(1.0\times10^{1})\times10^{-10}$
    - Not normalized: 0.1×10^{-8}, 10.0×10^{-10}

Scientific Notation (Binary)



$$1.01_2 \times 2^{-1}$$

- Computer arithmetic that supports this called floating point due to the "floating" of the binary point
    - Declare such variable in C as `float` (or `double`)

Scientific Notation Translation

$2^{-1} = 0.5$
$2^{-2} = 0.25$
$2^{-3} = 0.125$
$2^{-4} = 0.0625$

- Convert from scientific notation to binary point
  - Perform the multiplication by shifting the decimal until the exponent disappears
    - Example: $1.011_2 \times 2^{\textcircled{4}} = \overset{4}{1}0\overset{2}{1}\overset{1}{1}0_2 = 22_{10}$
    - Example: $1.011_2 \times 2^{\textcircled{-2}} = 0.0\overset{-2}{1}0\overset{-4}{1}\overset{-5}{1}_2 = 0.34375_{10}$
- Convert from binary point to *normalized* scientific notation
  - Distribute out exponents until binary point is to the right of a single digit
    - Example: $1101.001_2 = 1.101001_2 \times 2^{\underline{3}}$
- **Practice:** Convert 11.375_{10} to binary scientific notation (normalized)

$8+2+1+0.25+0.125$

$2^3+2^1+2^0+2^{-2}+2^{-3} = 1011.011_2 = \boxed{1.011011 \times 2^3}$

- **Practice:** Convert 1/5 to binary

$\frac{1}{5} - \frac{1}{8} = \frac{3}{40}$, $\frac{3}{40} - \frac{1}{16} = \frac{1}{80} = \frac{1}{16}\left(\frac{1}{5}\right)$

$\frac{1}{5} = \frac{1}{8} + \frac{1}{16} + \frac{1}{16}\left(\frac{1}{5}\right)$

same number, but shifted right by 4 bits

$\boxed{0.\overline{0011}}$

Floating Point Topics

- Fractional binary numbers
- **IEEE floating-point standard**
- Floating-point operations and rounding
- Floating-point in C

- There are many more details that we won't cover
    - It's a 58-page standard…

IEEE Floating Point

- IEEE 754
  - Established in 1985 as uniform standard for floating point arithmetic
  - Main idea: make numerically sensitive programs portable
  - Specifies two things: representation and result of floating operations
  - Now supported by all major CPUs

- Driven by numerical concerns
  - **Scientists**/numerical analysts want them to be as **real** as possible
  - **Engineers** want them to be **easy to implement** and **fast**
  - In the end:
    - Scientists mostly won out
    - Nice standards for rounding, overflow, underflow, but...
    - Hard to make fast in hardware
    - **Float operations can be an order of magnitude slower than integer ops**

competing goals

FLOPs

used in computer benchmarks

Floating Point Encoding

- Use normalized, base 2 scientific notation:
  - Value: $\pm 1 \times \text{Mantissa} \times 2^{\text{Exponent}}$
  - Bit Fields: $(-1)^{S} \times 1.M \times 2^{(E-\text{bias})}$
- Representation Scheme:
  - Sign bit (0 is positive, 1 is negative)
  - Mantissa (a.k.a. significand) is the fractional part of the number in normalized form and encoded in bit vector **M**
  - Exponent weights the value by a (possibly negative) power of 2 and encoded in the bit vector **E**

31 30 23 22 0

S **E** **M**

1 bit **8 bits** **23 bits**

3 separate fields within 32 bits

The Exponent Field

- Use biased notation

  $w=8$

  - Read exponent as unsigned, but with *bias of* $2^{w-1}-1$ = 127
  - Representable exponents roughly ½ positive and ½ negative
  - Exponent 0 (Exp = 0) is represented as E = 0b 0111 1111 $= 2^7 - 1$

    $- \text{bias}$

    $\text{Exp} = 0$
- Why biased?
  - Makes floating point arithmetic easier
  - Makes somewhat compatible with two's complement
- **Practice:** To encode in biased notation, add the bias then encode in unsigned:
  - Exp = 1 $\xrightarrow{+\text{bias}}$ 128 $\xrightarrow{\text{encode}}$ E = 0b 1000 0000
  - Exp = 127 → 254 → E = 0b 1111 1110
  - Exp = -63 → 64 → E = 0b 0100 0000

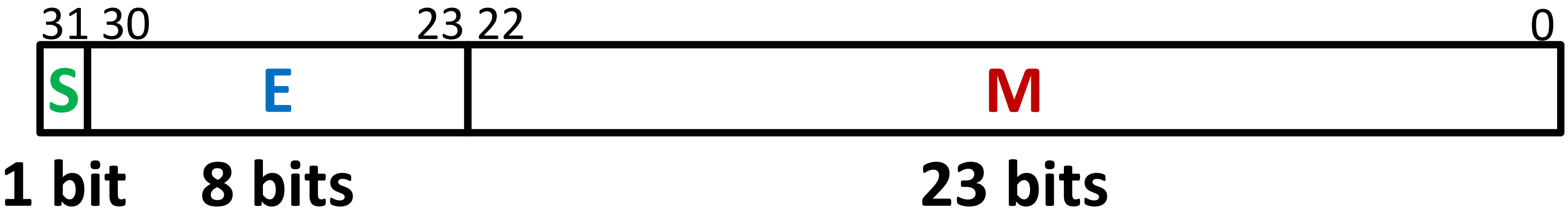
2^7

2^8

$256-2 = (256-1) - 1$

2^6

The Mantissa (Fraction) Field



$$(-1)^{S} \times (1 \, . \, M) \times 2^{(E-\text{bias})}$$

- Note the implicit 1 in front of the M bit vector
  - ⊕ Exp = 0 Mantissa = 1.1
  - Example: 0b 0011 1111 1100 0000 0000 0000 0000 0000 is read as $1.1_2 = 1.5_{10}$, *not* $0.1_2 = 0.5_{10}$
  - Gives us an extra bit of *precision*
- Mantissa "limits"
  - Low values near M = 0b0…0 are close to 2^{Exp} → $1.0\ldots00 \times 2^{\text{Exp}}$
  - High values near M = 0b1…1 are close to $2^{\text{Exp}+1}$ → $1.1\ldots11 \times 2^{\text{Exp}} = 2^{\text{Exp}}(2-2^{-23})$

Peer Instruction Question

- What is the correct value encoded by the following floating point number? bias = 127
  - S E M
  - 0b 0 10000000 11000000000000000000000
  - 128 - 127
  - ⊕ Exp = 1 Man = 1.110...0
  - Vote at http://PollEv.com/justinh ↖ implicit

A. + 0.75

B. + 1.5

C. + 2.75

[✓ **D.** + 3.5]

E. We're lost…

$+1\underset{\curvearrowright}{.1}1_2 \times 2^1$

$11.1_2 = 2^1+2^0+2^{-1} = 3.5$

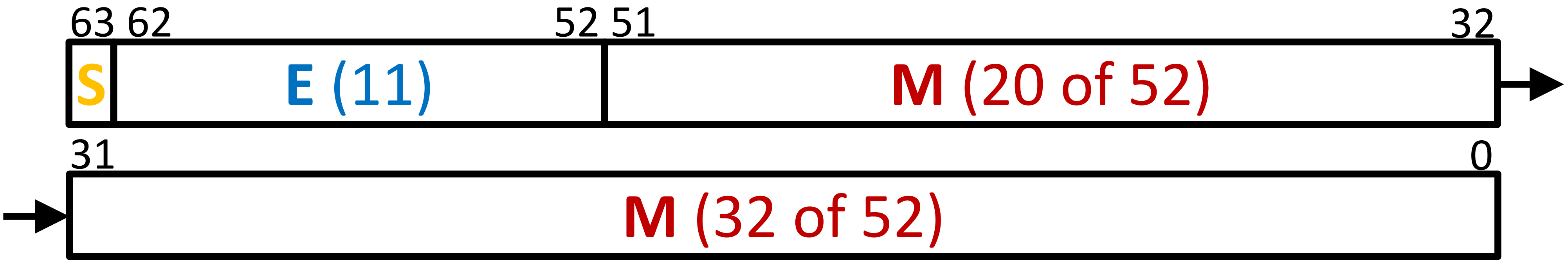
Precision and Accuracy

- Precision is a count of the number of bits in a computer word used to represent a value
  - Capacity for accuracy
- Accuracy is a measure of the difference between the *actual value of a number* and its computer representation

  - *High precision permits high accuracy but doesn't guarantee it. It is possible to have high precision but low accuracy.*
  - **Example:** `float pi = 3.14;`
    - `pi` will be represented using all 24 bits of the mantissa (highly precise), but is only an approximation (not accurate)

Need Greater Precision?

- Double Precision (vs. Single Precision) in 64 bits



  - C variable declared as `double`
  - Exponent bias is now $2^{10}–1 = 1023$, bias $= 2^{w-1}-1$
  - **Advantages:** greater precision (larger mantissa), greater range (larger exponent)
  - **Disadvantages:** more bits used, slower to manipulate

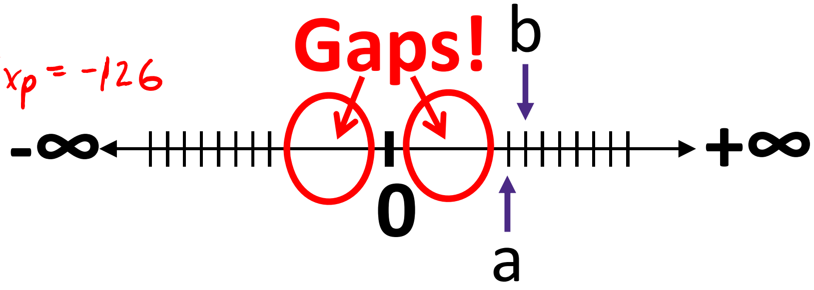
Representing Very Small Numbers

- But wait… what happened to zero?
  - Using standard encoding 0x00000000 = $1.0 \times 2^{-127} \neq 0$

    S=0, E=-127, M=0
  - *Special case:* E and M all zeros = 0
    - Two zeros! But at least 0x00000000 = 0 like integers

      0x80000000 = -0
- New numbers closest to 0:

  (E = 0b 0000 0001) → Exp = -126
  - a = $1.0…0_2 \times 2^{-126} = 2^{-126}$
  - b = $1.0…01_2 \times 2^{-126} = 2^{-126} + 2^{-149}$
  - Normalization and implicit 1 are to blame
  - *Special case:* E = 0, M ≠ 0 are denormalized numbers



Denorm Numbers

This is extra (non-testable) material

- Denormalized numbers
  - $\boxed{\text{No leading 1}}$
  - Uses implicit exponent of –126 even though E = 0x00

- Denormalized numbers close the gap between zero and the smallest normalized number
  - Smallest norm: $\pm\ 1.0\ldots0_{two}\times 2^{-126} = \pm\ 2^{-126}$
  - Smallest denorm: $\pm\ \underline{0}.0\ldots01_{two}\times 2^{-126} = \pm\ 2^{-149}$
    - There is still a gap between zero and the smallest denormalized number

So much closer to 0

Other Special Cases

all 1's

- E = 0xFF, M = 0: ± ∞
  - *e.g.* division by 0
  - Still work in comparisons!
- E = 0xFF, M ≠ 0: Not a Number (NaN)
  - *e.g.* square root of negative number, 0/0, ∞–∞
  - NaN propagates through computations
  - Value of M can be useful in debugging (M tells you cause of NaN)
- New largest value (besides ∞)?
  - E = 0xFF has now been taken!
  - E = 0xFE has largest: $1.1\ldots1_2 \times 2^{127} = 2^{128} - 2^{104}$

Floating Point Encoding Summary

smallest E (all 0's)

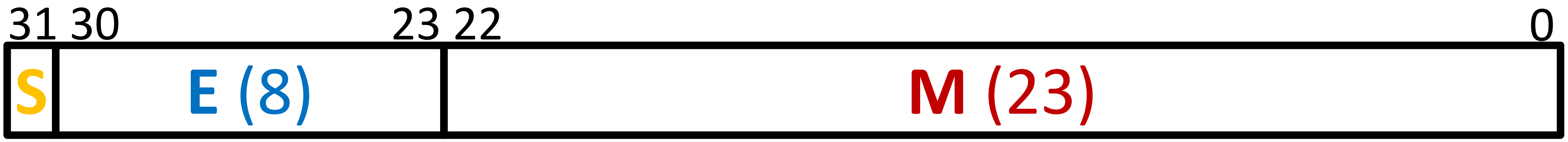
everything else

largest E (all 1's)

Exponent	Mantissa	Meaning
0x00	0	± 0
0x00	non-zero	± denorm num
0x01 – 0xFE	anything	± norm num
0xFF	0	± ∞
0xFF	non-zero	NaN

Summary

- Floating point approximates real numbers:



  - Handles large numbers, small numbers, special numbers
  - Exponent in biased notation (bias = $2^{w-1}–1$)
    - Outside of representable exponents is *overflow* and *underflow*
  - Mantissa approximates fractional portion of binary point
    - Implicit leading 1 (normalized) except in special cases
    - Exceeding length causes *rounding*

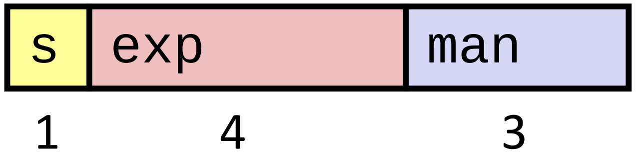
Exponent	Mantissa	Meaning
0x00	0	± 0
0x00	non-zero	± denorm num
0x01 – 0xFE	anything	± norm num
0xFF	0	± ∞
0xFF	non-zero	NaN

BONUS SLIDES

An example that applies the IEEE Floating Point concepts to a smaller (8-bit) representation scheme.

These slides expand on material covered today, so while you don’t need to read these, the information is “fair game.”

Tiny Floating Point Example



- 8-bit Floating Point Representation
  - The sign bit is in the most significant bit (MSB)
  - The next four bits are the exponent, with a bias of $2^{4-1}–1 = 7$
  - The last three bits are the mantissa

- Same general form as IEEE Format
  - Normalized binary scientific point notation
  - Similar special cases for 0, denormalized numbers, NaN, ∞

Dynamic Range (Positive Only)

	S	E	M	Exp	Value	
Denormalized numbers	0	0000	000	-6	0	
	0	0000	001	-6	1/8*1/64 = 1/512	closest to zero
	0	0000	010	-6	2/8*1/64 = 2/512	
	…					
	0	0000	110	-6	6/8*1/64 = 6/512	
	0	0000	111	-6	7/8*1/64 = 7/512	largest denorm
Normalized numbers	0	0001	000	-6	8/8*1/64 = 8/512	smallest norm
	0	0001	001	-6	9/8*1/64 = 9/512	
	…					
	0	0110	110	-1	14/8*1/2 = 14/16	
	0	0110	111	-1	15/8*1/2 = 15/16	closest to 1 below
	0	0111	000	0	8/8*1 = 1	
	0	0111	001	0	9/8*1 = 9/8	closest to 1 above
	0	0111	010	0	10/8*1 = 10/8	
	…					
	0	1110	110	7	14/8*128 = 224	
	0	1110	111	7	15/8*128 = 240	largest norm
	0	1111	000	n/a	inf	

Special Properties of Encoding

- Floating point zero (0^+) exactly the same bits as integer zero
  - All bits = 0
- Can (Almost) Use Unsigned Integer Comparison
  - Must first compare sign bits
  - Must consider $0^- = 0^+ = 0$
  - NaNs problematic
    - Will be greater than any other values
    - What should comparison yield?
  - Otherwise OK
    - Denorm vs. normalized
    - Normalized vs. infinity