Floating Point

CSE 351 Autumn 2017

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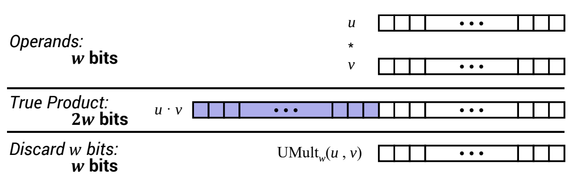
Administrivia

- Lab 1 Prelim due tonight at 11:59pm
  - Only submit `bits.c`
- Lab 1 due Friday (10/13)
  - Submit `bits.c`, `pointer.c`, `lab1reflect.txt`
- Homework 2 released tomorrow, due 10/20
  - On Integers, Floating Point, and x86-64

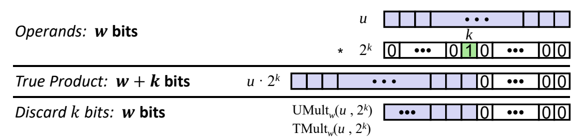
Unsigned Multiplication in C



- Standard Multiplication Function
  - Ignores high order *w* bits
- Implements Modular Arithmetic
  - $\text{UMult}_w(u, v) = u \cdot v \mod 2^w$

Multiplication with shift and add

- Operation `u<<k` gives `u*`2^k
  - Both signed and unsigned



- Examples:
  - `u<<3 == u * 8`
  - `u<<5 - u<<3 == u * 24`
  - Most machines shift and add faster than multiply
    - ***Compiler generates this code automatically***

Number Representation Revisited

- What can we represent in one word?
  - Signed and Unsigned Integers
  - Characters (ASCII)
  - Addresses
- How do we encode the following:
  - Real numbers (*e.g.* 3.14159)
  - Very large numbers (*e.g.* 6.02×10^{23})
  - Very small numbers (*e.g.* 6.626×10^{-34})
  - Special numbers (*e.g.* ∞, NaN)

} **Floating Point**

Floating Point Topics

- **Fractional binary numbers**
- IEEE floating-point standard
- Floating-point operations and rounding
- Floating-point in C

- There are many more details that we won't cover
  - It's a 58-page standard…

Representation of Fractions

- "Binary Point," like decimal point, signifies boundary between integer and fractional parts:

Example 6-bit representation: xx.yyyy

2^1 2^0 2^{-1} 2^{-2} 2^{-3} 2^{-4}

- Example: $10.1010_2 = 1\times2^1 + 1\times2^{-1} + 1\times2^{-3} = 2.625_{10}$
- Binary point numbers that match the 6-bit format above range from 0 (00.0000_2) to 3.9375 (11.1111_2)

Scientific Notation (Decimal)

mantissa → $6.02_{10} \times 10^{23}$ ← exponent

decimal point, radix (base)

- *Normalized form*: exactly one digit (non-zero) to left of decimal point
- Alternatives to representing 1/1,000,000,000
  - Normalized: 1.0×10^{-9}
  - Not normalized: 0.1×10^{-8}, 10.0×10^{-10}

Scientific Notation (Binary)

mantissa → $1.01_2 \times 2^{-1}$ ← exponent

binary point, radix (base)

- Computer arithmetic that supports this called floating point due to the "floating" of the binary point
  - Declare such variable in C as `float` (or `double`)

Scientific Notation Translation

- Convert from scientific notation to binary point
  - Perform the multiplication by shifting the decimal until the exponent disappears
    - Example: $1.011_2\times2^4 = 10110_2 = 22_{10}$
    - Example: $1.011_2\times2^{-2} = 0.01011_2 = 0.34375_{10}$
- Convert from binary point to *normalized* scientific notation
  - Distribute out exponents until binary point is to the right of a single digit
    - Example: $1101.001_2 = 1.101001_2\times2^3$
- **Practice:** Convert 11.375_{10} to binary scientific notation
- **Practice:** Convert 1/5 to binary

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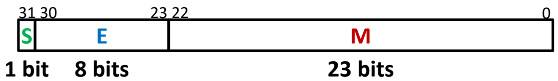
IEEE Floating Point

- IEEE 754
  - Established in 1985 as uniform standard for floating point arithmetic
  - Main idea: make numerically sensitive programs portable
  - Specifies two things: representation and result of floating operations
  - Now supported by all major CPUs
- Driven by numerical concerns
  - **Scientists**/numerical analysts want them to be as **real** as possible
  - **Engineers** want them to be **easy to implement** and **fast**
  - In the end:
    - Scientists mostly won out
    - Nice standards for rounding, overflow, underflow, but...
    - Hard to make fast in hardware
    - **Float operations can be an order of magnitude slower than integer ops**

Floating Point Encoding

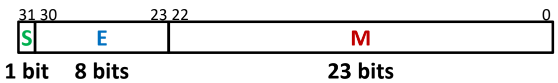
- Use normalized, base 2 scientific notation:
  - Value: $\pm 1 \times \text{Mantissa} \times 2^{\text{Exponent}}$
  - Bit Fields: $(-1)^{S} \times 1.M \times 2^{(E-\text{bias})}$
- Representation Scheme:
  - Sign bit (0 is positive, 1 is negative)
  - Mantissa (a.k.a. significand) is the fractional part of the number in normalized form and encoded in bit vector **M**
  - Exponent weights the value by a (possibly negative) power of 2 and encoded in the bit vector **E**



The Exponent Field

- Use biased notation
  - Read exponent as unsigned, but with *bias* of $2^{w-1}-1 = 127$
  - Representable exponents roughly ½ positive and ½ negative
  - Exponent 0 (Exp = 0) is represented as E = 0b 0111 1111
- Why biased?
  - Makes floating point arithmetic easier
  - Makes somewhat compatible with two's complement
- **Practice:** To encode in biased notation, add the bias then encode in unsigned:
  - Exp = 1 → → E = 0b
  - Exp = 127 → → E = 0b
  - Exp = -63 → → E = 0b

The Mantissa (Fraction) Field



$$(-1)^{S} \times (1\ .\ M) \times 2^{(E-\text{bias})}$$

- Note the implicit 1 in front of the M bit vector
  - <u>Example</u>: 0b 0011 1111 1100 0000 0000 0000 0000 0000 is read as $1.1_2 = 1.5_{10}$, *not* $0.1_2 = 0.5_{10}$
  - Gives us an extra bit of *precision*
- Mantissa "limits"
  - Low values near M = 0b0…0 are close to 2^{Exp}
  - High values near M = 0b1…1 are close to $2^{\text{Exp}+1}$

Peer Instruction Question

- What is the correct value encoded by the following floating point number?
  - 0b 0 10000000 11000000000000000000000
  - Vote at http://PollEv.com/justinh

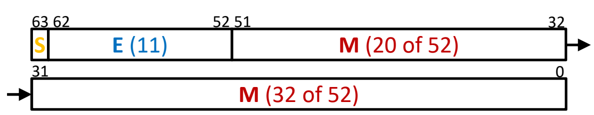
A. + 0.75
B. + 1.5
C. + 2.75
D. + 3.5
E. We're lost…

Precision and Accuracy

- Precision is a count of the number of bits in a computer word used to represent a value
  - Capacity for accuracy
- Accuracy is a measure of the difference between the *actual value of a number* and its computer representation
  - *High precision permits high accuracy but doesn't guarantee it. It is possible to have high precision but low accuracy.*
  - **Example:** `float pi = 3.14;`
    - `pi` will be represented using all 24 bits of the mantissa (highly precise), but is only an approximation (not accurate)

Need Greater Precision?

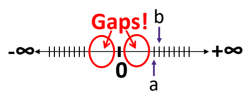
- Double Precision (vs. Single Precision) in 64 bits



  - C variable declared as `double`
  - Exponent bias is now $2^{10}-1 = 1023$
  - **Advantages:** greater precision (larger mantissa), greater range (larger exponent)
  - **Disadvantages:** more bits used, slower to manipulate

Representing Very Small Numbers

- But wait… what happened to zero?
  - Using standard encoding 0x00000000 =
  - *Special case:* E and M all zeros = 0
    - Two zeros! But at least 0x00000000 = 0 like integers
- New numbers closest to 0:
  - a = $1.0…0_2 \times 2^{-126} = 2^{-126}$
  - b = $1.0…01_2 \times 2^{-126} = 2^{-126} + 2^{-149}$
  - Normalization and implicit 1 are to blame
  - *Special case:* E = 0, M ≠ 0 are denormalized numbers



Denorm Numbers

This is extra (non-testable) material

- Denormalized numbers
  - No leading 1
  - Uses implicit exponent of –126 even though E = 0x00
- Denormalized numbers close the gap between zero and the smallest normalized number
  - Smallest norm: $\pm 1.0…0_{two} \times 2^{-126} = \pm 2^{-126}$
  - Smallest denorm: $\pm 0.0…01_{two} \times 2^{-126} = \pm 2^{-149}$
    - There is still a gap between zero and the smallest denormalized number

So much closer to 0

Other Special Cases

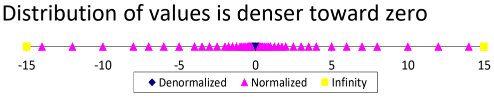
- E = 0xFF, M = 0: ± ∞
  - *e.g.* division by 0
  - Still work in comparisons!
- E = 0xFF, M ≠ 0: Not a Number (NaN)
  - *e.g.* square root of negative number, 0/0, ∞–∞
  - NaN propagates through computations
  - Value of M can be useful in debugging
- New largest value (besides ∞)?
  - E = 0xFF has now been taken!
  - E = 0xFE has largest: $1.1…1_2 \times 2^{127} = 2^{128} – 2^{104}$

Floating Point Encoding Summary

Exponent	Mantissa	Meaning
0x00	0	± 0
0x00	non-zero	± denorm num
0x01 – 0xFE	anything	± norm num
0xFF	0	± ∞
0xFF	non-zero	NaN

Distribution of Values

- What ranges are NOT representable?
  - Between largest norm and infinity — Overflow
  - Between zero and smallest denorm — Underflow
  - Between norm numbers? — Rounding
- Given a FP number, what's the bit pattern of the next largest representable number?
  - What is this "step" when Exp = 0?
  - What is this "step" when Exp = 100?
- Distribution of values is denser toward zero



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Floating Point Operations: Basic Idea

Value = $(-1)^S \times$Mantissa$\times 2^{\text{Exponent}}$

S	E	M

- `x +f y = Round(x + y)`
- `x *f y = Round(x * y)`

- Basic idea for floating point operations:
  - First, compute the exact result
  - Then *round* the result to make it fit into desired precision:
    - Possibly over/underflow if exponent outside of range
    - Possibly drop least-significant bits of mantissa to fit into M bit vector

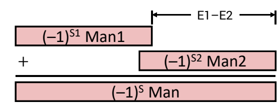
Floating Point Addition

Line up the binary points!

- $(-1)^{S1}\times$Man1$\times 2^{\text{Exp1}}$ + $(-1)^{S2}\times$Man2$\times 2^{\text{Exp2}}$
  - Assume E1 > E2

```
  1.010*2^2        1.0100*2^2
+ 1.000*2^-1     + 0.0001*2^2
  ???              1.0101*2^2
```

- Exact Result: $(-1)^S\times$Man$\times 2^{\text{Exp}}$
  - Sign S, mantissa Man:
    - Result of signed align & add
  - Exponent E: E1



- Adjustments:
  - If Man ≥ 2, shift Man right, increment E
  - if Man < 1, shift Man left k positions, decrement E by k
  - Over/underflow if E out of range
  - Round Man to fit mantissa precision

Floating Point Multiplication

- $(-1)^{S1}\times M1\times 2^{E1}$ × $(-1)^{S2}\times M2\times 2^{E2}$

- Exact Result: $(-1)^S\times M\times 2^E$
  - Sign S: s1 ^ s2
  - Mantissa Man: M1 × M2
  - Exponent E: E1 + E2

- Adjustments:
  - If Man ≥ 2, shift Man right, increment E
  - Over/underflow if E out of range
  - Round Man to fit mantissa precision

Mathematical Properties of FP Operations

- Exponent overflow yields +∞ or -∞
- Floats with value +∞, -∞, and NaN can be used in operations
  - Result usually still +∞, -∞, or NaN; but not always intuitive
- Floating point operations do not work like real math, due to rounding
  - Not associative: `(3.14+1e100)–1e100 != 3.14+(1e100–1e100)`
    - `0` vs `3.14`
  - Not distributive: `100*(0.1+0.2) != 100*0.1+100*0.2`
    - `30.000000000000003553` vs `30`
  - Not cumulative
    - Repeatedly adding a very small number to a large one may do nothing

Summary

- Floating point approximates real numbers:

S (31)	E (8) (30–23)	M (23) (22–0)

  - Handles large numbers, small numbers, special numbers
  - Exponent in biased notation (bias = 2^{w-1}–1)
    - Outside of representable exponents is *overflow* and *underflow*
  - Mantissa approximates fractional portion of binary point
    - Implicit leading 1 (normalized) except in special cases
    - Exceeding length causes *rounding*

Exponent	Mantissa	Meaning
0x00	0	± 0
0x00	non-zero	± denorm num
0x01 – 0xFE	anything	± norm num
0xFF	0	± ∞
0xFF	non-zero	NaN

BONUS SLIDES

An example that applies the IEEE Floating Point concepts to a smaller (8-bit) representation scheme.
These slides expand on material covered today, so while you don't need to read these, the information is "fair game."

Tiny Floating Point Example

s	exp	man
1	4	3

- 8-bit Floating Point Representation
  - The sign bit is in the most significant bit (MSB)
  - The next four bits are the exponent, with a bias of $2^{4-1}–1 = 7$
  - The last three bits are the mantissa
- Same general form as IEEE Format
  - Normalized binary scientific point notation
  - Similar special cases for 0, denormalized numbers, NaN, ∞

Dynamic Range (Positive Only)

	S E M	Exp	Value	
Denormalized numbers	0 0000 000	-6	0	
	0 0000 001	-6	1/8*1/64 = 1/512	closest to zero
	0 0000 010	-6	2/8*1/64 = 2/512	
	…			
	0 0000 110	-6	6/8*1/64 = 6/512	
	0 0000 111	-6	7/8*1/64 = 7/512	largest denorm
Normalized numbers	0 0001 000	-6	8/8*1/64 = 8/512	smallest norm
	0 0001 001	-6	9/8*1/64 = 9/512	
	…			
	0 0110 110	-1	14/8*1/2 = 14/16	
	0 0110 111	-1	15/8*1/2 = 15/16	closest to 1 below
	0 0111 000	0	8/8*1 = 1	
	0 0111 001	0	9/8*1 = 9/8	closest to 1 above
	0 0111 010	0	10/8*1 = 10/8	
	…			
	0 1110 110	7	14/8*128 = 224	
	0 1110 111	7	15/8*128 = 240	largest norm
	0 1111 000	n/a	inf	

Special Properties of Encoding

- Floating point zero (0^+) exactly the same bits as integer zero
  - All bits = 0
- Can (Almost) Use Unsigned Integer Comparison
  - Must first compare sign bits
  - Must consider $0^- = 0^+ = 0$
  - NaNs problematic
    - Will be greater than any other values
    - What should comparison yield?
  - Otherwise OK
    - Denorm vs. normalized
    - Normalized vs. infinity