Number Representation & Operators

CSE 351

Section 2

January 14, 2016

Number Bases

- Any numerical value can be represented as a linear combination of powers of base n, where n is an integer greater than 1
- Example: decimal (`n=10`)
  - Decimal numbers are just linear combinations of 1, 10, 100, 1000, etc.
  - E.g.: $1234 = 1 \times 1000 + 2 \times 100 + 3 \times 10 + 4 \times 1$
- We can also use the base `n=2` (binary) or `n=16` (hexadecimal)

Binary Numbers

- Each digit is either a `1` or a `0`
- Each digit corresponds to a power of 2
- Why use binary?
  - Easy to physically represent two states in memory, registers, across wires, etc.
  - High/Low voltage levels
  - This can scale to much larger numbers by using more hardware to store more bits

Decimal to Binary Conversion

To convert the decimal number `d` to binary, do the following:

1. Compute `(d % 2)`. This will give you the lowest-order bit.
2. Divide `d` by 2, round down to the nearest integer, and continue the process to get the higher order bits.

Example: Convert 25_{10} to binary.

First bit:	`25 % 2 = 1`	`(25 / 2) = 12`
Second bit:	`12 % 2 = 0`	`(12 / 2) = 6`
Third bit:	`6 % 2 = 0`	`(6 / 2) = 3`
Fourth bit:	`3 % 2 = 1`	`(3 / 2) = 1`
Fifth bit:	`1 % 2 = 1`	`(1 / 2) = 0`

Since we hit 0, we're done! $25_{10} = 11001_2$.

Hexadecimal Numbers

- Same concept as decimal and binary, but the base is 16
- Why use hexadecimal?
  - Easy to convert between hex and binary (1 hex digit represents 4 bits)
  - Much more compact than binary
  - And best of all, you can make fun words with letters A-F!

Decimal to Hexadecimal Conversion

To convert a decimal number to hexadecimal, use the same technique we used for binary, but divide/mod by 16 instead of 2.

- Hexadecimal numbers have a prefix of "0x"

Example: Convert 1234_{10} to hexadecimal

First digit:	1234 % 16 = 2	(1234 / 16) = 77
Second digit:	77 % 16 = 13_{10} = D_{16}	(77 / 16) = 4
Third digit:	4 % 16 = 4	(4 / 16) = **0**

We're done because we hit **0**!

1234_{10} = 0x4D2 = $4D2_{16}$

Binary to Hexadecimal Conversion

Converting between binary and hexadecimal is both the easiest and most useful conversion for 351. Hex and Binary have a direct correlation, where 1 hex digit maps to 4 bits.

Example: Convert 0xA5E2 to binary.

We can convert this number digit by digit:

```
  A     5     E     2
1010  0101  1110  0010
```

Converting back to hex is the exact same process; break the bit vector into groups of 4 and convert to hex.

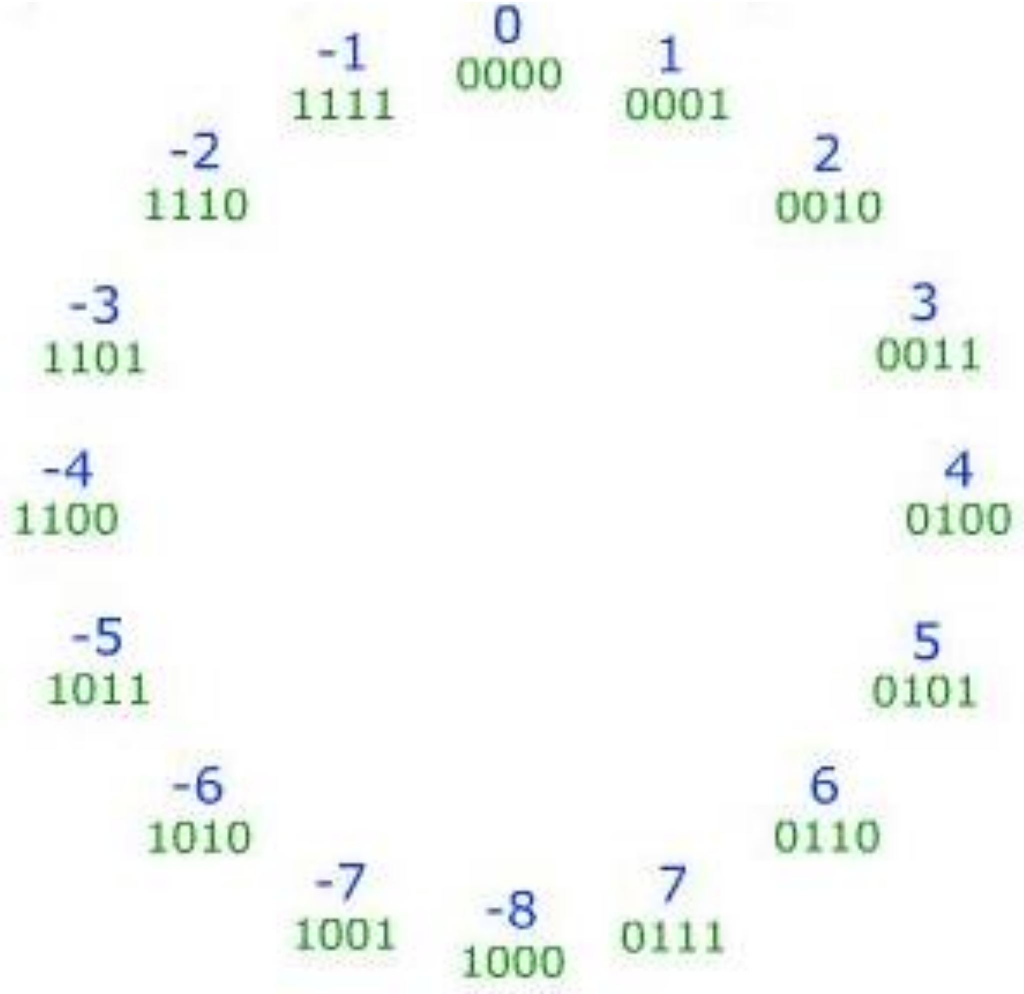
0	0	0000
1	1	0001
2	2	0010
3	3	0011
4	4	0100
5	5	0101
6	6	0110
7	7	0111
8	8	1000
9	9	1001
A	10	1010
B	11	1011
C	12	1100
D	13	1101
E	14	1110
F	15	1111

Representing Signed Integers

- Two common ways:
  - <u>Sign & Magnitude</u>
    - Use 1 bit for the sign, remaining bits for magnitude
    - Works OK, but:
      - There are 2 ways to represent zero (`-0` and `0`)
      - Arithmetic is tricky ($4 - 3 \neq 4 + (-3)$)
  - <u>Two's Complement</u>
    - For positives, similar to regular binary representation
    - But, highest bit has a negative weight
    - Solves Sign-and-Magnitude's problems!

Two's Complement

- This is an example of the range of numbers that can be represented by a 4-bit two's complement number
- An `n`-bit, two's complement number can represent the range $[-2^{n-1}, 2^{n-1} - 1]$.
  - Note the asymmetry of this range about 0 – there's one more negative number than positive
- Note what happens when you overflow
- If you still don't understand it, speak up!
  - Very confusing concept



Understanding Two's Complement Numbers

- Understanding the two's complement representation of integers is much like converting binary to decimal, but with one catch.
- Because of the cyclic nature of two's complement, you **subtract** the value of the most significant bit.

Example: In 8-bit land, what signed int does `0b11010110` represent?

- We follow the same process as binary to decimal conversion. The less significant 7 bits give us:
$$(1 \times 2^1) + (1 \times 2^2) + (1 \times 2^4) + (1 \times 2^6) = 86$$
- Now, however, we **subtract** the highest bit value.
$$86 - (1 \times 2^7) = -42_{10}$$

Understanding Two's Complement
[A Handy Trick]

- There's a simpler way to find the value of a two's complement number, using the handy formula:

$$\sim x + 1 = -x.$$

- We can rewrite this as `x = ~(-x - 1)`, i.e. subtract 1 from the given number, and flip the bits to get the positive portion of the number.

Example: `0b11010110`

- Subtract 1: `0b110101`**`10`** `- 1 = 0b110101`**`01`**
- Flip the bits: `0b00101010` $= (32+8+2)_{10} = 42_{10}$
- So the original number we had was -42_{10}.

Bitwise Operators

- NOT: `~`
  - This will flip all bits in the operand
- AND: `&`
  - This will perform a bitwise AND on every pair of bits
- OR: `|`
  - This will perform a bitwise OR on every pair of bits
- XOR: `^`
  - This will perform a bitwise XOR on every pair of bits
- SHIFT: `<<`, `>>`
  - This will shift the bits right or left
    - logical vs. arithmetic

Logical Operators

- NOT: `!`
  - Evaluates the entire operand, rather than each bit
  - Produces a 1 if `== 0`, produces 0 if nonzero
- AND: `&&`
  - Produces 1 if both operands are nonzero
- OR: `||`
  - Produces 1 if either operand is nonzero

Common Operator Uses

- A double bang (`!!`) is useful when normalizing values to 0 or 1
  - Imitates Boolean types
- Shifts are useful for multiplying/dividing quickly
  - Most multiplications are reduced to shifts when possible by GCC already
  - When writing assembly routines, shifts will be more useful
  - Shifts are also consistent for negative numbers (thanks to sign extension)
- DeMorgan's Laws:
  - `~(A | B) == (~A & ~B)`
  - `~(A & B) == (~A | ~B)`

Masks

- These are usually strings of 1s that are used to isolate a subset of bits in an operand
  - Example: the mask `0xFF = ...0011111111` will "mask" the first byte of an integer
- Once you have created a mask, you can shift it left or right
  - Example: the mask `0xFF << 8` will "mask" the second byte of an integer
- You can apply a mask in different ways
  - To set bits in x, you can do `x = x | MASK`
  - To invert bits in x, you can do `x = x ^ MASK`
  - To erase everything but the masked bits in x, do `x = x & MASK`

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Application: Symmetric Encryption

- This is an example that shows how XOR can be used to encrypt data.
- Say Alice wishes to communicate message `M` to Bob
  - Let M be the bit string: **`0b11011010`**
- Both Alice and Bob have a secret cipher key `C`
  - Let C be the bit string: `0b01100010`
- Alice sends Bob the encrypted message `M' = M ^ C`
  - `M' =` `0b10111000`
- Bob applies C to M' to retrieve M, since `(M ^ C) ^ C = M`
  - `M' ^ C =` **`0b11011010`**
- XOR ciphers are not very secure by themselves, but the XOR operation is used in some modes of AES encryption

Application: Gray Codes

- Gray Codes encode numbers such that consecutive numbers only differ in their representations by 1 bit
  - Useful when trying to transfer counter values across different clock domains (common in FIFOs)
  - If each wire represents one binary digit, we want to ensure that when the counter increments, the voltage level changes only on one wire
- Let n be our counter output
  - `(n >> 1) ^ n` will produce a gray coded version of n
- If we receive the gray code g, we need to convert it to n:

```
for (int mask = g >> 1; mask != 0; mask >> 1) {
        g = g ^ mask;
}
```

For an example, compile and run `gray_code.c`

Lab 1

- Worksheet in class
- Tips:
  - Work on 8-bit versions first, then scale your solution to work for 32-bit inputs
  - Save intermediate results in variables for clarity
  - SHIFTING BY MORE THAN 31 BITS IS UNDEFINED! This will not yield 0.
- Any questions for the good of the class?

Example Problems

- Create 0xFFFFFFFF using only one operator
  - Limited to constants from `0x00` to 0xFF
  - Naïve approach:

```
0xFF + (0xFF << 8) + (0xFF << 16) …
```

  - Smart approach:

```
~0x00 = 0xFFFFFFFF
```

Example Problems

- Replace the leftmost byte of a 32-bit integer with `0xAB`
  - Let our integer be x
  - First, we want to create a mask for the lower 24 bits of the image
    - `~(0xFF << 24)` will do that using just two operations
  - `(x & mask)` will zero out the leftmost 8 bits
  - Now, we want to OR in `0xAB` to those zeroed-out bits
  - Final result:

    ```
    (x & mask) | (0xAB << 24)
    ```
  - Total operators: 5!