Roadmap

C:

```
car *c = malloc(sizeof(car));
c->miles = 100;
c->gals = 17;
float mpg = get_mpg(c);
free(c);
```

Java:

```
Car c = new Car();
c.setMiles(100);
c.setGals(17);
float mpg =
    c.getMPG();
```

Assembly language:

```
get_mpg:
    pushq   %rbp
    movq    %rsp, %rbp
    ...
    popq    %rbp
    ret
```

Machine code:

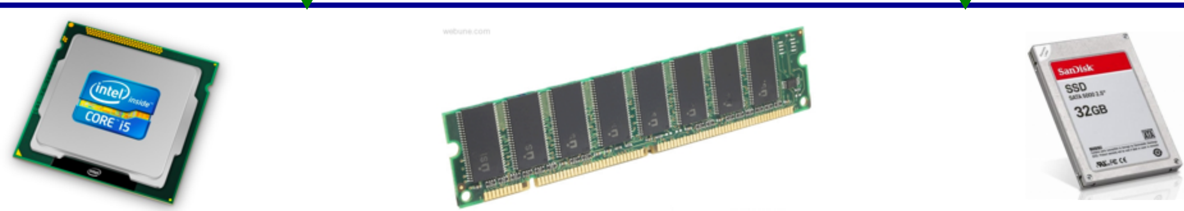
```
0111010000011000
100011010000010000000010
1000100111000010
110000011111101000011111
```

OS:

Windows 8

Mac

Computer system:



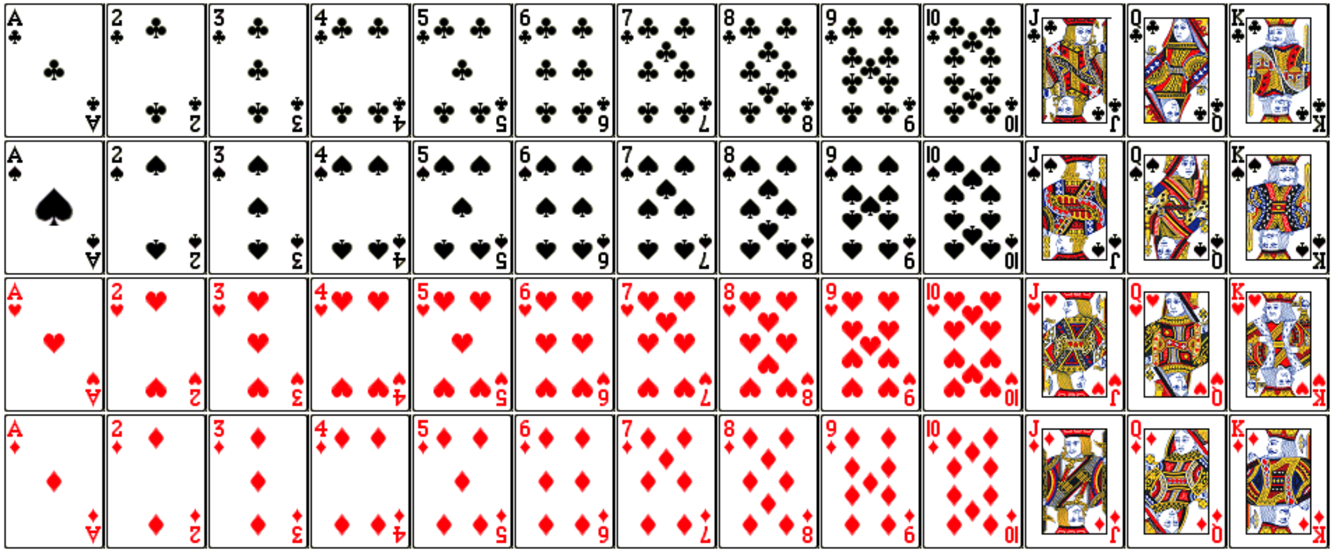
- Memory & data
- **Integers & floats**
- Machine code & C
- x86 assembly
- Procedures & stacks
- Arrays & structs
- Memory & caches
- Processes
- Virtual memory
- Memory allocation
- Java vs. C

Integers

- **Representation of integers: unsigned and signed**
- **Casting**
- **Arithmetic and shifting**
- **Sign extension**

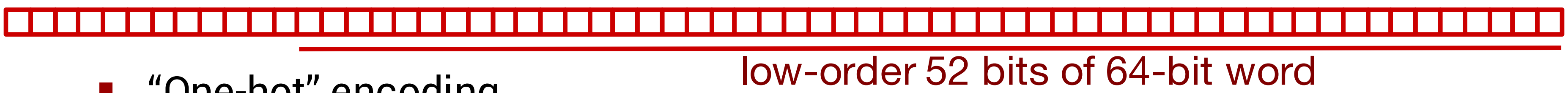
But before we get to integers….

- **Encode a standard deck of playing cards.**
- **52 cards in 4 suits**
  - How do we encode suits, face cards?
- **What operations do we want to make easy to implement?**
  - Which is the higher value card?
  - Are they the same suit?



Two possible representations

- **52 cards – 52 bits with bit corresponding to card set to 1**



  - "One-hot" encoding
  - Drawbacks:
    - Hard to compare values and suits
    - Large number of bits required

Two possible representations

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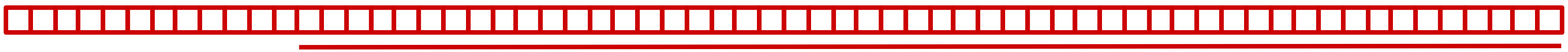
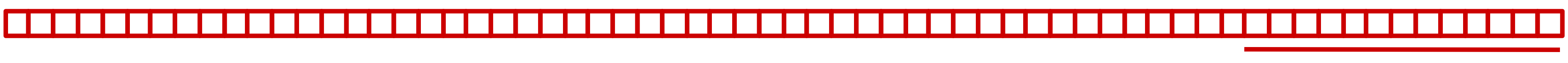


low-order 52 bits of 64-bit word

  - "One-hot" encoding
  - Drawbacks:
    - Hard to compare values and suits
    - Large number of bits required

- **4 bits for suit, 13 bits for card value – 17 bits with two set to 1**

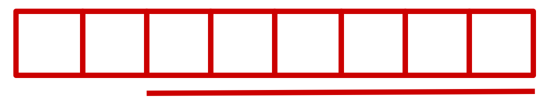


  - Pair of one-hot encoded values
  - Easier to compare suits and values
    - Still an excessive number of bits

- **Can we do better?**

Two better representations

- **Binary encoding of all 52 cards – only 6 bits needed**

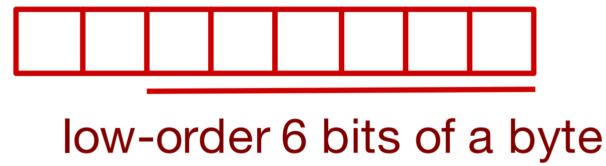


low-order 6 bits of a byte

  - Fits in one byte
  - Smaller than one-hot encodings.
  - How can we make value and suit comparisons easier?

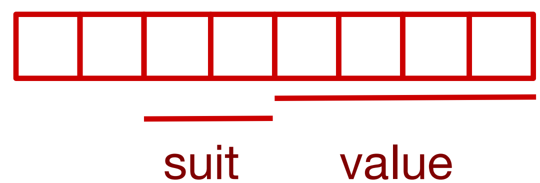
Two better representations

- **Binary encoding of all 52 cards – only 6 bits needed**



  - Fits in one byte
  - Smaller than one-hot encodings.
  - How can we make value and suit comparisons easier?

- **Binary encoding of suit (2 bits) and value (4 bits) separately**



  - Also fits in one byte, and easy to do comparisons

K	Q	J	...	3	2	A
1101	1100	1011	...	0011	0010	0001

♠	00
♣	01
♦	10
♥	11

Compare Card Suits

mask: a bit vector that, when bitwise ANDed with another bit vector v, turns all but the bits of interest in v to 0

```
#define SUIT_MASK  0x30

int sameSuitP(char card1, char card2) {
  return (!((card1 & SUIT_MASK) ^ (card2 & SUIT_MASK)));
  // return (card1 & SUIT_MASK) == (card2 & SUIT_MASK);
}
```

returns `int`

equivalent

SUIT_MASK = 0x30 =

0	0	1	1	0	0	0	0

suit value

use `char` for a single byte

```
char hand[5];        // represents a 5-card hand
char card1, card2;   // two cards to compare
card1 = hand[0];
card2 = hand[1];
...
if ( sameSuitP(card1, card2) ) { ... }
```

Compare Card Suits

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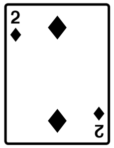

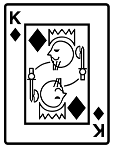

0	0	1	0	0	0	1	0

&

SUIT_MASK:

0	0	1	1	0	0	0	0

=

0	0	1	0	0	0	0	0

0	0	1	0	1	1	0	1

&

0	0	1	1	0	0	0	0

=

0	0	1	0	0	0	0	0

^

0	0	0	0	0	0	0	0

!

0	0	0	0	0	0	0	1

`!(x^y)` equivalent to `x==y`

Compare Card Values

mask: a bit vector that, when bitwise ANDed with another bit vector v, turns all but the bits of interest in v to 0

```
#define VALUE_MASK  0x0F

int greaterValue(char card1, char card2) {
  return ((unsigned int)(card1 & VALUE_MASK) >
          (unsigned int)(card2 & VALUE_MASK));
}
```

VALUE_MASK = 0x0F =

0	0	0	0	1	1	1	1
		suit		value			

```
char hand[5];        // represents a 5-card hand
char card1, card2;   // two cards to compare
card1 = hand[0];
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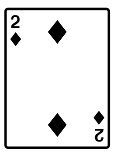
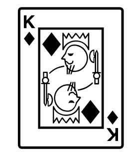


	0	0	1	0	0	0	1	0
&								
VALUE_MASK:	0	0	0	0	1	1	1	1
=								
	0	0	0	0	0	0	1	0

	0	0	1	0	1	1	0	1
&								
	0	0	0	0	1	1	1	1
=								
	0	0	0	0	1	1	0	1



$2_{10} > 13_{10} = 0$ (false)

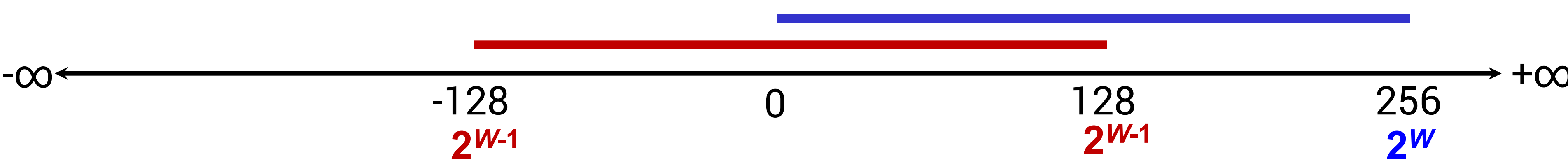
April 6

Announcements

Thurs 10:30am Section moved to **EEB 045**.

Encoding Integers

- **The hardware (and C) supports two flavors of integers:**
  - *unsigned* – only the non-negatives
  - *signed* – both negatives and non-negatives
- **Cannot represent all the integers**
  - There are only 2^W distinct bit patterns of W bits
  - **Unsigned values: 0 ... 2^W-1**
  - **Signed values: -2^{W-1} ... 2^{W-1}-1**



- **Reminder: terminology for binary representations**

"Most-significant" or "high-order" bit(s) *(MSB)*

0110010110101001

"Least-significant" or "low-order" bit(s)

Unsigned Integers

- **Unsigned values are just what you expect**
  - $b_7b_6b_5b_4b_3b_2b_1b_0 = b_72^7 + b_62^6 + b_52^5 + \ldots + b_12^1 + b_02^0$
    - Useful formula: $1+2+4+8+\ldots+2^{N-1} = 2^N - 1$

- **Add and subtract using the normal "carry" and "borrow" rules, just in binary.**

```
  63
+  8
  71
```

```
 00111111
+00001000
 01000111
```

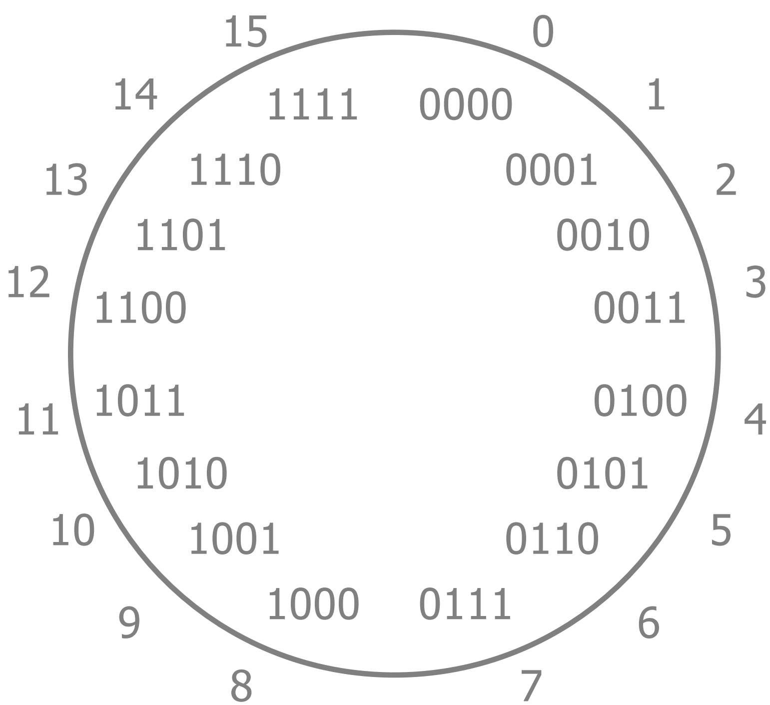
- **How would you make *signed* integers?**

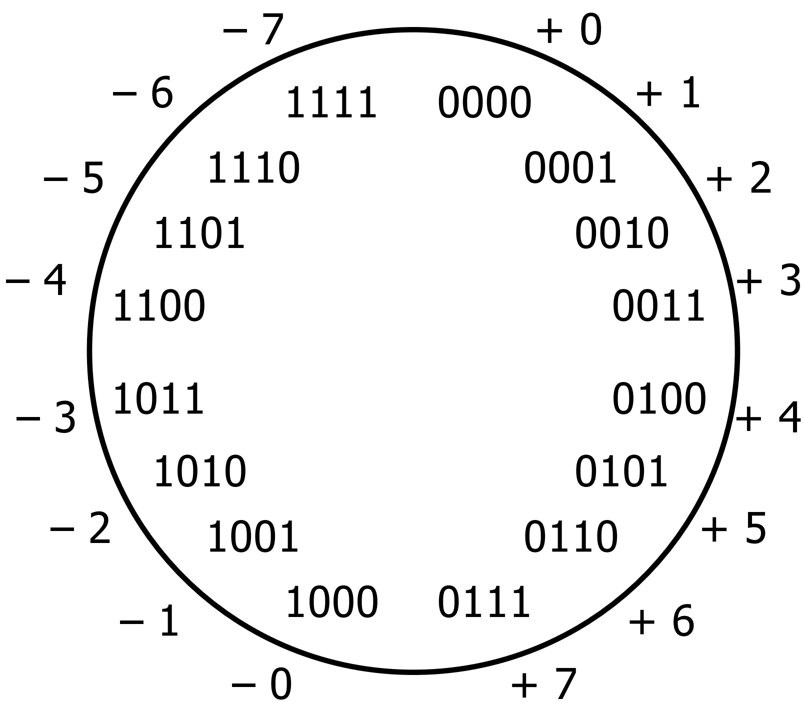
Sign-and-Magnitude

- **Use high-order bit to indicate positive/negative (the "sign bit")**
- **Positive numbers: sign = 0**
  - Does the natural thing, same as for unsigned
- **Negative numbers: sign = 1**
- **Examples (8 bits):**
  - 0x00 = 00000000_2 is non-negative, because the sign bit is 0
  - 0x7F = 01111111_2 is non-negative ($+127_{10}$)
  - 0x85 = 10000101_2 is negative (-5_{10})
  - 0x80 = 10000000_2 is negative...
    - Negative zero!

Sign-and-Magnitude

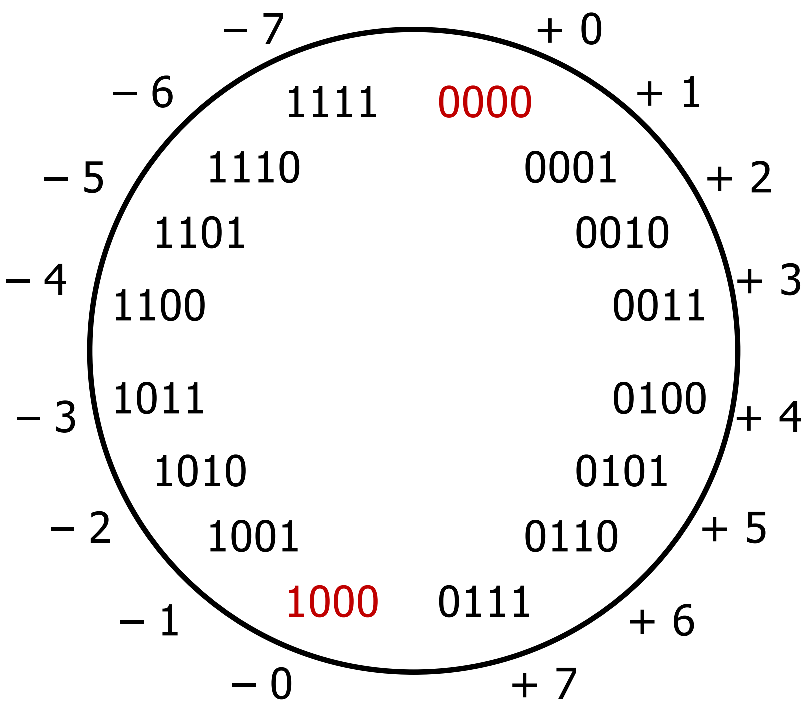
- **High-order bit (MSB)** flips the sign, rest of the bits are **magnitude**





Sign-and-Magnitude

- **High-order bit (MSB)** flips the sign, rest of the bits are **magnitude**
- **Downsides**
  - There are two representations of 0! (bad for checking equality)

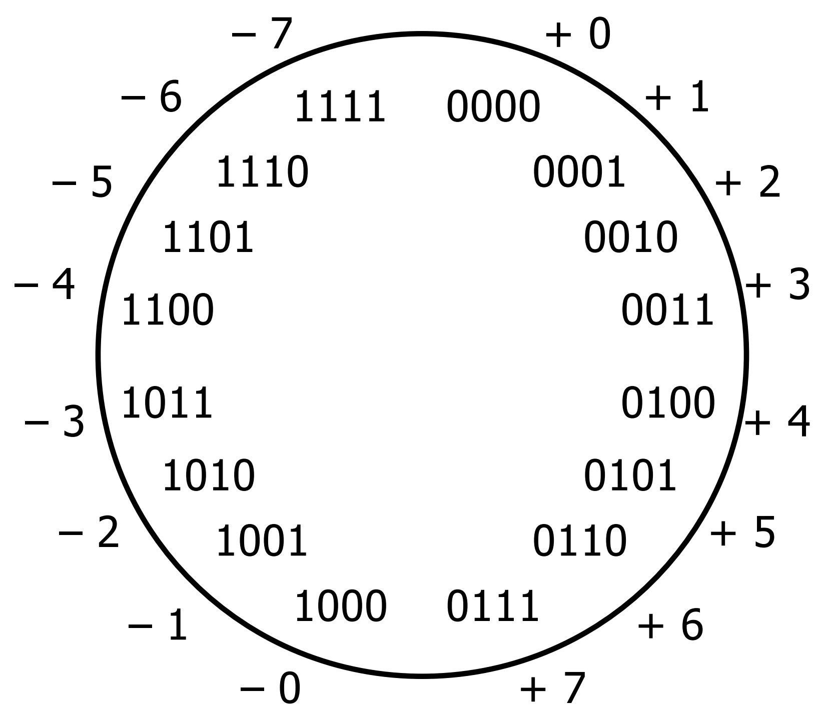


Sign-and-Magnitude

- **High-order bit (MSB)** flips the sign, rest of the bits are **magnitude**
- **Downsides**
  - There are two representations of 0! (bad for checking equality)
  - **Arithmetic is cumbersome.**
    - Example:
      4 - 3 != 4 + (-3)

4	0100
– 3	– 0011
1	0001

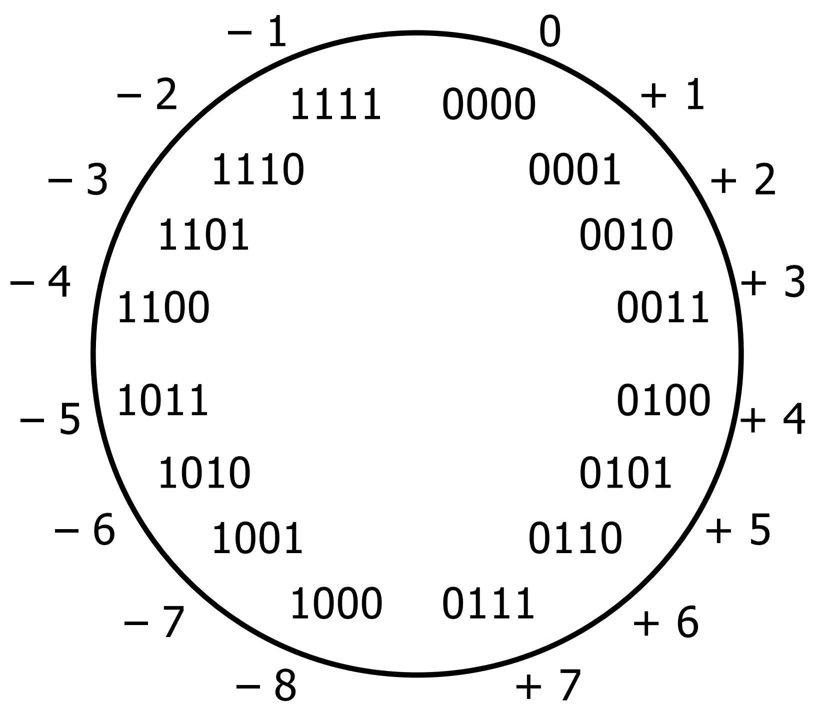
4	0100
+ –3	+ 1011
–7	1111



How do we solve these problems?

Two's Complement

- High-order bit (MSB) *still* indicates that the value is *negative*

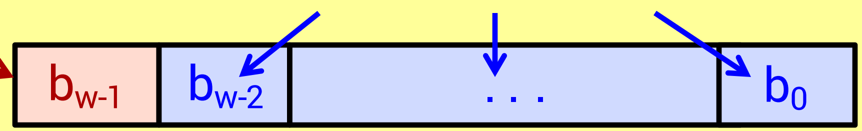


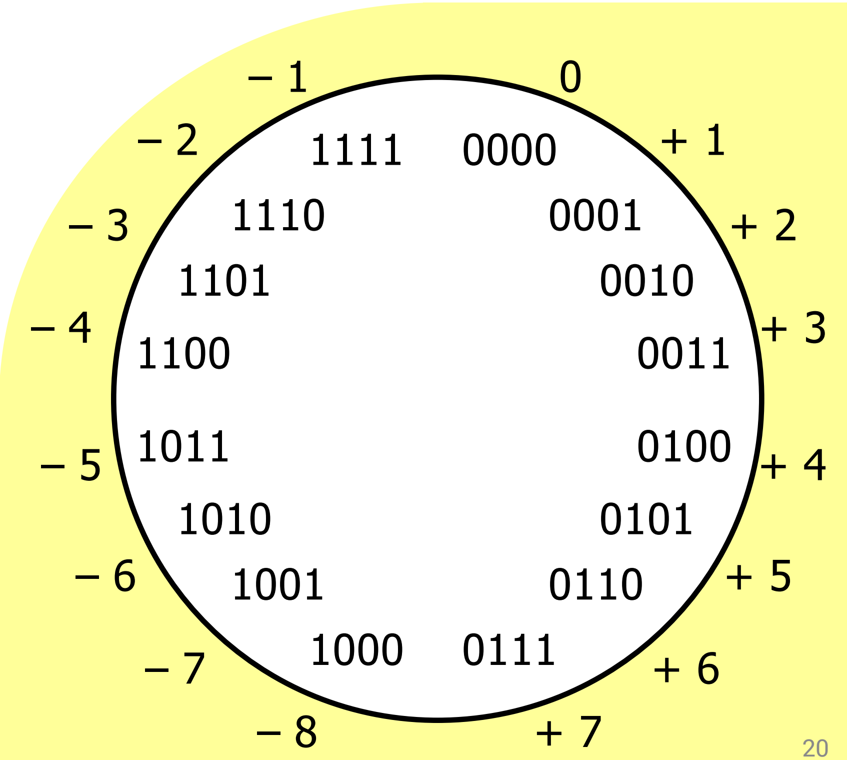
Two's Complement Negatives

- High-order bit (MSB) *still* indicates that the value is *negative*
  - But instead, let MSB have ***same value***, but ***negative weight.***

b_{w-1} = 1 adds -2^{w-1} to the value

for i < w-1: b_i = 1 adds $+2^i$ to the value.





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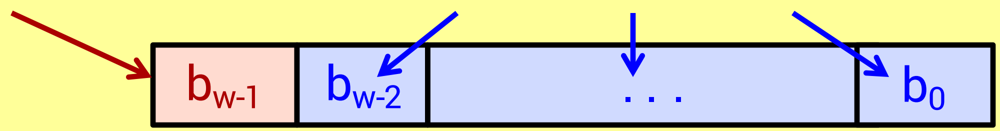
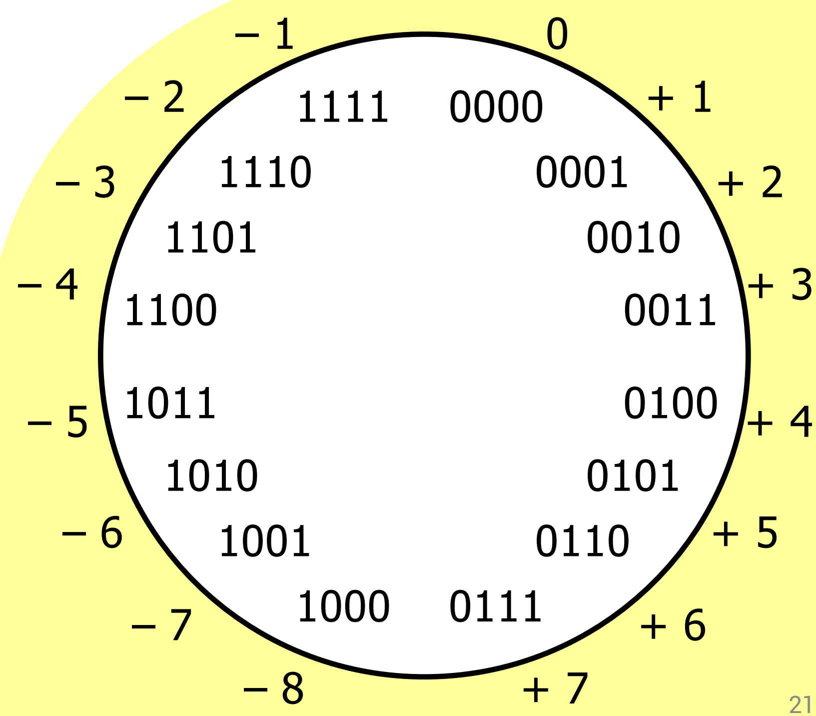


e.g. **unsigned** 1010_2:

$1*2^3 + 0*2^2 + 1*2^1 + 0*2^0 = 10_{10}$

2's compl. 1010_2:

$-1*2^3 + 0*2^2 + 1*2^1 + 0*2^0 = -6_{10}$



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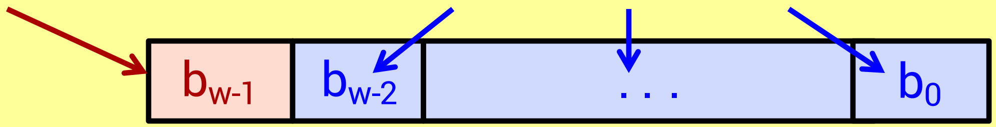

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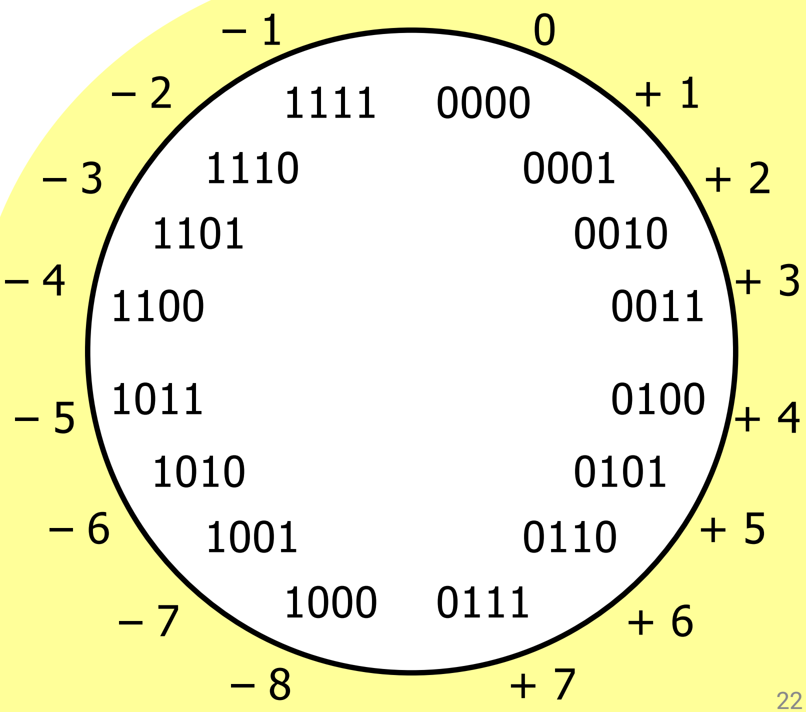
2's compl. 1010_2:

$-1*2^3 + 0*2^2 + 1*2^1 + 0*2^0 = -6_{10}$

- -1 is represented as $1111_2 = -2^3 + (2^3 - 1)$
  - MSB makes it super negative, add up all the other bits to get back up to -1



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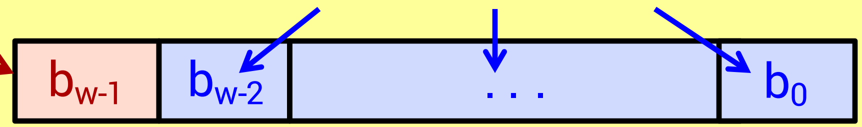

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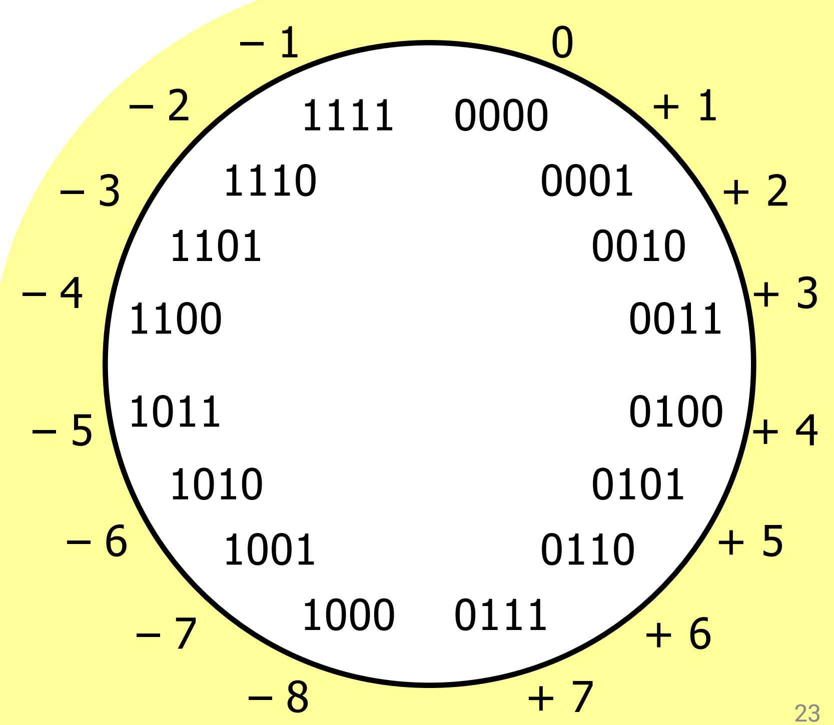
- -1 is represented as $1111_2 = -2^3 + (2^3 - 1)$
- **Advantages:**
  - Single zero
  - Simple arithmetic



4-bit Unsigned vs. Two's Complement

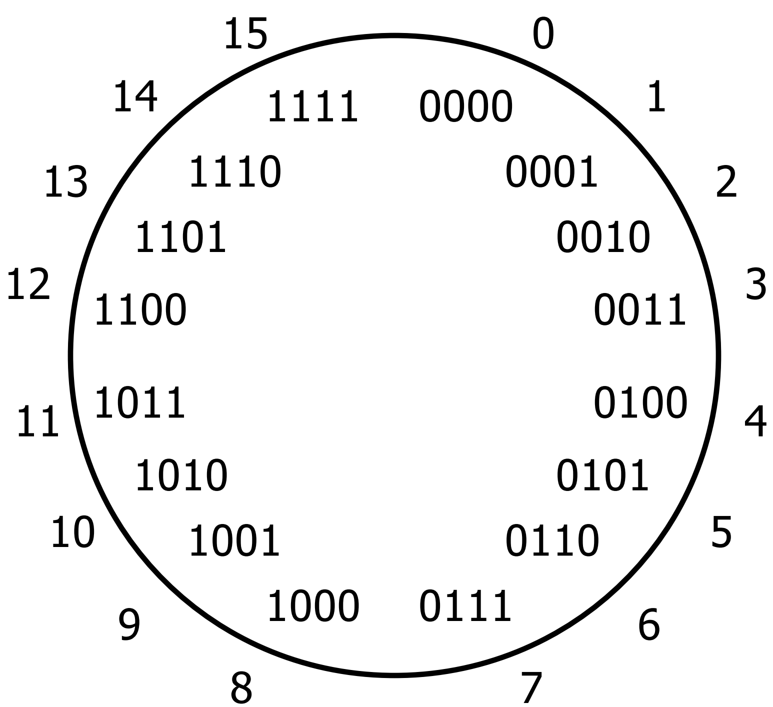
1 0 1 1

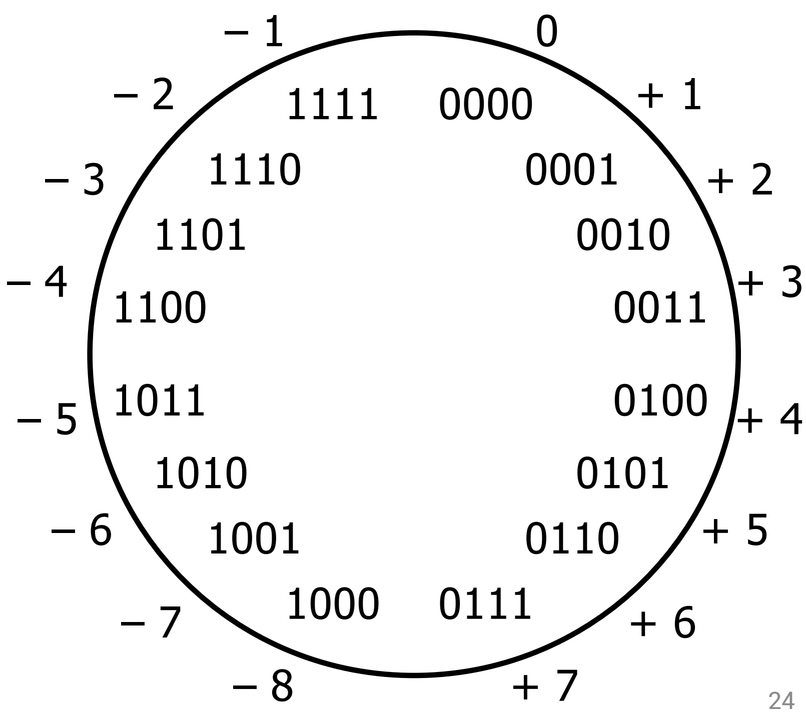
$2^3 \times 1 + 2^2 \times 0 + 2^1 \times 1 + 2^0 \times 1$

$-2^3 \times 1 + 2^2 \times 0 + 2^1 \times 1 + 2^0 \times 1$

11

-5





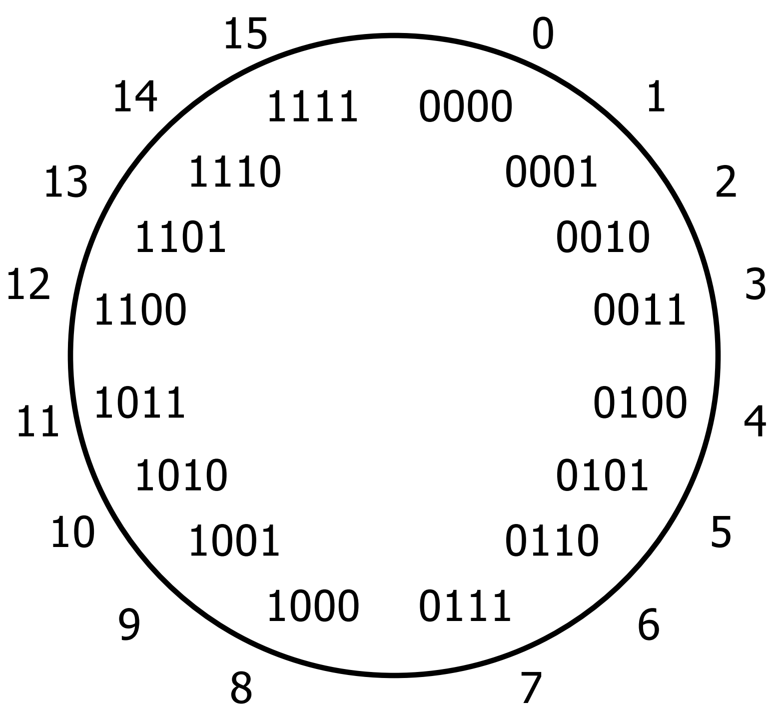
4-bit Unsigned vs. Two's Complement

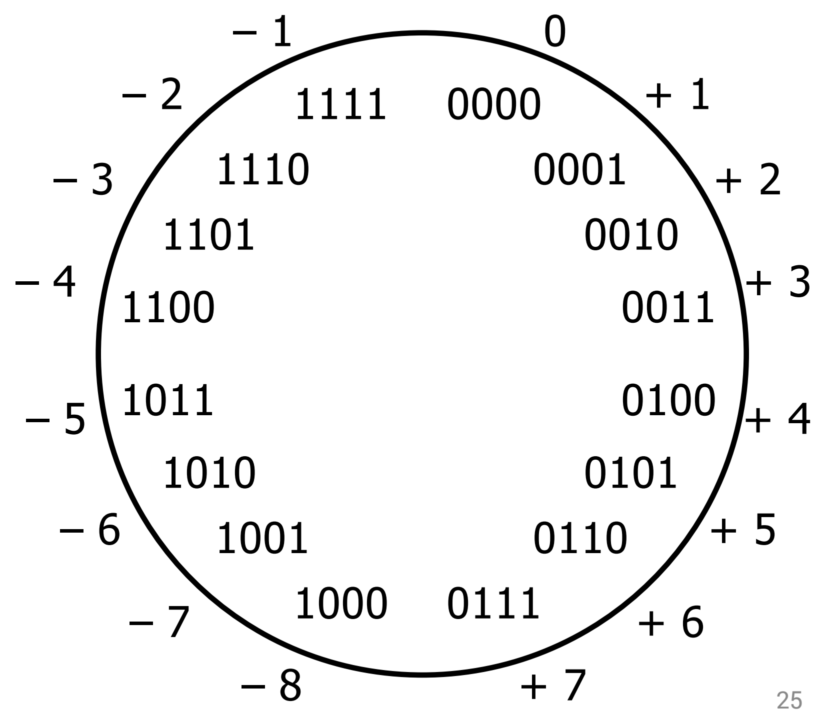
1 0 1 1

$2^3 \times 1 + 2^2 \times 0 + 2^1 \times 1 + 2^0 \times 1$

$-2^3 \times 1 + 2^2 \times 0 + 2^1 \times 1 + 2^0 \times 1$

11 ← (math) difference = 16 = 2^4 → **-5**





4-bit Unsigned vs. Two's Complement

1 0 1 1

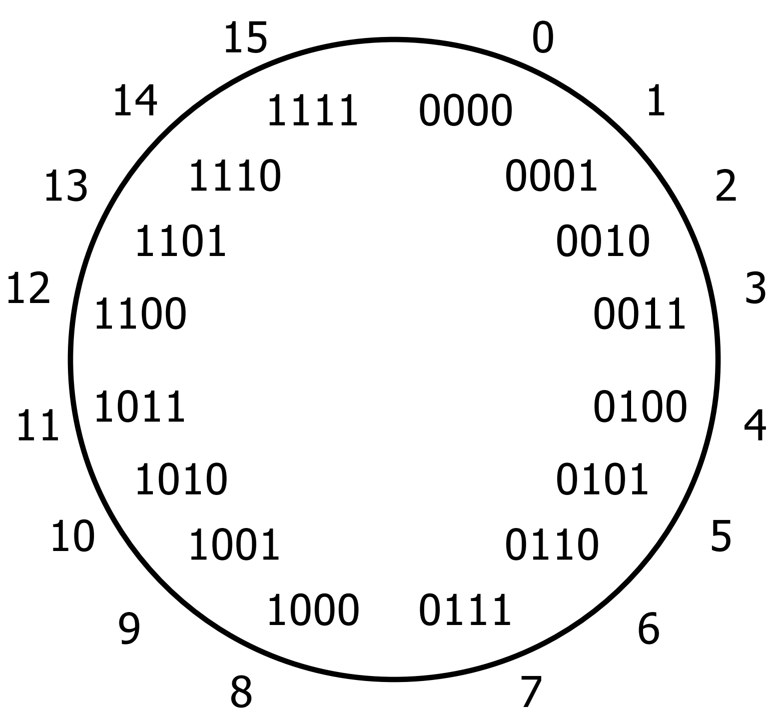
$2^3 \times 1 + 2^2 \times 0 + 2^1 \times 1 + 2^0 \times 1$

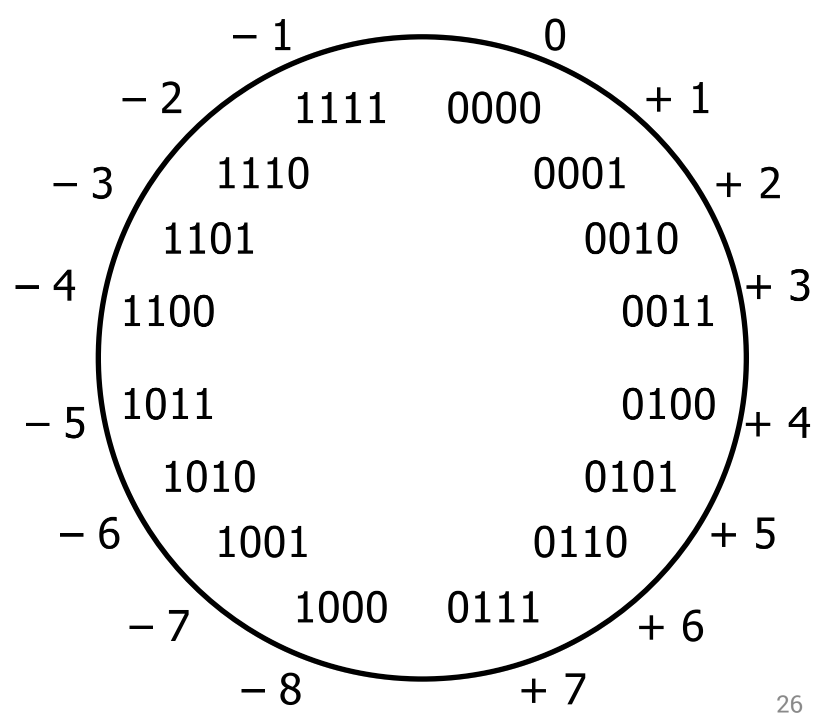
$-2^3 \times 1 + 2^2 \times 0 + 2^1 \times 1 + 2^0 \times 1$

11

(math) difference = 16 = 2^4

-5





Two’s Complement Arithmetic

- **The same addition procedure works for both unsigned and two’s complement integers**
  - Simplifies hardware: only one algorithm for addition
  - Algorithm: simple addition, discard the highest carry bit
    - Called “modular” addition: result is sum modulo 2W

- **Examples:**

4	0100	4	0100	– 4	1100
+ 3	+ 0011	– 3	+ 1101	+ 3	+ 0011

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- **Examples:**

4	0100	4	0100	– 4	1100
+ 3	+ 0011	– 3	+ 1101	+ 3	+ 0011
= 7	= 0111	= 1	1 0001	– 1	1111
			= 0001		

(drop carry)

Two's Complement

- **Why does it work?**
  - Put another way, for all positive integers x, we want:

```
   Bit representation of  x
 + Bit representation of –x
 ---------------------------
                          0   (ignoring the carry-out bit)
```

  - **What should the 8-bit representation of -1 be?**

```
    00000001
 +  ????????     (we want whichever bit string gives the right result)
 -----------
    00000000
```

  - Other examples:

```
    00000010             00000011
 +  ????????          +  ????????
 -----------          -----------
    00000000             00000000
```

Two's Complement

- **Why does it work?**
  - Put another way, for all positive integers x, we want:

$$\begin{array}{r} \text{Bit representation of } \;\; x \\ +\ \text{Bit representation of } -x \\ \hline 0 \end{array} \quad \textit{(ignoring the carry-out bit)}$$

  - **What should the 8-bit representation of -1 be?**

$$\begin{array}{r} \texttt{00000001} \\ +\ \texttt{11111111} \\ \hline \texttt{00000000} \end{array}$$

(we want whichever bit string gives the right result)

  - Other examples:

$$\begin{array}{r} \texttt{00000010} \\ +\ \texttt{11111110} \\ \hline \texttt{00000000} \end{array} \qquad \begin{array}{r} \texttt{00000011} \\ +\ \texttt{11111101} \\ \hline \texttt{00000000} \end{array}$$

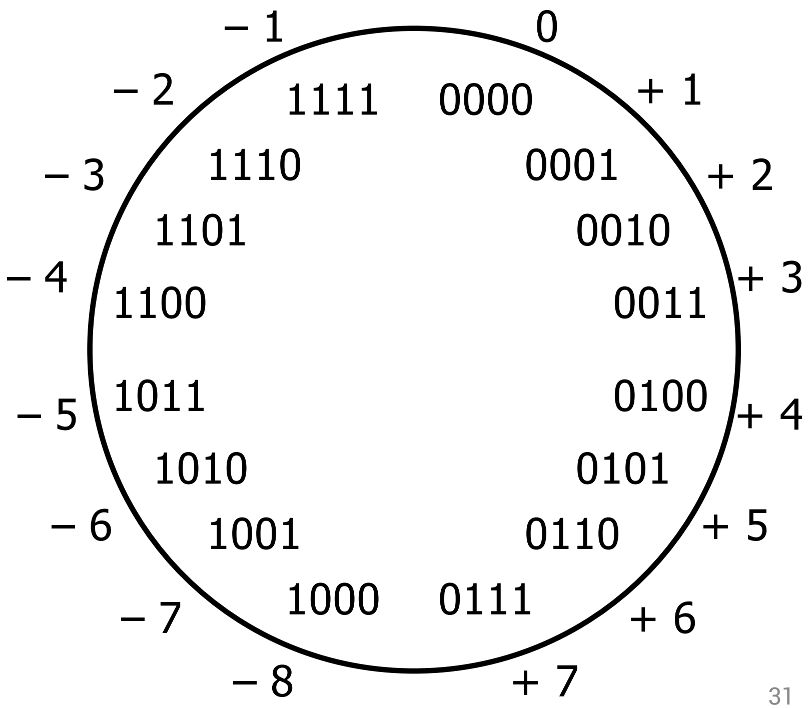
Turns out to be the bitwise complement plus 1!

Two's Complement

- **Negate any 2s-complement integer**
  - Take bitwise complement (flip all the bits) and then add one!

    `~x + 1 == -x`

```
~ 0101      5₁₀
  ----
  1010
+ 0001
  ----
  1011     -5₁₀
```



You can even do it again and it still works!

Unsigned & Signed Numeric Values

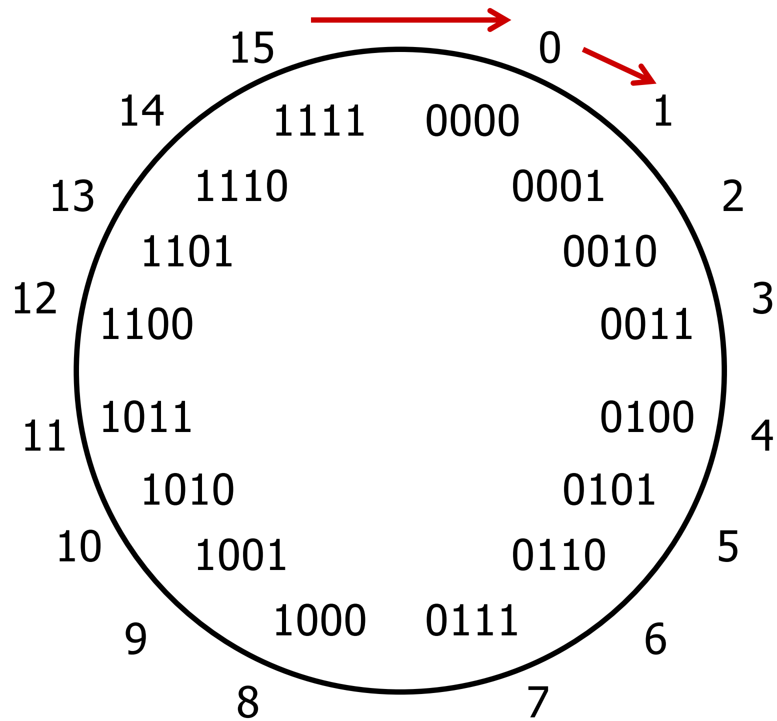
bits	Unsigned	Signed
0000	0	0
0001	1	1
0010	2	2
0011	3	3
0100	4	4
0101	5	5
0110	6	6
0111	7	7
1000	8	–8
1001	9	–7
1010	10	–6
1011	11	–5
1100	12	–4
1101	13	–3
1110	14	–2
1111	15	–1

- **<u>Signed</u> and <u>unsigned</u> integers have limits.**

Overflow/Wrapping: Unsigned

addition: drop the carry bit

$$\begin{array}{r} 15 \\ +\ 2 \\ \hline \cancel{17} \\ 1 \end{array} \qquad \begin{array}{r} 1111 \\ +\ 0010 \\ \hline 10001 \end{array}$$



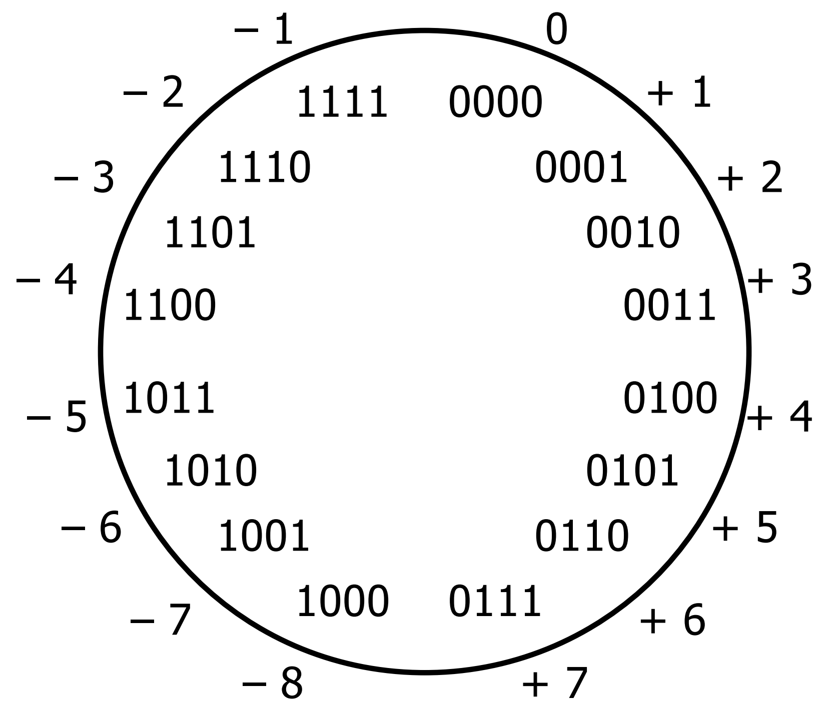
Modular Arithmetic

Overflow/Wrapping: Two's Complement

addition: drop the carry bit

$$\begin{array}{r} -1 \\ +\ 2 \\ \hline \end{array} \qquad \begin{array}{r} 1111 \\ +\ 0010 \\ \hline \end{array}$$

$$\begin{array}{r} 6 \\ +\ 3 \\ \hline \end{array} \qquad \begin{array}{r} 0110 \\ +\ 0011 \\ \hline \end{array}$$



Modular Arithmetic

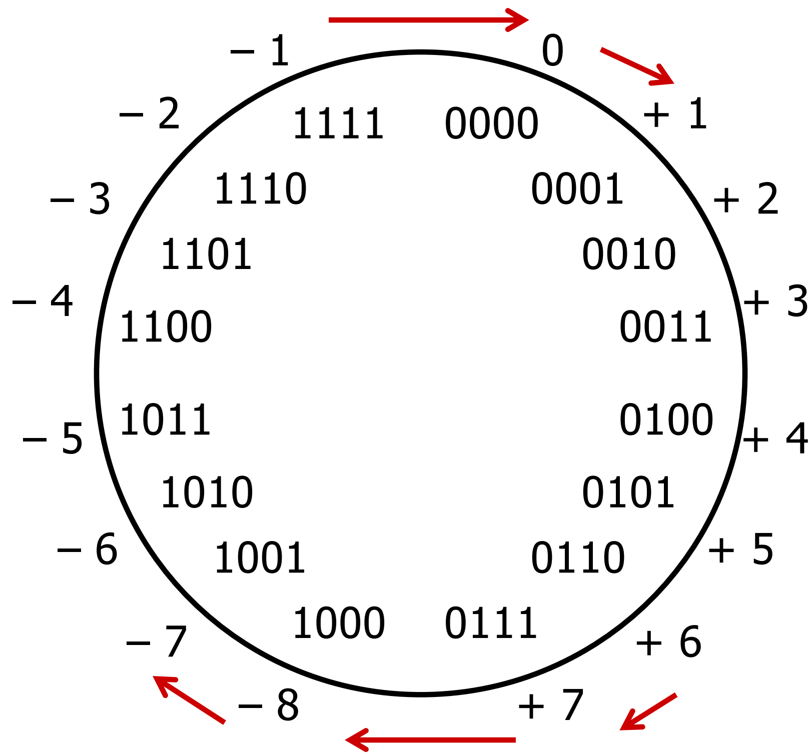
Overflow/Wrapping: Two's Complement

addition: drop the carry bit

For signed: overflow if operands have same sign and result's sign is different.

$$\begin{array}{r} -1 \\ +\ 2 \\ \hline 1 \end{array} \qquad \begin{array}{r} 1111 \\ +\ 0010 \\ \hline 10001 \end{array}$$

$$\begin{array}{r} 6 \\ +\ 3 \\ \hline \cancel{9} \\ -7 \end{array} \qquad \begin{array}{r} 0110 \\ +\ 0011 \\ \hline 1001 \end{array}$$



Modular Arithmetic

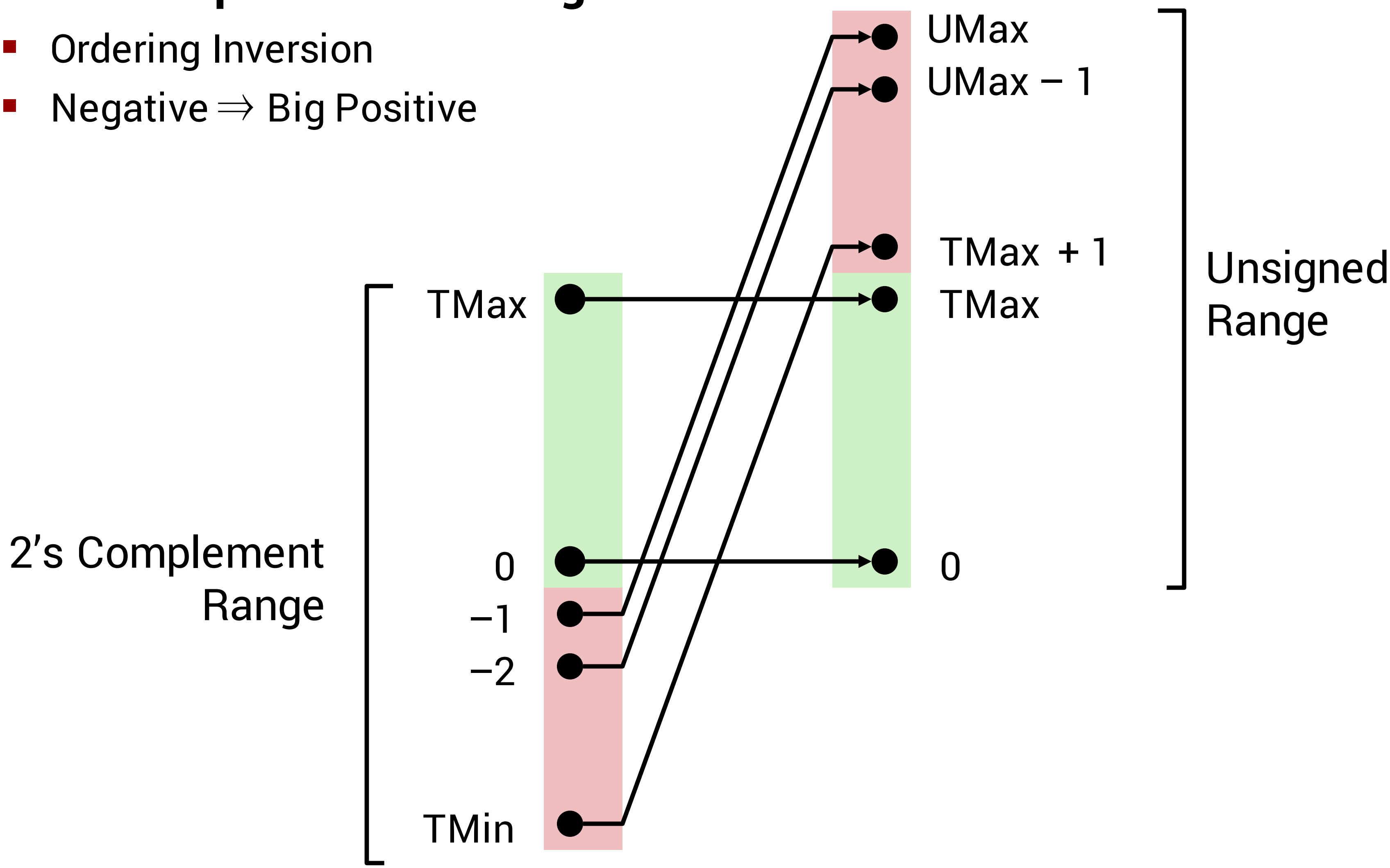
Unsigned & Signed Numeric Values

bits	Unsigned	Signed
0000	0	0
0001	1	1
0010	2	2
0011	3	3
0100	4	4
0101	5	5
0110	6	6
0111	7	7
1000	8	–8
1001	9	–7
1010	10	–6
1011	11	–5
1100	12	–4
1101	13	–3
1110	14	–2
1111	15	–1

- **Signed and unsigned integers have limits.**
  - If you compute a number that is too big (positive), it wraps.
  - If you compute a number that is too small (negative), it wraps.
- **The CPU *may be* capable of "throwing an exception" for overflow on signed values.**
  - It won't for unsigned.
- **But C and Java just cruise along silently when overflow occurs... Oops.**

Signed/Unsigned Conversion

- **Two's Complement ⇒ Unsigned**
  - Ordering Inversion
  - Negative ⇒ Big Positive



Values To Remember

- **Unsigned Values**
  - UMin = 0
    - 000…0
  - UMax = 2w – 1
    - 111…1
- Two's Complement Values
  - TMin = -2^{w-1}
    - 100…0
  - TMax = $2^{w-1} - 1$
    - 011…1
  - Negative one
    - 111…1 0xF…F

Values for W = 32

	Decimal	Hex	Binary
UMax	4,294,967,296	FF FF FF FF	11111111 11111111 11111111 11111111
TMax	2,147,483,647	7F FF FF FF	01111111 11111111 11111111 11111111
TMin	-2,147,483,648	80 00 00 00	10000000 00000000 00000000 00000000
-1	-1	FF FF FF FF	11111111 11111111 11111111 11111111
0	0	00 00 00 00	00000000 00000000 00000000 00000000

Values for W = 64:

```
LONG_MIN  = -9223372036854775808
LONG_MAX  =  9223372036854775807
ULONG_MAX = 18446744073709551615
```

In C: Signed vs. Unsigned

- **Integer Literals (constants)**
  - By default are considered to be signed integers
  - Use "U" (or "u") suffix to force unsigned:
    - `0U`, `4294967259u`

In C: Signed vs. Unsigned

!!!

- **Casting**
  - `int tx, ty;`
  - `unsigned ux, uy;`
- **Explicit casting between signed & unsigned:**
  - `tx = (int) ux;`
  - `uy = (unsigned) ty;`
- **Implicit casting also occurs via assignments and function calls:**
  - `tx = ux;`
  - `uy = ty;`
  - The gcc flag *-Wsign-conversion* produces warnings for implicit casts, but *-Wall* does not!
- **How does casting between signed and unsigned work?**
- **What values are going to be produced?**

In C: Signed vs. Unsigned

!!!

- **Casting**
  - `int tx, ty;`
  - `unsigned ux, uy;`
- **Explicit casting between signed & unsigned:**
  - `tx = (int) ux;`
  - `uy = (unsigned) ty;`
- **Implicit casting also occurs via assignments and function calls:**
  - `tx = ux;`
  - `uy = ty;`
  - The gcc flag *-Wsign-conversion* produces warnings for implicit casts, but *-Wall* does not!
- **How does casting between signed and unsigned work?**
- **What values are going to be produced?**
  - ***Bits are unchanged*, just interpreted differently!**

Casting Surprises

!!!

- Expression Evaluation
  - If you mix unsigned and signed in a single expression, then ***signed values are implicitly cast to unsigned***.
  - Including comparison operations `<`, `>`, `==`, `<=`, `>=`

Casting Surprises

- Examples for W = 32:
  Reminder: TMIN = -2,147,483,648 TMAX = 2,147,483,647

$Constant_1$	$Constant_2$	Relation	Interpret the bits as:
`0`	`0U`	==	Unsigned
`-1`	`0`	<	Signed
`-1`	`0U`	>	Unsigned
`2147483647`	`-2147483648`	>	Signed
`2147483647U`	`-2147483648`	<	Unsigned
`-1`	`-2`	>	Signed
`(unsigned int)-1`	`-2`	>	Unsigned
`2147483647`	`2147483648U`	<	Unsigned
`2147483647`	`(int) 2147483648U`	>	Signed

Casting Surprises

- Examples for W = 32:
  Reminder: TMIN = -2,147,483,648   TMAX = 2,147,483,647

$Constant_1$	$Constant_2$	Relation	Interpret the bits as:
`0` `0000 0000 0000 0000 0000 0000 0000 0000`	`0U` `0000 0000 0000 0000 0000 0000 0000 0000`	==	Unsigned
`-1` `1111 1111 1111 1111 1111 1111 1111 1111`	`0` `0000 0000 0000 0000 0000 0000 0000 0000`	<	Signed
`-1` `1111 1111 1111 1111 1111 1111 1111 1111`	`0U` `0000 0000 0000 0000 0000 0000 0000 0000`	>	Unsigned
`2147483647` `0111 1111 1111 1111 1111 1111 1111 1111`	`-2147483648` `1000 0000 0000 0000 0000 0000 0000 0000`	>	Signed
`2147483647U` `0111 1111 1111 1111 1111 1111 1111 1111`	`-2147483648` `1000 0000 0000 0000 0000 0000 0000 0000`	<	Unsigned
`-1` `1111 1111 1111 1111 1111 1111 1111 1111`	`-2` `1111 1111 1111 1111 1111 1111 1111 1110`	>	Signed
`(unsigned) -1` `1111 1111 1111 1111 1111 1111 1111 1111`	`-2` `1111 1111 1111 1111 1111 1111 1111 1110`	>	Unsigned
`2147483647` `0111 1111 1111 1111 1111 1111 1111 1111`	`2147483648U` `1000 0000 0000 0000 0000 0000 0000 0000`	<	Unsigned
`2147483647` `0111 1111 1111 1111 1111 1111 1111 1111`	`(int) 2147483648U` `1000 0000 0000 0000 0000 0000 0000 0000`	>	Signed

Sign Extension

- **What happens if you convert a 32-bit signed integer to a 64-bit signed integer?**

Sign Extension

0 0 1 0	4-bit 2
0 0 0 0 0 0 1 0	8-bit 2
1 1 0 0	4-bit -4
_ _ _ _ 1 1 0 0	8-bit -4

Sign Extension

0 0 1 0 4-bit 2

0 0 0 0 0 0 1 0 8-bit 2

1 1 0 0 4-bit -4

0 0 0 0 1 1 0 0 8-bit 12

Just adding zeroes to the front does not work

Sign Extension

0 0 1 0	4-bit 2
0 0 0 0 0 0 1 0	8-bit 2
1 1 0 0	4-bit -4
1 0 0 0 1 1 0 0	8-bit -116

Just making the first bit=1 also does not work

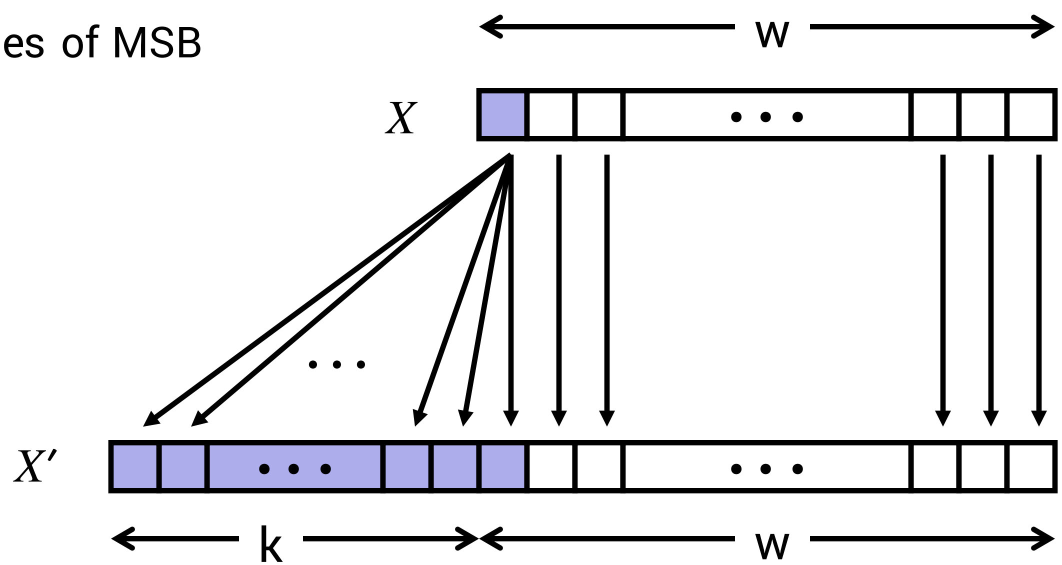
Sign Extension

0 0 1 0	4-bit 2
0 0 0 0 0 0 1 0	8-bit 2
1 1 0 0	4-bit -4
1 1 1 1 1 1 0 0	8-bit -4

Need to extend the sign bit to all “new” locations

Sign Extension

- **Task:**
  - Given w-bit signed integer x
  - Convert it to w+k-bit integer *with same value*
- **Rule:**
  - Make k copies of sign bit:
  - X ' = $\underbrace{x_{w-1},\ldots, x_{w-1}}_{\text{k copies of MSB}}, x_{w-1}, x_{w-2},\ldots, x_0$



Sign Extension Example

- **Converting from smaller to larger integer data type**
- **C automatically performs sign extension (Java too)**

```
short int x =  12345;
int      ix = (int) x;
short int y = -12345;
int      iy = (int) y;
```

	Decimal	Hex	Binary
x	12345	30 39	00110000 01101101
ix	12345	00 00 30 39	00000000 00000000 00110000 01101101
y	-12345	CF C7	11001111 11000111
iy	-12345	FF FF CF C7	11111111 11111111 11001111 11000111

Shift Operations

- **Left shift: x << n**
  - Shift bit vector x left by n positions
    - Throw away extra bits on left
    - Fill with 0s on right
- **Right shift: x >> n**
  - Shift bit-vector x right by n positions
    - Throw away extra bits on right
  - Logical shift (for unsigned values)
    - Fill with 0s on left
  - Arithmetic shift (for signed values)
    - Replicate most significant bit on left
    - Maintains sign of x

Shift Operations

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    - Fill with 0s on left
  - Arithmetic shift (for signed values)
    - Replicate most significant bit on left
    - Maintains sign of x

Argument x	00100010
x << 3	00010000
Logical: x >> 2	00001000
Arithmetic: x >> 2	00001000

Argument x	10100010
x << 3	00010000
Logical: x >> 2	00101000
Arithmetic: x >> 2	11101000

- **Note**
  - Shifts by n < 0 or n >= size of x (in bits) are **undefined**
  - **In C:** Behavior of >> depends on the compiler!
    - In GCC/Clang: it depends on if **x** is signed/unsigned
  - **In Java:** >>> is logical shift; >> is arithmetic

What are these computing?

- `x >> n`: **divide by 2^n**

- `x << n`: **multiply by 2^n**

Shifting is faster than general multiply or divide operations

Shifting and Arithmetic Example #1

General Form:
x << n
x >> n

```
x = 27;
y = x << 2;
y == 108
```

0 0 0 1 1 0 1 1

0 0 0 1 1 0 1 1 0 0

$x*2^n$

logical shift left:

shift in zeros from the right

unsigned

$x/2^n$

```
x = 237u;
y = x >> 2;
y == 59
```

logical shift right:

shift in zeros from the left

1 1 1 0 1 1 0 1

0 0 1 1 1 0 1 1 0 1

rounding (down)

Shifting and Arithmetic Example #2

General Form:
x << n
x >> n

signed

```
x = -101;
y = x << 2;
y == 108
```

1 0 0 1 1 0 1 1

1 0 0 1 1 0 1 1 0 0

overflow

$x*2^n$

logical shift left:

shift in zeros from the right

signed

```
x = -19;
y = x >> 2;
y == -5
```

$x/2^n$

arithmetic shift right:

shift in copies of most significant bit from the left

1 1 1 0 1 1 0 1

1 1 1 1 1 0 1 1 0 1

rounding (down)

Shifts by n < 0 or n >= size of x are undefined

Using Shifts and Masks

- **Extract the 2nd most significant byte of an integer?**

x	01100001	01100010	01100011	01100100

Using Shifts and Masks

- **Extract the 2nd most significant byte of an integer:**
  - First shift, then mask: (x >> 16) & 0xFF

`x`	01100001	01100010	01100011	01100100	
`x >> 16`	00000000	00000000	01100001	01100010	
`0xFF`	00000000	00000000	00000000	11111111	mask
`(x>>16) & 0xFF`	00000000	00000000	00000000	01100010	result

- **Extract the sign bit of a signed integer?**

Using Shifts and Masks

This picture is assuming arithmetic shifts, but process works in either case

- **Extract the sign bit of a signed integer:**
  - (x >> 31) & 1 - need the "& 1" to clear out all other bits except LSB

`x`	11100001 01100010 01100011 01100100	
`x >> 31`	11111111 11111111 11111111 11111111	
`(x >> 31) & 0x1`	00000000 00000000 00000000 00000001 00000000 00000000 00000000 00000001	mask result

`x`	01100001 01100010 01100011 01100100	
`x >> 31`	00000000 00000000 00000000 00000000	
`(x >> 31) & 0x1`	00000000 00000000 00000000 00000001 00000000 00000000 00000000 00000000	mask result

Using Shifts and Masks

- **Conditionals as Boolean expressions (assuming x is 0 or 1)**
  - In C: `if (x) a=y else a=z;` which is the same as `a = x ? y : z;`
    - If `x == 1` then `a = y`, otherwise `x == 0` and `a = z`
  - Can be re-written (assuming arithmetic right shift) as:

```
a = (((x << 31) >> 31) & y) | (((!x) << 31) >> 31) & z);
```

`x = 1`	00000000 00000000 00000000 00000001
`x << 31`	10000000 00000000 00000000 00000000
`((x << 31) >> 31)`	11111111 11111111 11111111 11111111
`y = 257`	00000000 00000000 00000001 00000001
`(((x << 31) >> 31) & y)`	00000000 00000000 00000001 00000001

If `x == 1`, then `!x == 0` and `((!x) << 31) >> 31) = 00..0`,
so: `(00..0 & z) = 0` and
`a = (00000000 00000000 00000001 00000001) | ( 00…00)` (in other words **a = y**)
If `x == 0`, then `!x == 1` and instead **a = z**.

One of two sides of the | will always be all zeroes.

Multiplication

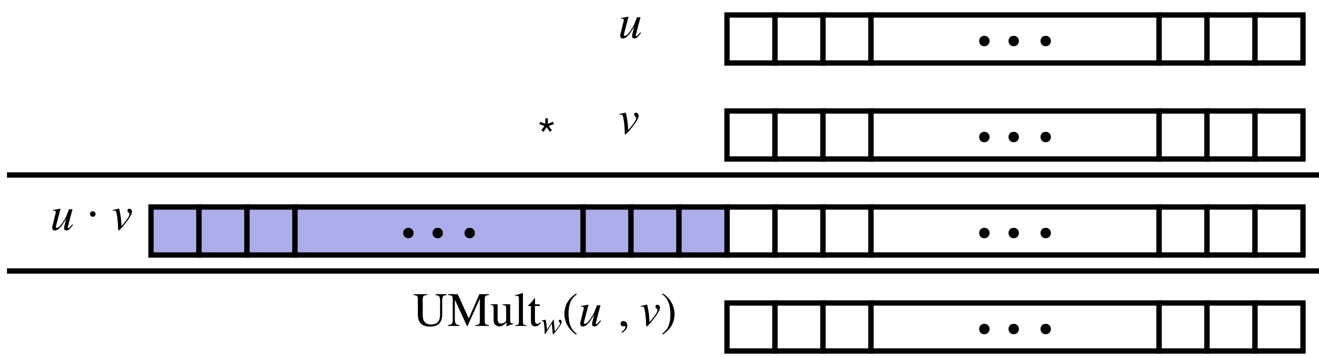
- **What do you get when you multiply 9 x 9?**
- **What about 2^{30} x 3?**
- **2^{30} x 5?**
- **-2^{31} x -2^{31}?**

Unsigned Multiplication in C

Operands: w bits

True Product: 2*w bits

Discard w bits: w bits



- **Standard Multiplication Function**
  - Ignores high order *w* bits
- **Implements Modular Arithmetic**

$\text{UMult}_w(u\,,v) \;=\; u \cdot v \;\bmod 2^w$

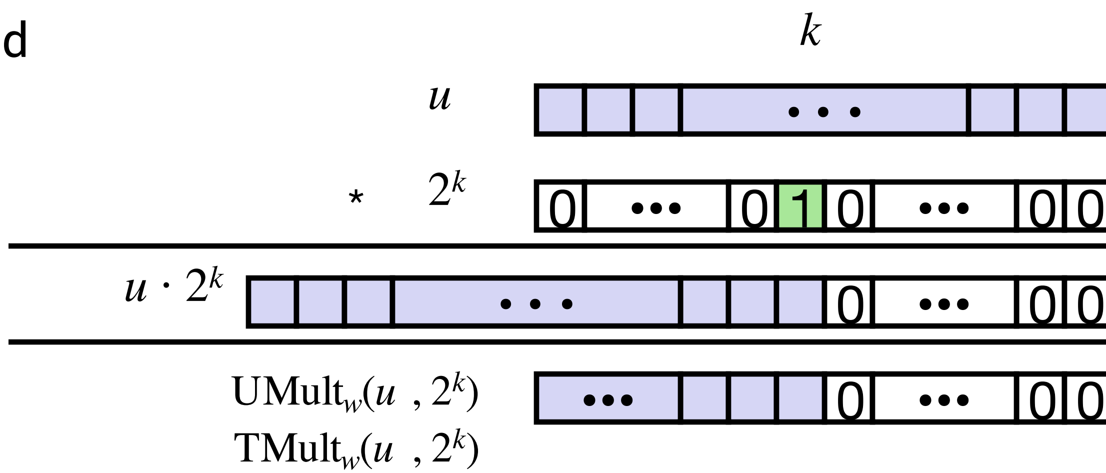
Multiplication with **shift** and **add**

- **Operation**
  - `u << k` gives `u * `***2^k***
  - Both signed and unsigned

Operands: w bits

True Product: w+k bits

Discard k bits: w bits



- **Examples**
  - `u << 3` `==` `u * 8`
  - `u << 5 - u << 3` `==` `u * 24`
  - Most machines shift and add faster than multiply
    - <u>**Compiler generates this code automatically**</u>

Code Security Example

```
/* Kernel memory region holding user-accessible data */
#define KSIZE 1024
char kbuf[KSIZE];

/* Copy at most maxlen bytes from kernel region to user buffer */
int copy_from_kernel(void* user_dest, int maxlen) {
    /* Byte count len is minimum of buffer size and maxlen */
    int len = KSIZE < maxlen ? KSIZE : maxlen;
    memcpy(user_dest, kbuf, len);
    return len;
}
```

```
#define MSIZE 528

void getstuff() {
    char mybuf[MSIZE];
    copy_from_kernel(mybuf, MSIZE);
    printf("%s\n", mybuf);
}
```

Malicious Usage

```
/* Declaration of library function memcpy */
void* memcpy(void* dest, void* src, size_t n);
```

```
/* Kernel memory region holding user-accessible data */
#define KSIZE 1024
char kbuf[KSIZE];

/* Copy at most maxlen bytes from kernel region to user buffer */
int copy_from_kernel(void* user_dest, int maxlen) {
    /* Byte count len is minimum of buffer size and maxlen */
    int len = KSIZE < maxlen ? KSIZE : maxlen;
    memcpy(user_dest, kbuf, len);
    return len;
}
```

```
#define MSIZE 528

void getstuff() {
    char mybuf[MSIZE];
    copy_from_kernel(mybuf, -MSIZE);
    . . .
}
```

- `len` is computed by finding the minimum of the two, which will be `maxlen` if we passed a negative value.
- Because `memcpy` takes an unsigned integer (`size_t`), this allows a malicious caller to read more of the kernel memory than it should.

April 8

Announcements

Lab 1 Prelim due today at 5pm.

Q&A: THE PENTIUM FDIV BUG

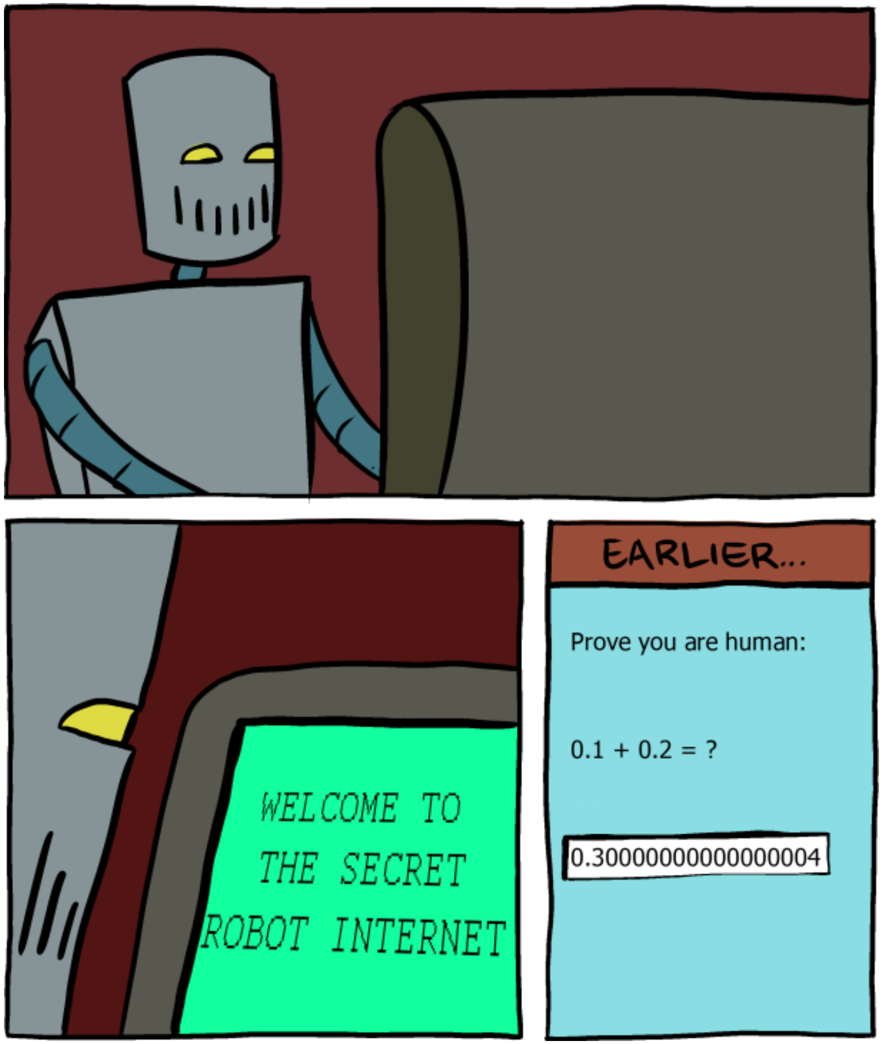
(floating point division)

Q: What do you get when you cross a Pentium PC with a research grant?
A: A mad scientist.

Q: Complete the following word analogy:
Add is to Subtract as Multiply is to:
1) Divide
2) ROUND
3) RANDOM
4) On a Pentium, all of the above
A: Number 4.

Q: What algorithm did Intel use in the Pentium's floating point divider?
A: "Life is like a box of chocolates."
(Source: F. Gump of Intel)

Q: According to Intel, the Pentium conforms to the IEEE standards 754 and 854 for floating point arithmetic. If you fly in aircraft designed using a Pentium, what is the correct pronunciation of "IEEE"?
A: Aaaaaaaiiiiiiiiieeeeeeeeeeeee!



http://www.smbc-comics.com/?id=2999

Source: http://www.columbia.edu/~sss31/rainbow/pentium.jokes.html

Floating point topics

- **Background: fractional binary numbers**
- **IEEE floating-point standard**
- **Floating-point operations and rounding**
- **Floating-point in C**

- **There are many more details that we won't cover**
  - It's a 58-page standard…

Summary

- **As with integers, floats suffer from the fixed number of bits available to represent them**
  - Can get overflow/underflow, just like ints
  - Some “simple fractions” have no exact representation (e.g., 0.2)
  - Can also ***lose precision***, unlike ints
    - “Every operation gets a slightly wrong result”
- **Mathematically equivalent ways of writing an expression may compute different results**
  - Violates associativity/commutativity/distributivity…
- **Never test floating point values for equality!**
- **Careful when converting between ints and floats!**

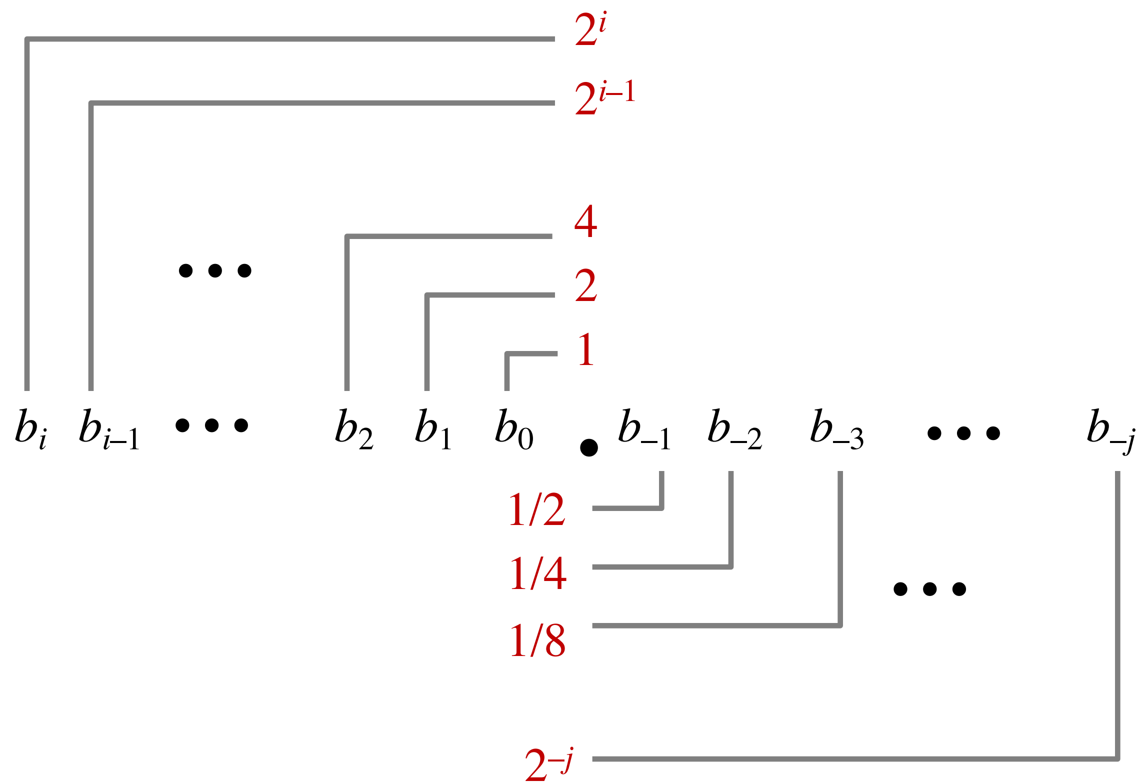
Fractional Binary Numbers

$$1\ 0\ 1\ 1\ .\ 1\ 0\ 1_{2}$$

8	4	2	1		½	¼	⅛
2^3	2^2	2^1	2^0		2^{-1}	2^{-2}	2^{-3}

$$8+2+1+\tfrac{1}{2}+\tfrac{1}{8} = 11.625_{10}$$

Fractional Binary Numbers



- Representation
  - Bits to right of "binary point" represent fractional powers of 2
  - Represents rational number: $\sum_{k=-j}^{i} b_k \cdot 2^k$

Fractional Binary Numbers

- **Value** **Binary:**
  - 5.75 $5\,\frac{3}{4}$
  - 2 and 7/8
  - 47/64

0 1 0 . 1 1 1

4 2 1 1/2 1/4 1/8

4/8 2/8

1 0 1 1 1 1 .

$\frac{32\ \ 16\ \ 8\ \ 4\ \ 2\ \ 1}{64}$

Fractional Binary Numbers

- **Value** **Binary:**
  - 5.75 101.11_2
  - 2 and 7/8 10.111_2
  - 47/64 0.101111_2
- **Observations**
  - Shift left = multiply by power of 2
  - Shift right = divide by power of 2
  - Numbers of the form $0.111111\ldots_2$ are just below 1.0
    - $1/2 + 1/4 + 1/8 + \ldots + 1/2^i + \ldots \rightarrow 1.0$
    - Use notation $1.0 - \varepsilon$

$0.\overline{9999}$

Limits of Representation

- **Limitations:**
  - Even given an arbitrary number of bits, can only **exactly** represent numbers of the form $x * 2^y$ (y can be negative)
  - Other rational numbers have repeating bit representations

Value:		**Binary Representation:**
1/3	= 0.333333…$_{10}$ =	0.01010101[01]…$_2$
1/5	=	0.001100110011[0011]…$_2$
1/10	=	0.0001100110011[0011]…$_2$

Fixed Point Representation

- **Binary point has a <u>fixed</u> position**
  - Position = number of binary digits before and after
- **Implied binary point. Two example schemes:**

  #1: the binary point is between bits 2 and 3

  < 32 $b_7\ b_6\ b_5\ b_4\ b_3$ [.] $b_2\ b_1\ b_0$

  #2: the binary point is between bits 4 and 5

  < 8 $b_7\ b_6\ b_5$ [.] $b_4\ b_3\ b_2\ b_1\ b_0$

- Wherever we put the binary point, with fixed point representations there is a **trade off** between the amount of **range** and **precision**
- **Fixed point = fixed *range* and fixed *precision***
  - range: difference between largest and smallest numbers possible
  - precision: smallest possible difference between any two numbers
- **Hard to pick how much you need of each!**

Rarely used in practice. Not built-in.

Floating Point

- **Analogous to scientific notation**
  - In Decimal:
    - Not **12000000**, but **1.2 x 10^7** In C: **1.2e7**
    - Not **0.0000012**, but **1.2 x 10^{-6}** In C: **1.2e-6**
  - In Binary:
    - Not **11000.000**, but **1.1 x 2^4**
    - Not **0.000101**, but **1.01 x 2^{-4}**

- **We have to divvy up the bits we have (e.g., 32) among:**
  - the sign (1 bit)
  - the significand / mantissa
  - the exponent

IEEE Floating Point

- **IEEE 754**
  - Established in 1985 as uniform standard for floating point arithmetic
  - Main idea: make numerically sensitive programs portable
  - Specifies two things: representation and result of floating operations
  - Now supported by all major CPUs
    - Some cheat! (looking at you, GPUs…)

- **Driven by numerical concerns**
  - **Scientists**/numerical analysts want them to be as **real** as possible
  - **Engineers** want them to be **easy to implement** and **fast**
  - In the end:
    - Scientists mostly won out
    - Nice standards for rounding, overflow, underflow, but…
    - Hard to make fast in hardware
    - **Float operations can be an order of magnitude slower than integer**

Floating Point Representation

- **Numerical form:**

$$V_{10} = (-1)^{s} * M * 2^{E}$$

  - Sign bit ***s*** determines whether number is negative or positive
  - Significand (mantissa) ***M*** normally a fractional value in range **[1.0, 2.0)**
  - Exponent ***E*** weights value by a (possibly negative) power of two

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  - Significand (mantissa) ***M*** normally a fractional value in range **[1.0, 2.0)**
  - Exponent ***E*** weights value by a (possibly negative) power of two

- **Representation in memory:**
  - MSB `s` is sign bit ***s***
  - `exp` field encodes ***E*** (but is *not equal* to E)
  - `frac` field encodes ***M*** (but is *not equal* to M)

s	exp	frac

Precisions

- **Single precision: 32 bits**

s	exp	frac
1 bit	8 bits	23 bits

- **Double precision: 64 bits**

s	exp	frac
1 bit	11 bits	52 bits

- **Finite representation means not all values can be represented exactly. Some will be approximated.**

Normalization and Special Values

$V = (-1)^{s} * M * 2^{E}$

s	exp	frac

- **"Normalized" = M has the form 1.xxxxx**
  - As in scientific notation, but in binary
  - 0.011 x 2^5 and 1.1 x 2^3 represent the same number, but the latter makes better use of the available bits
  - Since we know the mantissa starts with a 1, we don't bother to store it

- **How do we represent 0.0?**
  Or special or undefined values like 1.0/0.0?

Normalization and Special Values

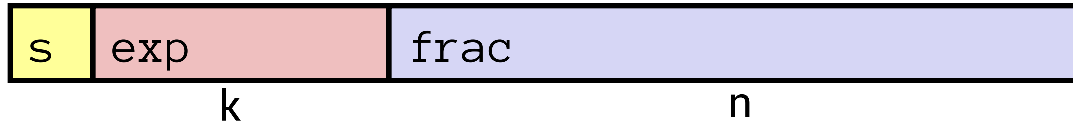
$V = (-1)^{S} * M * 2^{E}$

s	exp	frac

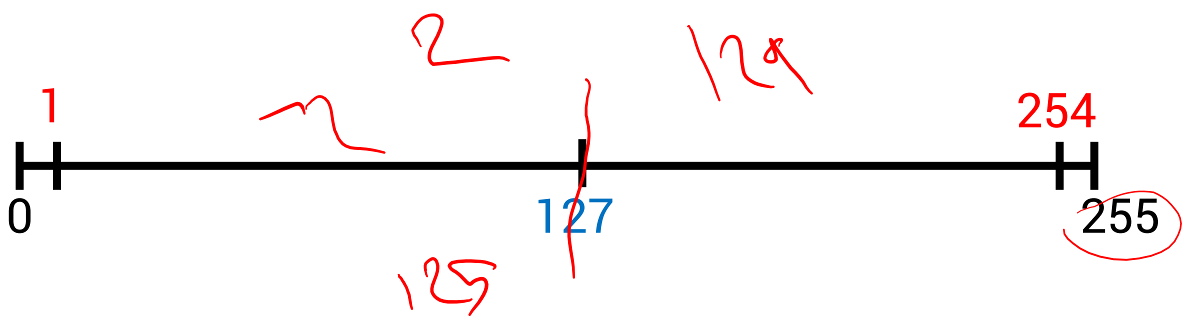
- **"Normalized" = M has the form 1.xxxxx**
  - As in scientific notation, but in binary
  - 0.011 x 2^5 and 1.1 x 2^3 represent the same number, but the latter makes better use of the available bits
  - Since we know the mantissa starts with a 1, we don't bother to store it.
- **Special values ("denormalized"):**
  - **Zero (`0`)**: `exp` == 00...0, `frac` == 00...0
  - **+∞, -∞:** `exp` == 11...1, `frac` == 00...0
    1.0/0.0 = -1.0/-0.0 = +∞, 1.0/-0.0 = -1.0/0.0 = -∞
  - NaN ("Not a Number"): `exp` == 11...1 `frac` != 00...0
    Results from operations with undefined result:
    sqrt(-1), ∞−∞, ∞•0, ...
  - **Note**: **exp=11…1 and exp=00…0 are reserved**, limiting exp range…

Normalized Values

$V = (-1)^{s} * M * 2^{E}$



- **Condition:** `exp` $\neq$ `000…0` and `exp` $\neq$ `111…1`
- **Exponent coded as *biased* value: *E* = *exp* - *Bias***
  - `exp` is an *unsigned* value ranging from 1 to 2^k-2 (k == # bits in `exp`)
  - *Bias* = 2^{k-1} - 1
    - Single precision: 127 (so *exp*: 1…254, *E*: -126…127)
    - Double precision: 1023 (so *exp*: 1…2046, *E*: -1022…1023)
  - These enable negative values for E, for representing very small values
    - Could have encoded with 2's complement or sign-and-magnitude
    - This just made it easier for HW to do float-exponent operations



Normalized Values

$V = (-1)^{s} * M * 2^{E}$

s	exp	frac
	k	n

- **Condition:** `exp` $\neq$ `000…0` and `exp` $\neq$ `111…1`
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  - These enable negative values for E, for representing very small values
    - Could have encoded with 2's complement or sign-and-magnitude
    - This just made it easier for HW to do float-exponent operations
- **Mantissa coded with implied leading 1: *M* =** $\mathtt{1.xxx...x}_2$
  - `xxx…x`: the n bits of `frac`
  - Minimum when `000…0` (*M* = 1.0)
  - Maximum when `111…1` (*M* = 2.0 – ε)
  - Get extra leading bit for "free"

Normalized Encoding Example

$V = (-1)^{S} * M * 2^{E}$

s	exp	frac
	k	n

- **Value:** `float f = 12345.0;`
  - $12345_{10} = 11000000111001_2$
    $= 1.1000000111001_2 \times 2^{13}$ (normalized form)

- **Mantissa:**

  $M = 1.\boxed{1000000111001}_2 = 1 + 1*2^{-1} + 1*2^{-8} + 1*2^{-9} + 1*2^{-10} + 1*2^{-13} = 1.5069580078125_{10}$

  $\text{frac} = 1000000111001\,0000000000_2$

- **Exponent:** E = exp - Bias, so exp = E + Bias

  $E = 13_{10}$

  $Bias = 127_{10}$

  $\text{exp} = 140_{10} = 10001100_2$

- **Result:**

0	10001100	10000001110010000000000
s	exp	frac

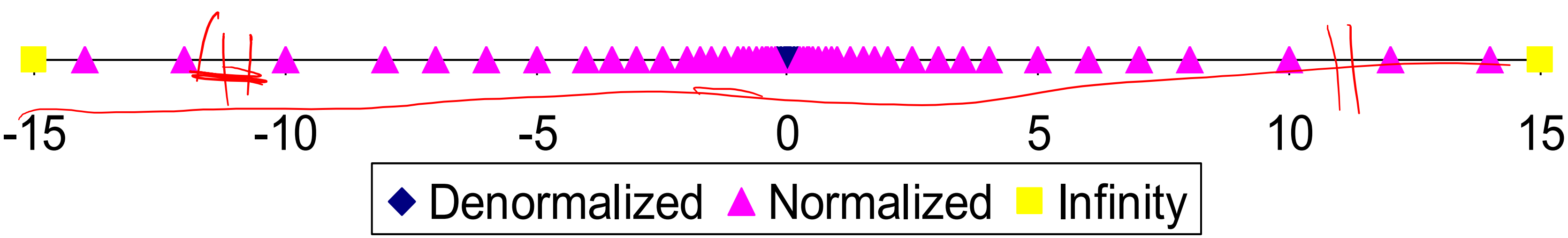
$V = (-1)^{S} * M * 2^{E} = (-1)^{0} * 1.5069580078125_{10} * 2^{13_{10}}$

Distribution of Values

- **6-bit IEEE-like format**
  - e = 3 exponent bits
  - f = 2 fraction bits
  - Bias is $2^{3-1}-1 = 3$

s	exp	frac
1	3	2

- **Notice how the distribution gets denser toward zero.**



Floating Point Operations

- **Unlike the representation for integers, the representation for floating-point numbers is <u>not exact</u>**
- **We have to know how to round from the real value**

Floating Point Operations: Basic Idea

$V = (-1)^{s} * M * 2^{E}$

s	exp	frac

- $x +_f y$ = Round(x + y)

- $x *_f y$ = Round(x * y)

- **Basic idea for floating point operations:**
  - First, compute the exact result
  - Then, *round* the result to make it fit into desired precision:
    - Possibly overflow if exponent too large
    - Possibly drop least-significant bits of mantissa to fit into `frac`

Floating Point Addition

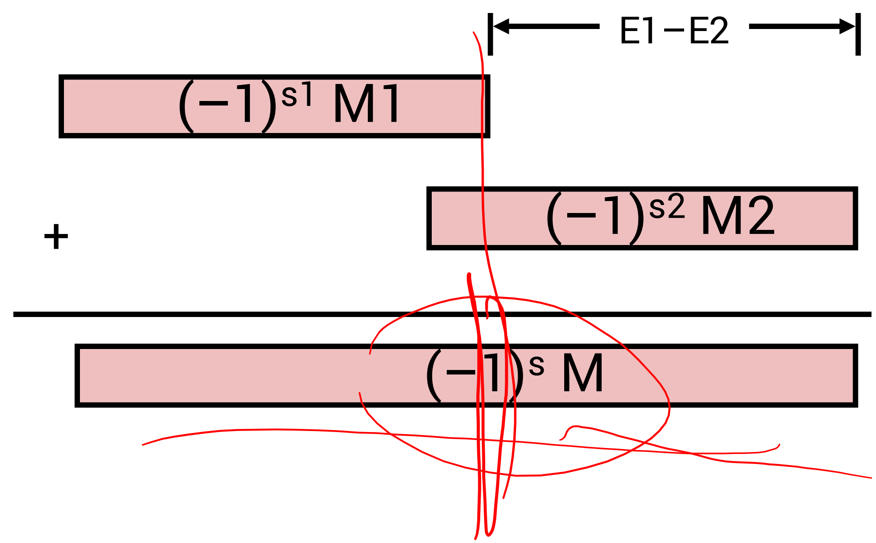
Line up the binary points

$$(-1)^{s1}*M1*2^{E1} + (-1)^{s2}*M2*2^{E2}$$

Assume *E1* > *E2*

```
  1.010*2^2
+ 1.000*2^-1
  ???
```

```
  1.0100*2^2
+ 0.0001*2^2
  1.0101*2^2
```



- **Exact Result:** $(-1)^s*M*2^E$
  - Sign `s`, mantissa `M`:
    - Result of signed align & add
  - Exponent `E`: `E1`
- **Fixing**
  - If *M* ≥ 2, shift *M* right, increment *E*
  - if *M* < 1, shift *M* left *k* positions, decrement *E* by *k*
  - Overflow if *E* out of range
  - Round *M* to fit `frac` precision

Floating Point Multiplication

$(-1)^{s1}*M1*2^{E1} \quad * \quad (-1)^{s2}*M2*2^{E2}$

- **Exact Result:** $(-1)^{s}*M*2^{E}$
  - Sign `s`: `s1 ^ s2`
  - Mantissa `M`: `M1 * M2`
  - Exponent `E`: `E1 + E2`
- **Fixing**
  - If $M \geq 2$, shift M right, increment E
  - If E out of range, overflow
  - Round M to fit `frac` precision

Rounding modes

- **Possible rounding modes** (illustrated with dollar rounding):

	\$1.40	**\$1.60**	**\$1.50**	**\$2.50**	**–\$1.50**
Round-toward-zero	\$1	\$1	\$1	\$2	–\$1
Round-down (-∞)	\$1	\$1	\$1	\$2	–\$2
Round-up (+∞)	\$2	\$2	\$2	\$3	–\$1
Round-to-nearest	\$1	\$2	??	??	??
Round-to-even	\$1	\$2	\$2	\$2	–\$2

- **Round-to-even avoids statistical bias in repeated rounding.**
  - Rounds up about half the time, down about half the time.
  - Default rounding mode for IEEE floating-point

Mathematical Properties of FP Operations

- **Exponent overflow yields +∞ or -∞**

- **Floats with value +∞, -∞, and NaN can be used in operations**
  - Result usually still +∞, -∞, or NaN; sometimes intuitive, sometimes not

- **Floating point ops do not work like real math, due to rounding!**
  - Not associative: `(3.14 + 1e100) – 1e100 != 3.14 + (1e100 – 1e100)`
  - Not distributive: `100 * (0.1 + 0.2) != 100 * 0.1 + 100 * 0.2`
    `30.000000000000003553` `30`
  - Not cumulative
    - Repeatedly adding a very small number to a large one may do nothing

Floating Point in C

!!!

- **C offers two (well, 3) levels of precision**

`float`	`1.0f`	single precision (32-bit)
`double`	`1.0`	double precision (64-bit)
`long double`	`1.0L`	*(double double, quadruple, or "extended")* precision (64-128 bits)

- **`#include <math.h>` to get `INFINITY` and `NAN` constants**
- **Equality (==) comparisons between floating point numbers are tricky, and often return unexpected results**
  - Just avoid them!

Floating Point in C

!!!

- **Conversions between data types:**
  - Casting between `int`, `float`, and `double` **changes** the bit representation.
  - `int` → `float`
    - May be rounded (not enough bits in mantissa: 23)
    - Overflow impossible
  - `int` → `double` or `float` → `double`
    - Exact conversion (32-bit ints; 52-bit frac + 1-bit sign)
  - `long` → `double`
    - Rounded or exact, depending on word size (64-bit → 52 bit mantissa ⇒ round)
  - `double` or `float` → `int`
    - Truncates fractional part (rounded toward zero)
      - E.g. 1.999 → 1, -1.99 → -1
    - "Not defined" when out of range or NaN: generally sets to Tmin (even if the value is a very big positive)

Number Representation Really Matters

- **1991: Patriot missile targeting error**
  - clock skew due to conversion from integer to floating point
- **1996: Ariane 5 rocket exploded ($1 billion)**
  - overflow converting 64-bit floating point to 16-bit integer
- **2000: Y2K problem**
  - limited (decimal) representation: overflow, wrap-around
- **2038: Unix epoch rollover**
  - Unix epoch = seconds since 12am, January 1, 1970
  - signed 32-bit integer representation rolls over to TMin in 2038
- **other related bugs**
  - 1982: Vancouver Stock Exchange 10% error in less than 2 years
  - 1994: Intel Pentium FDIV (floating point division) HW bug ($475 million)
  - 1997: USS Yorktown "smart" warship stranded: divide by zero
  - 1998: Mars Climate Orbiter crashed: unit mismatch ($193 million)

Floating Point and the Programmer

```
#include <stdio.h>

int main(int argc, char* argv[]) {

  float f1 = 1.0;
  float f2 = 0.0;
  int i;
  for (i = 0; i < 10; i++) {
    f2 += 1.0/10.0;
  }

  printf("0x%08x  0x%08x\n", *(int*)&f1, *(int*)&f2);
  printf("f1 = %10.8f\n", f1);
  printf("f2 = %10.8f\n\n", f2);

  f1 = 1E30;
  f2 = 1E-30;
  float f3 = f1 + f2;
  printf("f1 == f3? %s\n", f1 == f3 ? "yes" : "no" );

  return 0;
}
```

```
$ ./a.out
0x3f800000  0x3f800001
f1 = 1.000000000
f2 = 1.000000119

f1 == f3? yes
```

Summary

- **As with integers, floats suffer from the fixed number of bits available to represent them**
  - Can get overflow/underflow, just like ints
  - Some “simple fractions” have no exact representation (e.g., 0.2)
  - Can also lose precision, unlike ints
    - “Every operation gets a slightly wrong result”
- **Mathematically equivalent ways of writing an expression may compute different results**
  - Violates associativity/distributivity
- **Never test floating point values for equality!**
- **Careful when converting between ints and floats!**

Many more details for the curious...

- **Denormalized values – to get finer precision near zero**
- **Distribution of representable values**
- **Floating point multiplication & addition algorithms**
- **Rounding strategies**

- **We won’t be using or testing you on any of these extras in 351.**

Denormalized Values

- **Condition: `exp` = `000…0`**

- **Exponent value: *E* = exp – *Bias* + 1 (instead of E = exp – Bias)**
- **Significand coded with implied leading 0: *M* = `0.xxx…x`$_2$**
    - `xxx…x`: bits of `frac`
- **Cases**
    - `exp` = `000…0`, `frac` = `000…0`
        - Represents value 0
        - Note distinct values: +0 and –0 (why?)
    - `exp = 000…0, frac ≠ 000…0`
        - Numbers very close to 0.0
        - Lose precision as get smaller
        - Equispaced

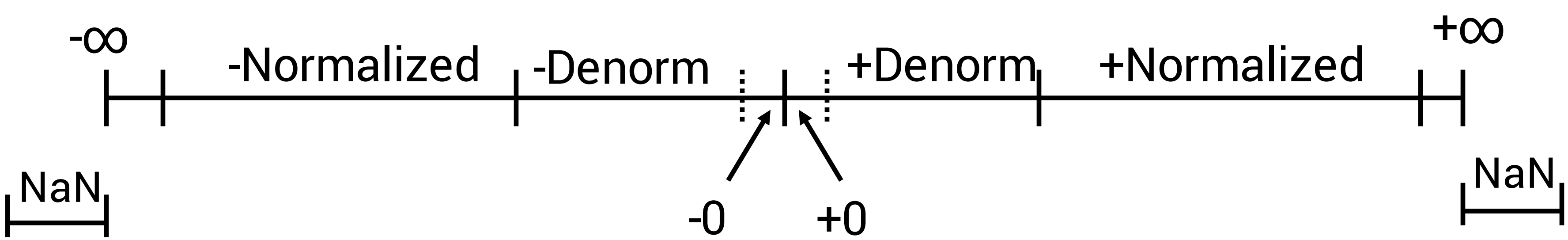
Special Values

- **Condition:** `exp = 111…1`

- **Case:** `exp = 111…1, frac = 000…0`
  - Represents value ∞ (infinity)
  - Operation that overflows
  - Both positive and negative
  - E.g., 1.0/0.0 = -1.0/-0.0 = +∞, 1.0/-0.0 = -1.0/0.0 = -∞

- **Case:** `exp = 111…1, frac -000…0`
  - Not-a-Number (NaN)
  - Represents case when no numeric value can be determined
  - E.g., sqrt(–1), ∞–∞, ∞•0, ...

Visualization: Floating Point Encodings



Tiny Floating Point Example

s	exp	frac
1	4	3

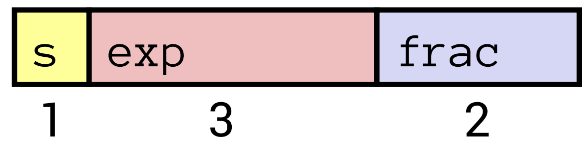
- **8-bit Floating Point Representation**
  - the sign bit is in the most significant bit.
  - the next four bits are the exponent, with a bias of 7.
  - the last three bits are the `frac`
- **Same general form as IEEE Format**
  - normalized, denormalized
  - representation of 0, NaN, infinity

Dynamic Range (Positive Only)

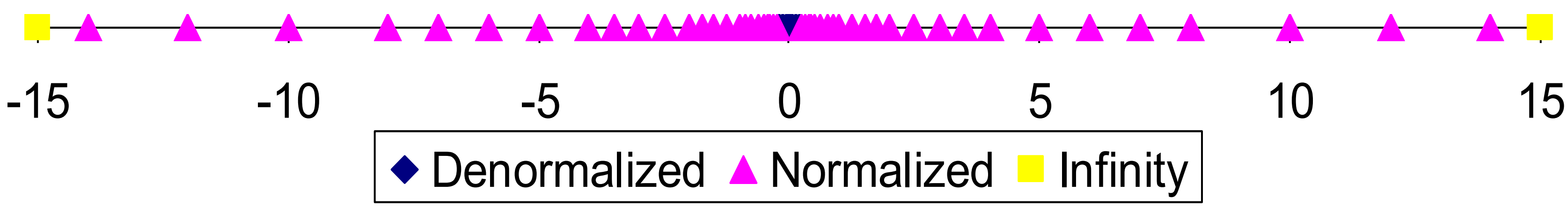
	s	exp	frac	E	Value	
Denormalized numbers	0	0000	000	-6	0	
	0	0000	001	-6	1/8*1/64 = 1/512	closest to zero
	0	0000	010	-6	2/8*1/64 = 2/512	
	…					
	0	0000	110	-6	6/8*1/64 = 6/512	
	0	0000	111	-6	7/8*1/64 = 7/512	largest denorm
Normalized numbers	0	0001	000	-6	8/8*1/64 = 8/512	smallest norm
	0	0001	001	-6	9/8*1/64 = 9/512	
	…					
	0	0110	110	-1	14/8*1/2 = 14/16	
	0	0110	111	-1	15/8*1/2 = 15/16	closest to 1 below
	0	0111	000	0	8/8*1 = 1	
	0	0111	001	0	9/8*1 = 9/8	closest to 1 above
	0	0111	010	0	10/8*1 = 10/8	
	…					
	0	1110	110	7	14/8*128 = 224	
	0	1110	111	7	15/8*128 = 240	largest norm
	0	1111	000	n/a	inf	

Distribution of Values

- **6-bit IEEE-like format**
  - e = 3 exponent bits
  - f = 2 fraction bits
  - Bias is 23-1-1 = 3

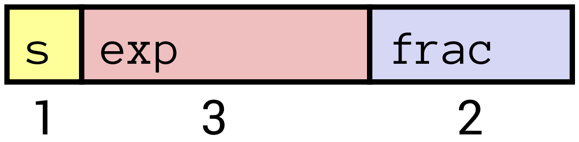


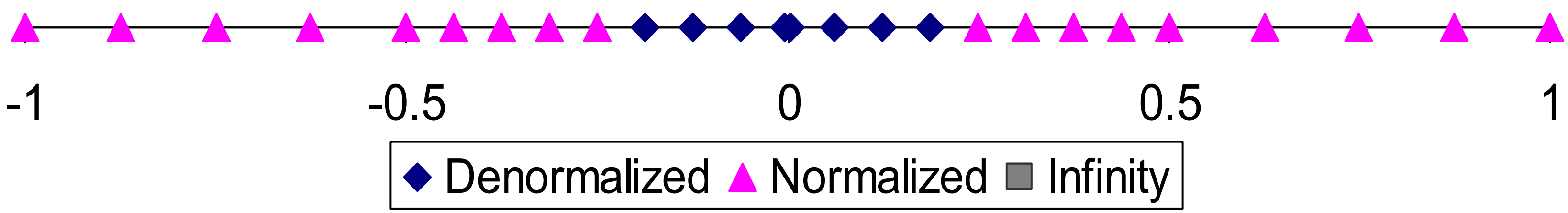
- **Notice how the distribution gets denser toward zero.**



Distribution of Values (close-up view)

- **6-bit IEEE-like format**
  - e = 3 exponent bits
  - f = 2 fraction bits
  - Bias is 3





Interesting Numbers

{single,double}

Description	exp	frac	Numeric Value
Zero	00…00	00…00	0.0
Smallest Pos. Denorm.	00…00	00…01	$2^{-\{23,52\}} * 2^{-\{126,1022\}}$
▪ Single ≈ $1.4 * 10^{-45}$			
▪ Double ≈ $4.9 * 10^{-324}$			
Largest Denormalized	00…00	11…11	$(1.0 - \varepsilon) * 2^{-\{126,1022\}}$
▪ Single ≈ $1.18 * 10^{-38}$			
▪ Double ≈ $2.2 * 10^{-308}$			
Smallest Pos. Norm.	00…01	00…00	$1.0 * 2^{-\{126,1022\}}$
▪ Just larger than largest denormalized			
One	01…11	00…00	1.0
Largest Normalized	11…10	11…11	$(2.0 - \varepsilon) * 2^{\{127,1023\}}$
▪ Single ≈ $3.4 * 10^{38}$			
▪ Double ≈ $1.8 * 10^{308}$			

Special Properties of Encoding

- **Floating point zero (0^+) exactly the same bits as integer zero**
  - All bits = 0

- **Can (Almost) Use Unsigned Integer Comparison**
  - Must first compare sign bits
  - Must consider $0^- = 0^+ = 0$
  - NaNs problematic
    - Will be greater than any other values
    - What should comparison yield?
  - Otherwise OK
    - Denorm vs. normalized
    - Normalized vs. infinity

Floating Point Multiplication

$$(-1)^{s1}\ M1\ \ 2^{E1} \ \ * \ \ (-1)^{s2}\ M2\ \ 2^{E2}$$

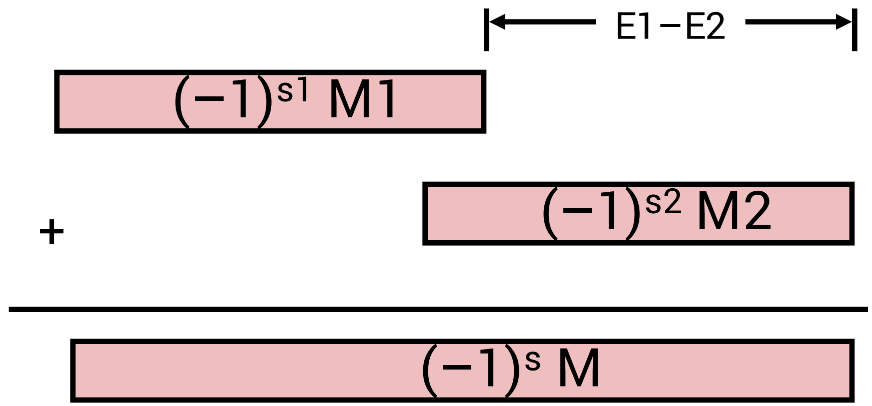
- **Exact Result:** $(-1)^{s}\ M\ \ 2^{E}$
  - Sign s: s1 ^ s2 // xor of s1 and s2
  - Significand M: M1 * M2
  - Exponent E: E1 + E2
- **Fixing**
  - If M ≥ 2, shift M right, increment E
  - If E out of range, overflow
  - Round M to fit frac precision

Floating Point Addition

$$(-1)^{s1}\,M1\;\;2^{E1}\;\;+\;\;(-1)^{s2}\,M2\;\;2^{E2}$$

Assume E1 > E2

- **Exact Result:** $(-1)^s\,M\;\;2^E$
  - Sign s, significand M:
    - Result of signed align & add
  - Exponent E: E1



- **Fixing**
  - If M ≥ 2, shift M right, increment E
  - if M < 1, shift M left k positions, decrement E by k
  - Overflow if E out of range
  - Round M to fit frac precision

Closer Look at Round-To-Even

- **Default Rounding Mode**
  - Hard to get any other kind without dropping into assembly
  - All others are statistically biased
    - Sum of set of positive numbers will consistently be over- or under- estimated

- **Applying to Other Decimal Places / Bit Positions**
  - When exactly halfway between two possible values
    - Round so that least significant digit is even
  - E.g., round to nearest hundredth

1.2349999	1.23	(Less than half way)
1.2350001	1.24	(Greater than half way)
1.2350000	1.24	(Half way—round up)
1.2450000	1.24	(Half way—round down)

Rounding Binary Numbers

- **Binary Fractional Numbers**
  - "Half way" when bits to right of rounding position = $100\ldots_2$

- **Examples**
  - Round to nearest 1/4 (2 bits right of binary point)

Value	Binary	Rounded	Action	Rounded Value
2 3/32	10.00011_2	10.00_2	(<1/2—down)	2
2 3/16	10.00110_2	10.01_2	(>1/2—up)	2 1/4
2 7/8	10.11100_2	11.00_2	(1/2—up)	3
2 5/8	10.10100_2	10.10_2	(1/2—down)	2 1/2

Floating Point Puzzles

s	exp	frac
1 bit	8 bits	23 bits

s	exp	frac
1 bit	11 bits	52 bits

- **For each of the following C expressions, either:**
  - Argue that it is true for all argument values
  - Explain why not true

```
int x = …;
float f = …;
double d = …;
double d2 = …;
```

Assume neither **`d`** nor **`f`** is NaN

```
1) x == (int)(float) x
2) x == (int)(double) x
3) f == (float)(double) f
4) d == (double)(float) d
5) f == -(-f);
6) 2/3 == 2/3.0
7) (d+d2)-d == d2
```