Floating Point

CSE 351 Autumn 2016

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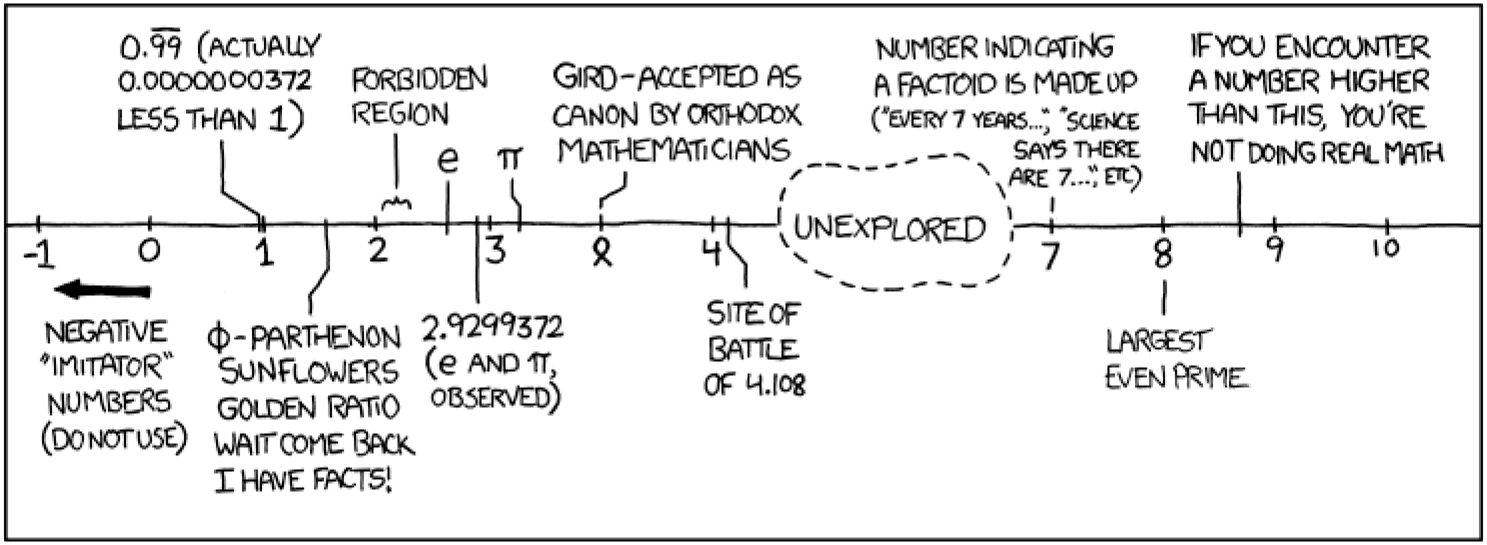


http://xkcd.com/899/

Administrivia

- Lab 1 due today at 5pm (prelim) and Friday at 5pm
  - Use Makefile and DLC and GDB to check & debug
- Homework 1 (written problems) released tomorrow
- Piazza
  - Response time from staff members often significantly slower on weekends
  - Would love to see more student participation!

Integers

- Binary representation of integers
  - Unsigned and signed
  - Casting in C
- Consequences of finite width representations
  - Overflow, sign extension
- Shifting and arithmetic operations
- Multiplication

Multiplication

- What do you get when you multiply 9 x 9?

81 → need extra digit

- What about 2^{30} x 3?

(2+1)

$2^{31}+2^{30}$ → representable only in unsigned

- 2^{30} x 5?

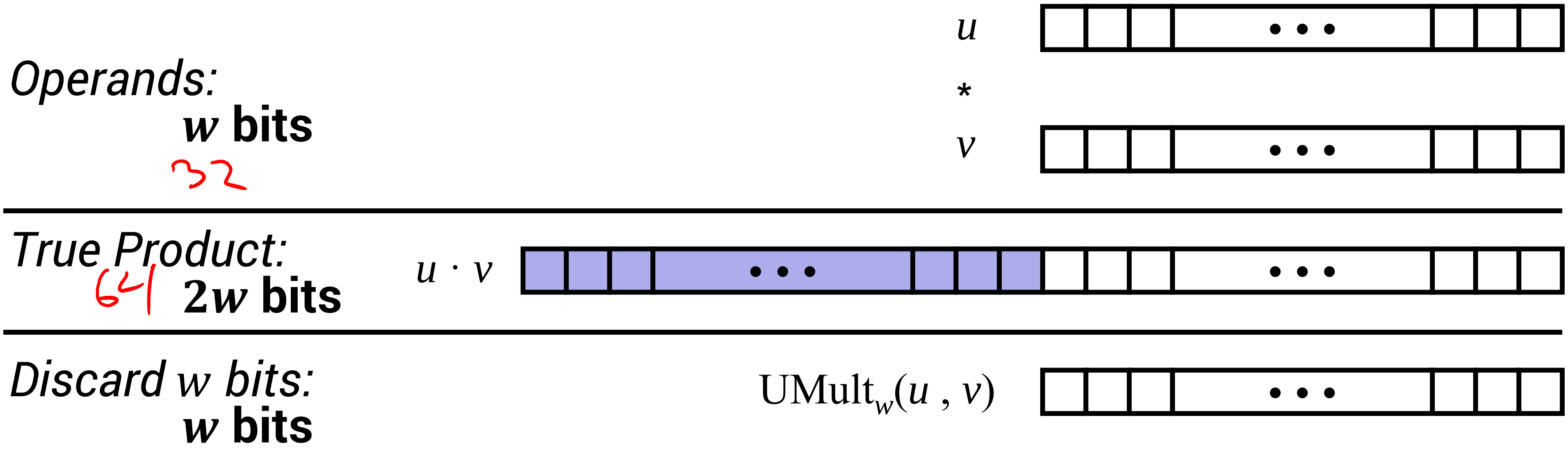
(4+1)

$2^{32}+2^{30}$ → not representable in 32-bit int

- -2^{31} x -2^{31}?

$+2^{62}$ → so large!

Unsigned Multiplication in C



Operands:
w bits
32

u
$*$
v

True Product:
64 **$2w$ bits**

$u \cdot v$

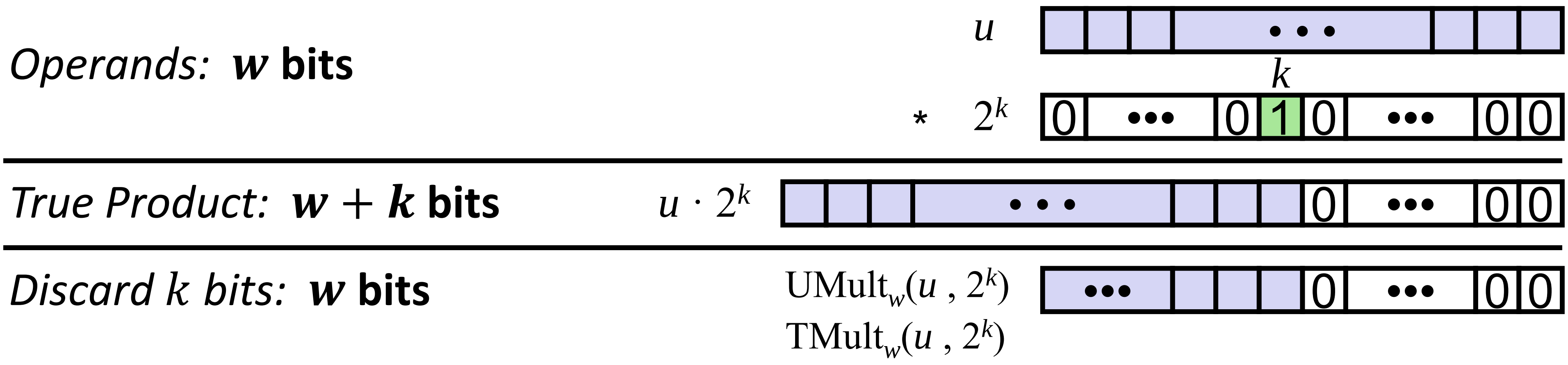
Discard w bits:
w bits

$\mathrm{UMult}_w(u , v)$

- Standard Multiplication Function
  - Ignores high order w bits
- Implements Modular Arithmetic
  - $\mathrm{UMult}_w(u , v) = u \cdot v \mod 2^w$

Multiplication with shift and add

- Operation u<<k gives u*2^k
  - Both signed and unsigned



Operands: ***w*** **bits**

True Product: ***w*** **+** ***k*** **bits**

Discard k bits: ***w*** **bits**

- Examples:
  - u<<3 == u * 8
  - u<<5 - u<<3 == u * 24 = u * (32-8)
  - Most machines shift and add faster than multiply
    - **Compiler generates this code automatically**

7 = 4+2+1

= u<<2 + u<<1 + u

Number Representation Revisited

- What can we represent in one word?
  - Signed and Unsigned Integers
  - Characters (ASCII)
  - Addresses
- How do we encode the following:
  - Real numbers (e.g. 3.14159) π
  - Very large numbers (e.g. 6.02×10^{23}) Avogadro's
  - Very small numbers (e.g. 6.626×10^{-34}) Planck's
  - Special numbers (e.g. ∞, NaN)

Floating Point

Goals of Floating Point

- Support a wide range of values
  - Both very small and very large
- Keep as much *precision* as possible
- Help programmer with errors in real arithmetic
  - Support +∞, -∞, Not-A-Number (NaN), exponent overflow and underflow
- Keep encoding that is somewhat compatible with two’s complement

Floating point topics

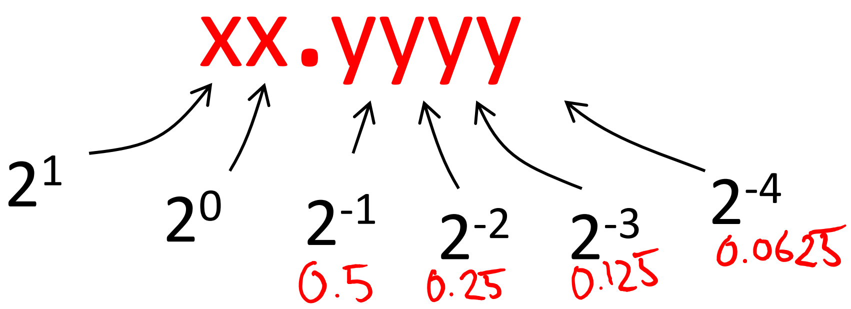
- Fractional binary numbers
- IEEE floating-point standard
- Floating-point operations and rounding
- Floating-point in C

- There are many more details that we won't cover
  - It's a 58-page standard…

Representation of Fractions

- "Binary Point," like decimal point, signifies boundary between integer and fractional parts:

Example 6-bit representation:



xx.yyyy

2^{1} 2^{0} 2^{-1} 2^{-2} 2^{-3} 2^{-4}

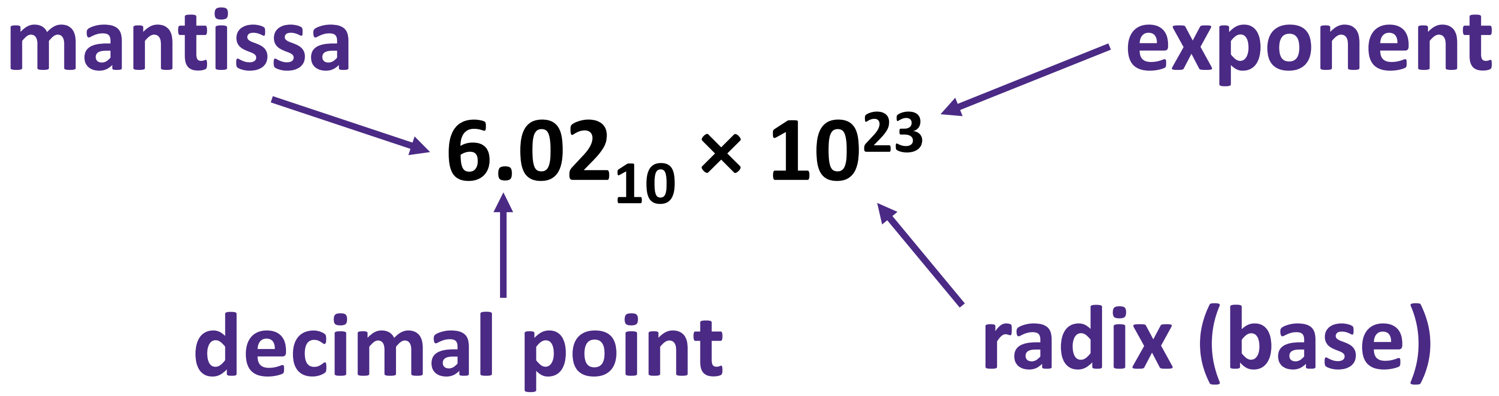
0.5 0.25 0.125 0.0625

- **Example:** $10.1010_2 = 1\times2^{1} + 1\times2^{-1} + 1\times2^{-3} = 2.625_{10}$

- Binary point numbers that match the 6-bit format above range from 0 (00.0000_2) to 3.9375 ($11.111\boxed{1}_2$) +1

$= 4 - 2^{-4}$

Scientific Notation (Decimal)



mantissa

exponent

$$6.02_{10} \times 10^{23}$$

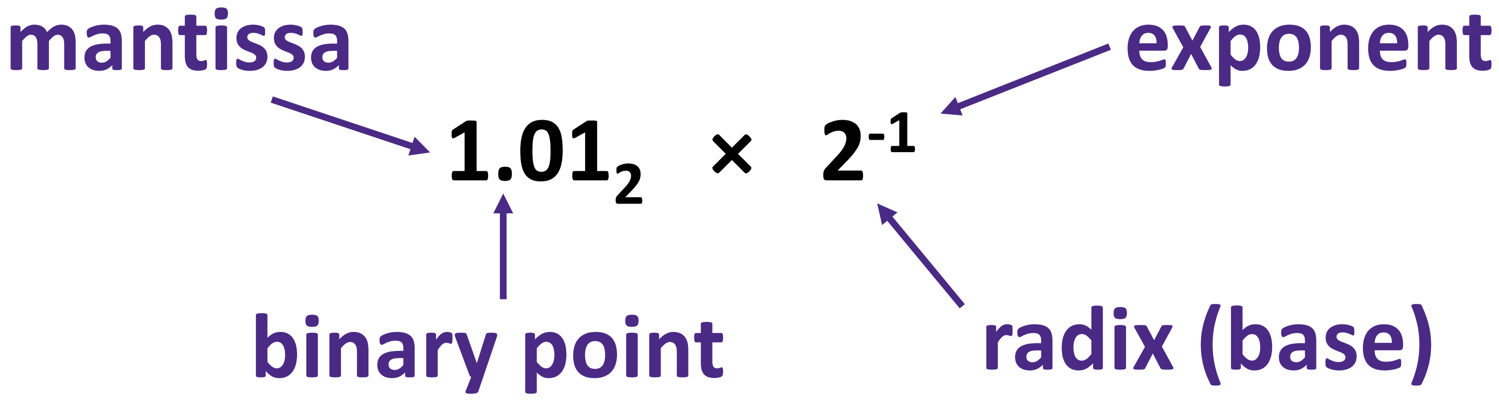
decimal point

radix (base)

- *Normalized form*: exactly one digit (non-zero) to left of decimal point

- Alternatives to representing 1/1,000,000,000
  - Normalized: 1.0×10^{-9}
  - Not normalized: 0.1×10^{-8}, 10.0×10^{-10}

Scientific Notation (Binary)



mantissa

exponent

$$1.01_2 \times 2^{-1}$$

binary point

radix (base)

- Computer arithmetic that supports this called floating point due to the "floating" of the binary point
  - Declare such variable in C as `float`

Scientific Notation Translation

shift point right: 10.11×2^3

shift point left: 0.1011×2^5

- Consider the number $1.011_2 \times 2^4$
  - To convert to ordinary number, shift the decimal to the right by 4
    - Result: $10110_2 = 22_{10}$
  - For negative exponents, shift decimal to the left
    - $1.011_2 \times 2^{-2} \Rightarrow 0.01011_2 = 0.34375_{10}$
  - Go from ordinary number to scientific notation by shifting until in *normalized* form
    - $1101.001_2 \rightarrow 1.101001_2 \times 2^3$

- **Practice:** Convert 11.375_{10} to binary scientific notation

$8+2+1+0.25+0.125$

$2^3+2^1+2^0+2^{-2}+2^{-3} = 1011.011_2 = \boxed{1.011011 \times 2^3}$

- **Practice:** Convert 1/5 to binary

0.2

$\frac{1}{5}-\frac{1}{8}=\frac{3}{40}, \frac{3}{40}-\frac{1}{16}=\frac{1}{80}=\frac{1}{5}\left(\frac{1}{16}\right)$

$\boxed{0.\overline{0011}_2}$

IEEE Floating Point

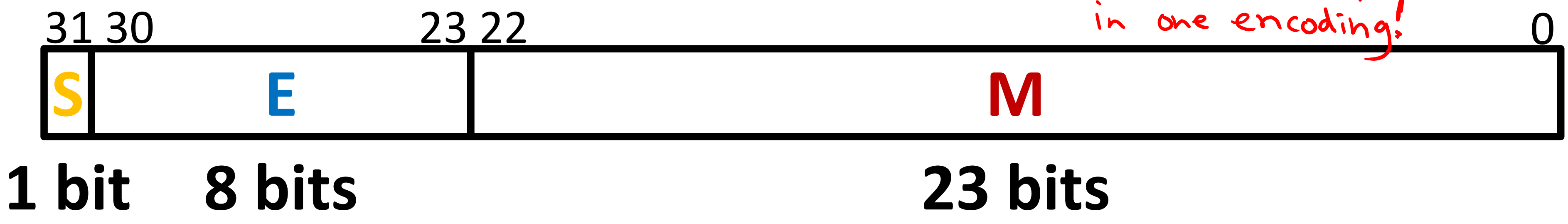
- IEEE 754
  - Established in 1985 as uniform standard for floating point arithmetic
  - Main idea: make numerically sensitive programs portable
  - Specifies two things: representation and result of floating operations
  - Now supported by all major CPUs

- Driven by numerical concerns
  - **Scientists**/numerical analysts want them to be as **real** as possible
  - **Engineers** want them to be **easy to implement** and **fast**
  - In the end:
    - Scientists mostly won out
    - Nice standards for rounding, overflow, underflow, but...
    - Hard to make fast in hardware
    - **Float operations can be an order of magnitude slower than integer ops**

Floating Point Encoding

- Use normalized, base 2 scientific notation:
  - Value: $\pm 1 \times \text{Mantissa} \times 2^{\text{Exponent}}$
  - Bit Fields: $(-1)^{S} \times 1.M \times 2^{(E+\text{bias})}$
- Representation Scheme:
  - Sign bit (0 is positive, 1 is negative)
  - Mantissa (a.k.a. significand) is the fractional part of the number in normalized form and encoded in bit vector **M**
  - Exponent weights the value by a (possibly negative) power of 2 and encoded in the bit vector **E**

3 separate fields put together in one encoding!

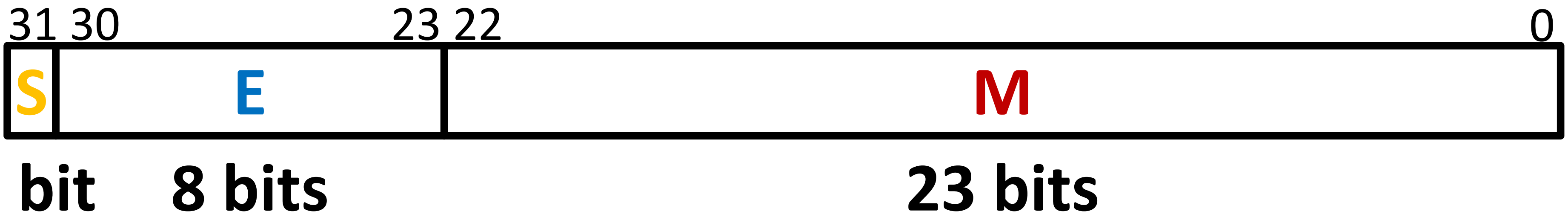


The Exponent Field

- Use biased notation
  - Read exponent as unsigned, but with *bias* of –(2^{w-1}-1) = –127
  - Representable exponents roughly ½ positive and ½ negative
  - Exponent 0 (Exp = 0) is represented as E = 0b 0111 1111
- Why biased?
  - Makes floating point arithmetic easier
  - Makes somewhat compatible with two's complement
- **Practice:** To encode in biased notation, subtract the bias (add 127) then encode in unsigned:
  - Exp = 1 → 128 → E = 0b 1000 0000
  - Exp = 127 → 254 → E = 0b 1111 1110
  - Exp = -63 → 64 → E = 0b 0100 0000

these 8 bits go in the 32-bit floating point encoding

The Mantissa Field



$$(-1)^{S} \times (1 . M) \times 2^{(E+bias)}$$

- Note the implicit 1 in front of the M bit vector
  - ⊕ Exp = 0
  - Example: 0b 0011 1111 1100 0000 0000 0000 0000 0000 is read as $1.1_2 = 1.5_{10}$, *not* $0.1_2 = 0.5_{10}$
  - Gives us an extra bit of *precision*
- Mantissa "limits"
  - Low values near M = 0b0…0 are close to 2^{Exp}
  - High values near M = 0b1…1 are close to 2^{Exp+1}

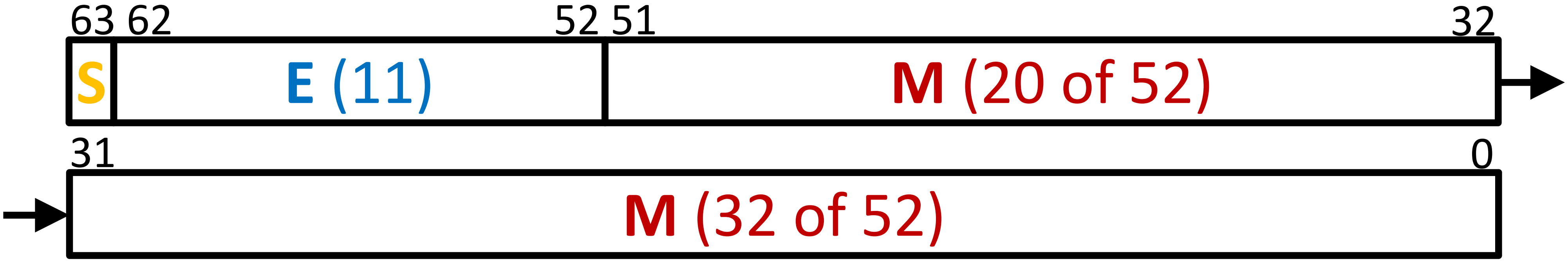
↑ +1 in this position causes overflow into E (Exp+1)

Precision and Accuracy

- Precision is a count of the number of bits in a computer word used to represent a value
  - Capability for accuracy
- Accuracy is a measure of the difference between the *actual value of a number* and its computer representation
  - *High precision permits high accuracy but doesn't guarantee it. It is possible to have high precision but low accuracy.*
  - **Example:** `float pi = 3.14;`
    - `pi` will be represented using all 24 bits of the significand (highly precise), but is only an approximation (not accurate)

Need Greater Precision?

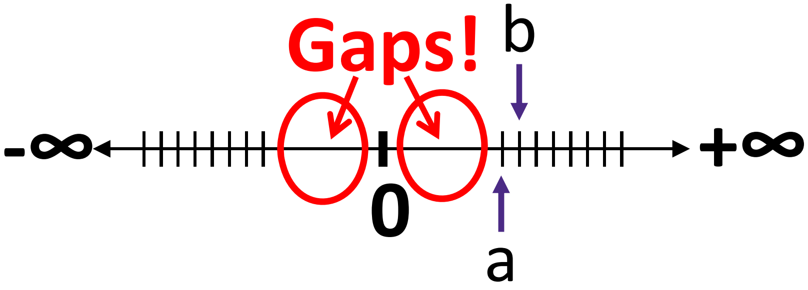
- Double Precision (vs. Single Precision) in 64 bits



  - C variable declared as `double`
  - Exponent bias is now –(2^{10}–1) = –1023
  - **Advantages:** greater precision (larger mantissa), greater range (larger exponent)
  - **Disadvantages:** more bits used, slower to manipulate

Representing Very Small Numbers

- But wait… what happened to zero?
  - Using standard encoding $\overset{S=0,\,E=0,\,M=0}{0x00000000}$ = $+1.0\times2^{0-127}=2^{-127}\neq0$
  - *Special case:* E and M all zeros = 0
    - Two zeros! But at least 0x00000000 = 0 like integers
- New numbers closest to 0:
  - $a = 1.0…0_2\times2^{-126} = 2^{-126}$
  - $b = 1.\underbrace{0…01}_{23\text{ bits}}{}_2\times2^{-126} = 2^{-126}+2^{-149}$
  - Normalization and implicit 1 are to blame
  - *Special case:* E = 0, M ≠ 0 are denormalized numbers



Denorm Numbers

- Denormalized numbers
  - No leading 1
  - Careful! Implicit exponent is –126 (not –127) even though E = 0x00 → normally $2^{0-127} = 2^{-127}$, but instead using $\underline{2^{-126}}$
- Now what do the gaps look like?
  - a = Smallest norm: $\pm\, 1.0\ldots0_{\text{two}} \times 2^{-126} = \pm\, 2^{-126}$
  - c = Largest denorm: $\pm\, 0.1\ldots1_{\text{two}} \times 2^{-126} = \pm\, (2^{-126} - 2^{-149})$
  - Smallest denorm: $\pm\, 0.0\ldots01_{\text{two}} \times 2^{-126} = \pm\, 2^{-149}$

No gap

So much closer to 0

Currently $c - a = 2^{-149}$

if we had used denorm exponent of 2^{-127},

then $\bar{c} = 2^{-127} - 2^{-150}$

and $\bar{c} - a = 2^{-126} - 2^{-127} + 2^{-150}$, so larger gap between $\bar{c}$ & a.

Other Special Cases

all 1's

- E = 0xFF, M = 0: ± ∞
  - e.g., division by 0
  - Still work in comparisons!
- E = 0xFF, M ≠ 0: Not a Number (NaN)
  - e.g., square root of negative number, 0/0, ∞–∞
  - NaN propagates through computations
  - Value of M can be useful in debugging
- Largest value (besides ∞)?
  - E = 0xFF has now been taken!
  - E = 0xFE has largest: $\underbrace{1.1\ldots1}_{\text{24 ones in a row}}{}_2\times2^{127} = 2^{128} - 2^{104}$

Exp = 254-127 = 127

$= (2^{24}-1)\times2^{-23}\times2^{127} = 2^{128}-2^{104}$ ☑

Floating Point Encoding Summary

Exponent	Mantissa	Meaning
0x00	0	± 0
0x00	non-zero	± denorm num
0x01 – 0xFE	anything	± norm num
0xFF	0	± ∞
0xFF	non-zero	NaN

smallest exponent

largest exponent

Distribution of Values

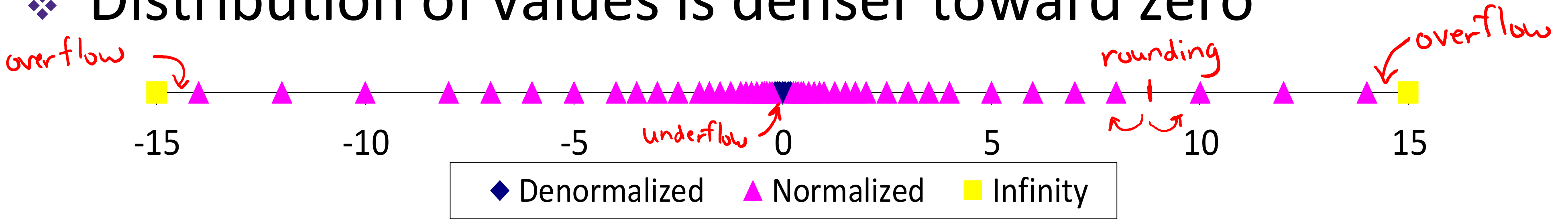
- What ranges are NOT representable?
  - Between largest norm and infinity Overflow
  - Between zero and smallest denorm Underflow
  - Between norm numbers? Rounding
- Given a FP number, what's the bit pattern of the next largest representable number?

  add 1 to LSB of mantissa

  from 0|0111 1111|0...0

  next num is 0|0111 1111|0...01
  - What is this "step" when Exp = 0? 2^{-23}
  - What is this "step" when Exp = 100? $2^{100-23} = 2^{77}$

  step gets larger when Exp is larger
- Distribution of values is denser toward zero



Peer Instruction Question

Let FP[1,2) = # of representable floats between 1 and 2

Let FP[2,3) = # of representable floats between 2 and 3

- Which of the following statements is true?
  - Vote at http://PollEv.com/justinh
  - **Extra:** what are the actual values of FP[1,2) and FP[2,3)?
    - Hint: Encode 1, 2, 3 into floating point

(A) FP[1,2) > FP[2,3)

(B) FP[1,2) == FP[2,3)

(C) FP[1,2) < FP[2,3)

(D) It depends

$1 = 1.0 \times 2^0 \xrightarrow{FP} 0 \,|\, 0111\ 1111 \,|\, 00\ldots0$

$2 = 1.0 \times 2^1 \rightarrow 0 \,|\, 1000\ 0000 \,|\, 00\ldots0$

$3 = 1.1 \times 2^1 \rightarrow 0 \,|\, 1000\ 0000 \,|\, 10\ldots0$

$FP[1,2) = 2^{23}$ (can toggle all bits of mantissa)

$FP[2,3) = 2^{22}$ (can toggle all but most significant bit of mantissa)

Floating Point Operations: Basic Idea

$$\text{Value} = (-1)^{S} \times \text{Mantissa} \times 2^{\text{Exponent}}$$

- $x +_f y = \text{Round}(x + y)$
- $x *_f y = \text{Round}(x * y)$

- Basic idea for floating point operations:
  - First, compute the exact result
  - Then *round* the result to make it fit into desired precision:
    - Possibly over/underflow if exponent outside of range
    - Possibly drop least-significant bits of mantissa to fit into M bit vector

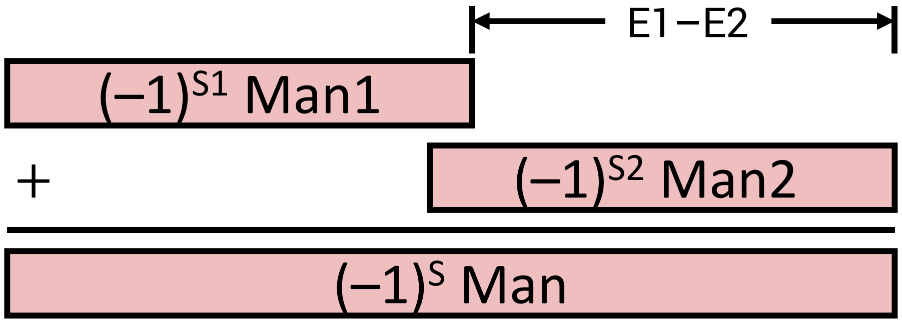
Floating Point Addition

Line up the binary points

- $(-1)^{S1} \times \text{Man1} \times 2^{\text{Exp1}} + (-1)^{S2} \times \text{Man2} \times 2^{\text{Exp2}}$
  - Assume E1 > E2

$$\begin{array}{r} 1.010*2^{2} \\ +\ 1.000*2^{-1} \\ \hline ??? \end{array} \rightarrow \begin{array}{r} 1.0100*2^{2} \\ +\ \mathbf{0.0001*2^{2}} \\ \hline 1.0101*2^{2} \end{array}$$

- Exact Result: $(-1)^{S} \times \text{Man} \times 2^{\text{Exp}}$
  - Sign S, mantissa Man:
    - Result of signed align & add
  - Exponent E: E1



E1 – E2

$(-1)^{S1}$ Man1

\+ $(-1)^{S2}$ Man2

$(-1)^{S}$ Man

- Adjustments:
  - If Man ≥ 2, shift Man right, increment E
  - if Man < 1, shift Man left k positions, decrement E by k
  - Over/underflow if E out of range
  - Round Man to fit mantissa precision

Floating Point Multiplication

- $(-1)^{S1} \times M1 \times 2^{E1} \times (-1)^{S2} \times M2 \times 2^{E2}$

- Exact Result: $(-1)^{S} \times M \times 2^{E}$
  - Sign S: s1 ^ s2
  - Mantissa Man: M1 × M2
  - Exponent E: E1 + E2

- Adjustments:
  - If Man ≥ 2, shift Man right, increment E
  - Over/underflow if E out of range
  - Round Man to fit mantissa precision

Summary

- Floating point approximates real numbers:

31 30 23 22 0

S E (8) M (23)

  - Handles large numbers, small numbers, special numbers
  - Exponent in biased notation (bias = $-(2^{w-1}-1)$)
    - Outside of representable exponents is *overflow* and *underflow*
  - Mantissa approximates fractional portion of binary point
    - Implicit leading 1 (normalized) except in special cases
    - Exceeding length causes *rounding*

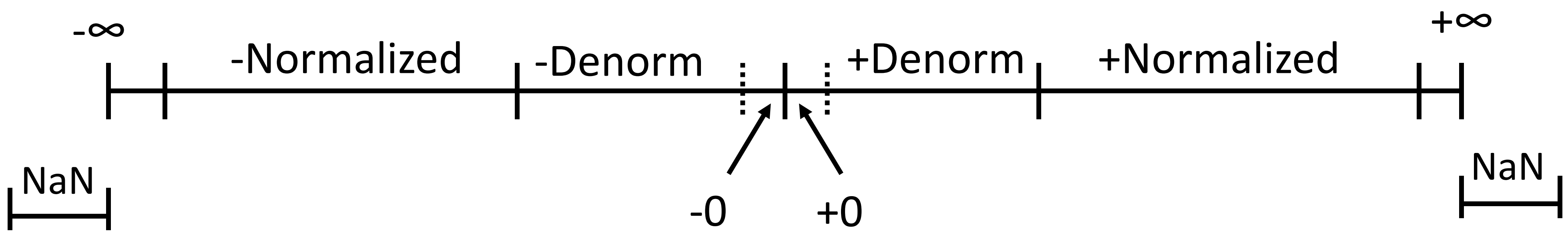
Exponent	Mantissa	Meaning
0x00	0	± 0
0x00	non-zero	± denorm num
0x01 – 0xFE	anything	± norm num
0xFF	0	± ∞
0xFF	non-zero	NaN

BONUS SLIDES

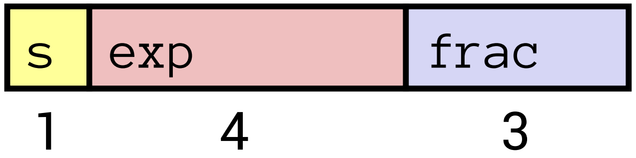
More details for the curious. These slides expand on material covered today, so while you don't need to read these, the information is "fair game."

- Tiny Floating Point Example
- Distribution of Values

Visualization: Floating Point Encodings



Tiny Floating Point Example



`s`	`exp`	`frac`
1	4	3

- 8-bit Floating Point Representation
  - the sign bit is in the most significant bit.
  - the next four bits are the exponent, with a bias of 7.
  - the last three bits are the `frac`

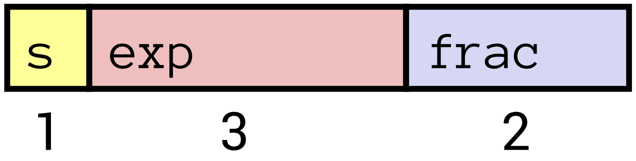
- Same general form as IEEE Format
  - normalized, denormalized
  - representation of 0, NaN, infinity

Dynamic Range (Positive Only)

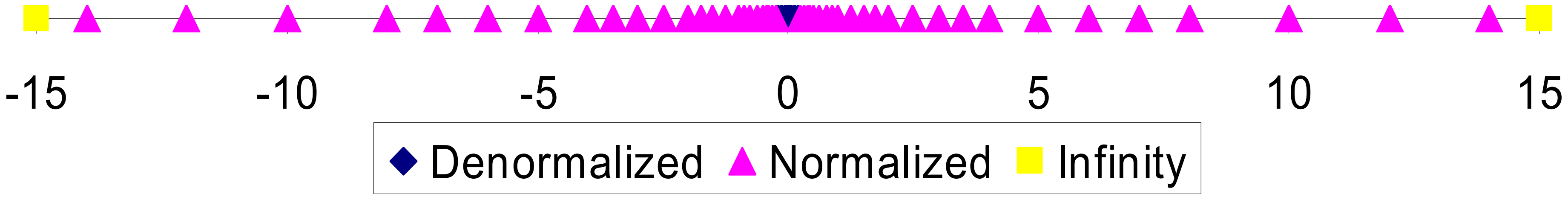
	s	exp	frac	E	Value	
Denormalized numbers	0	0000	000	-6	0	
	0	0000	001	-6	1/8*1/64 = 1/512	closest to zero
	0	0000	010	-6	2/8*1/64 = 2/512	
	…					
	0	0000	110	-6	6/8*1/64 = 6/512	
	0	0000	111	-6	7/8*1/64 = 7/512	largest denorm
Normalized numbers	0	0001	000	-6	8/8*1/64 = 8/512	smallest norm
	0	0001	001	-6	9/8*1/64 = 9/512	
	…					
	0	0110	110	-1	14/8*1/2 = 14/16	
	0	0110	111	-1	15/8*1/2 = 15/16	closest to 1 below
	0	0111	000	0	8/8*1 = 1	
	0	0111	001	0	9/8*1 = 9/8	closest to 1 above
	0	0111	010	0	10/8*1 = 10/8	
	…					
	0	1110	110	7	14/8*128 = 224	
	0	1110	111	7	15/8*128 = 240	largest norm
	0	1111	000	n/a	inf	

Distribution of Values

- 6-bit IEEE-like format
  - e = 3 exponent bits
  - f = 2 fraction bits
  - Bias is 23-1-1 = 3

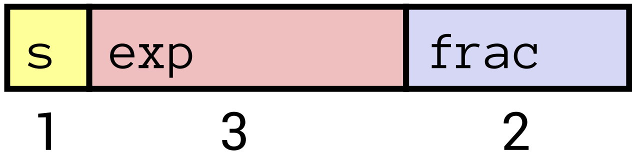


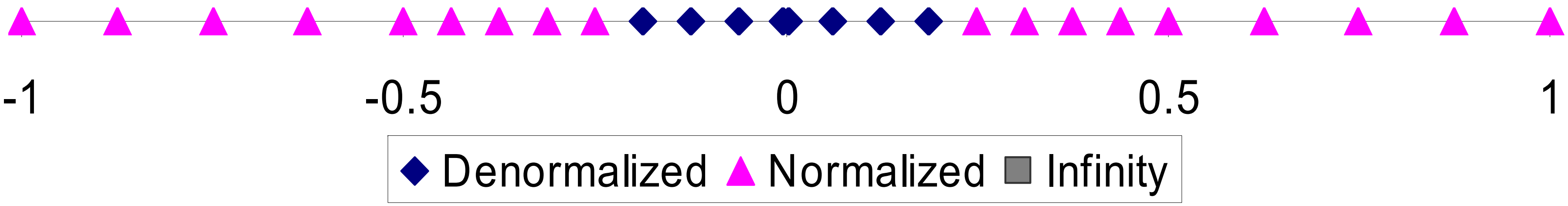
- Notice how the distribution gets denser toward zero.



Distribution of Values (close-up view)

- 6-bit IEEE-like format
  - e = 3 exponent bits
  - f = 2 fraction bits
  - Bias is 3





Interesting Numbers

`{single,double}`

Description	`exp`	`frac`	Numeric Value
Zero	00…00	00…00	0.0
Smallest Pos. Denorm. - Single ≈ $1.4 * 10^{-45}$ - Double ≈ $4.9 * 10^{-324}$	00…00	00…01	$2^{-\{23,52\}} * 2^{-\{126,1022\}}$
Largest Denormalized - Single ≈ $1.18 * 10^{-38}$ - Double ≈ $2.2 * 10^{-308}$	00…00	11…11	$(1.0 - \varepsilon) * 2^{-\{126,1022\}}$
Smallest Pos. Norm. - Just larger than largest denormalized	00…01	00…00	$1.0 * 2^{-\{126,1022\}}$
One	01…11	00…00	1.0
Largest Normalized - Single ≈ $3.4 * 10^{38}$ - Double ≈ $1.8 * 10^{308}$	11…10	11…11	$(2.0 - \varepsilon) * 2^{\{127,1023\}}$

Special Properties of Encoding

- Floating point zero (0^+) exactly the same bits as integer zero
  - All bits = 0

- Can (Almost) Use Unsigned Integer Comparison
  - Must first compare sign bits
  - Must consider $0^- = 0^+ = 0$
  - NaNs problematic
    - Will be greater than any other values
    - What should comparison yield?
  - Otherwise OK
    - Denorm vs. normalized
    - Normalized vs. infinity