Integers II

CSE 351 Autumn 2016

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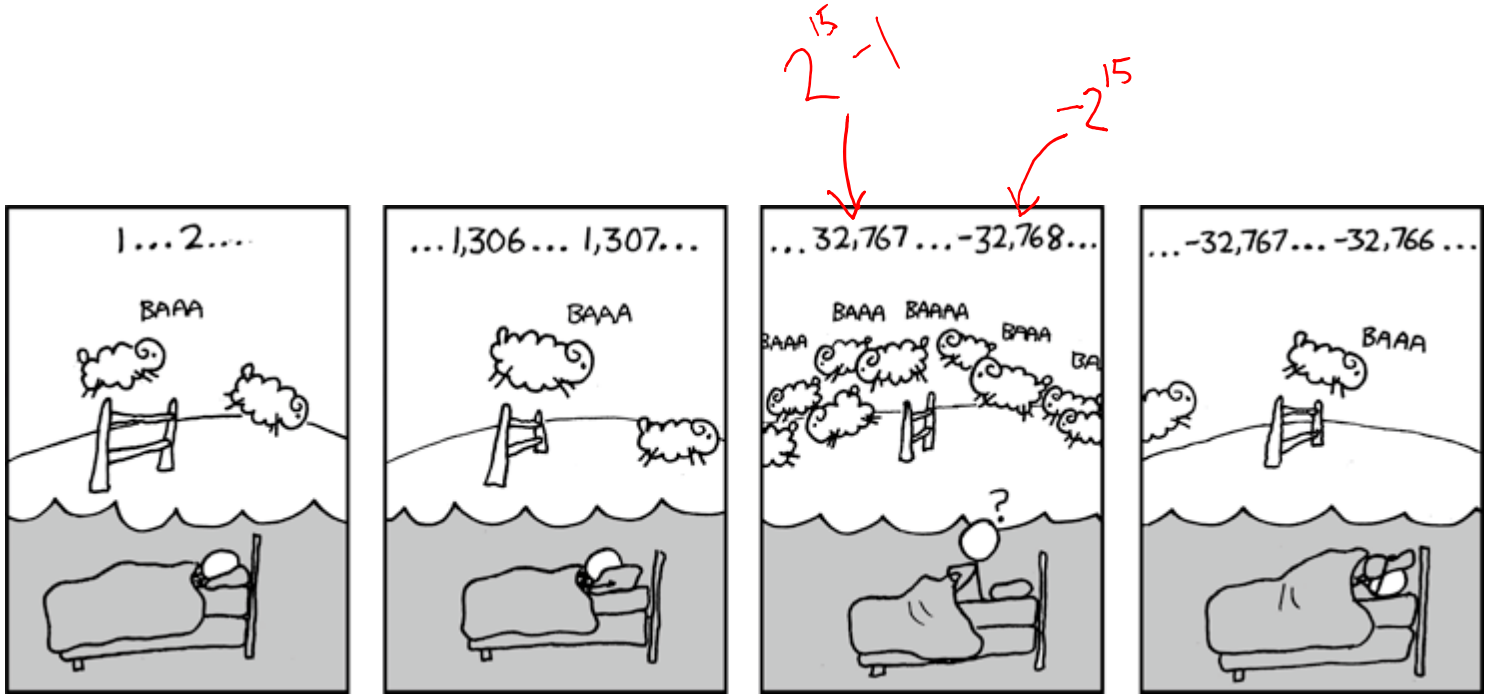


http://xkcd.com/571/

Administrivia

- Lab 1 due next Friday (10/14)
  - Partial due on Monday (10/10)
- Homework 1 will be released on Tuesday (10/11)
  - Timing overlaps a bit with Lab 1
- No Panopto for this course
  - Online lecture videos from 2013 at: https://courses.cs.washington.edu/courses/cse351/16au/videos.html

Integers

- Binary representation of integers
  - Unsigned and signed
  - Casting in C
- Consequences of finite width representations
  - Overflow, sign extension
- Shifting and arithmetic operations

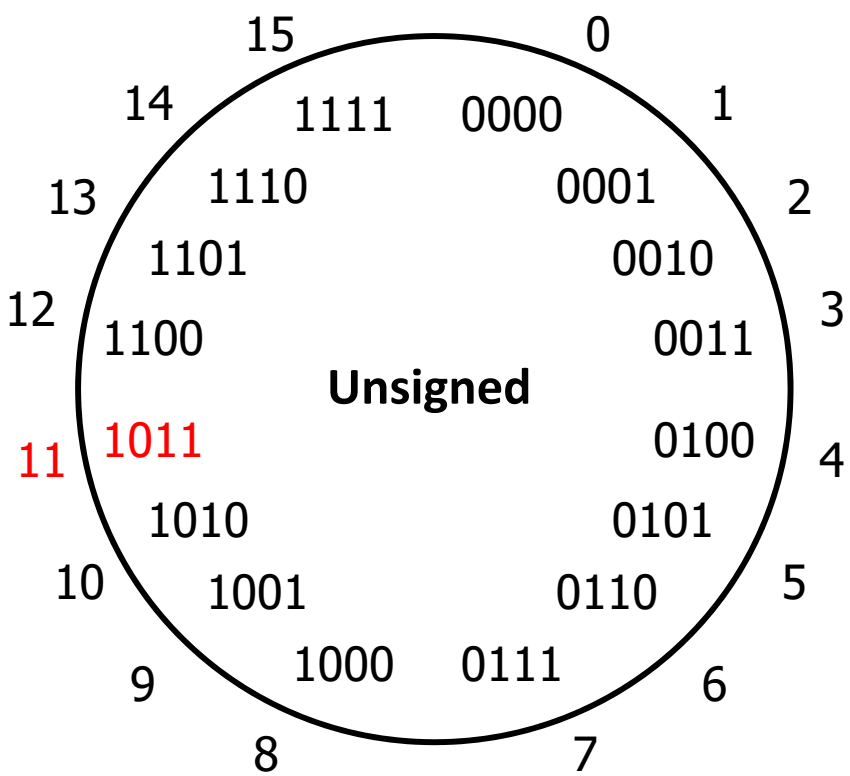
Unsigned vs. Two's Complement

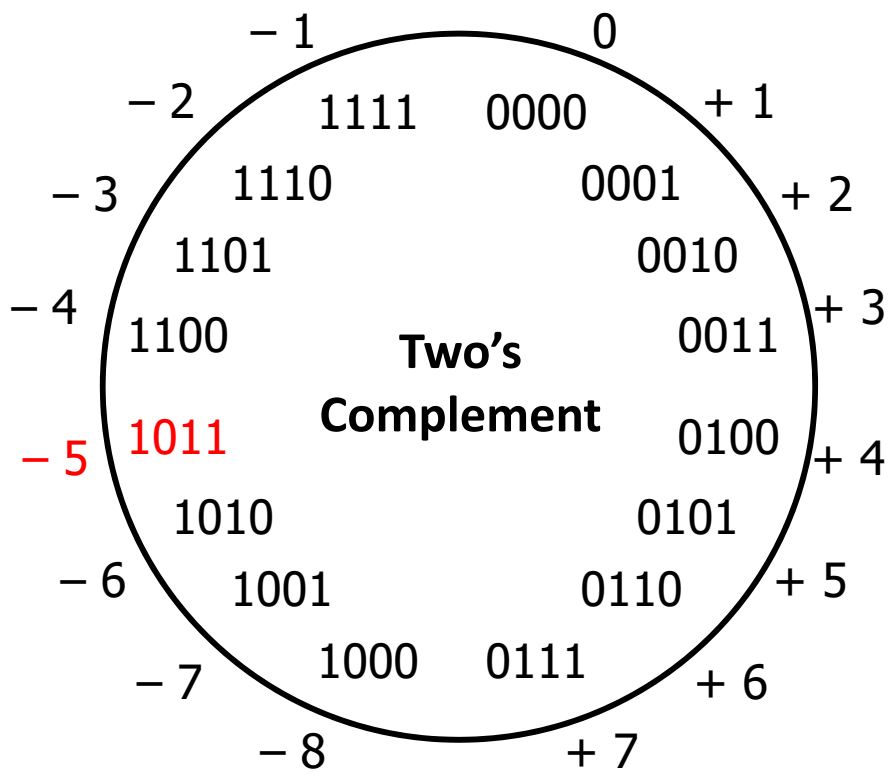
- 4-bit Example: **1011**

Unsigned: $1 \times 2^3 + 0 \times 2^2 + 1 \times 2^1 + 1 \times 2^0$

Two's Complement: $1 \times (-2^3) + 0 \times 2^2 + 1 \times 2^1 + 1 \times 2^0$

11 ← (math) difference = 16 = 2^4 → **-5**





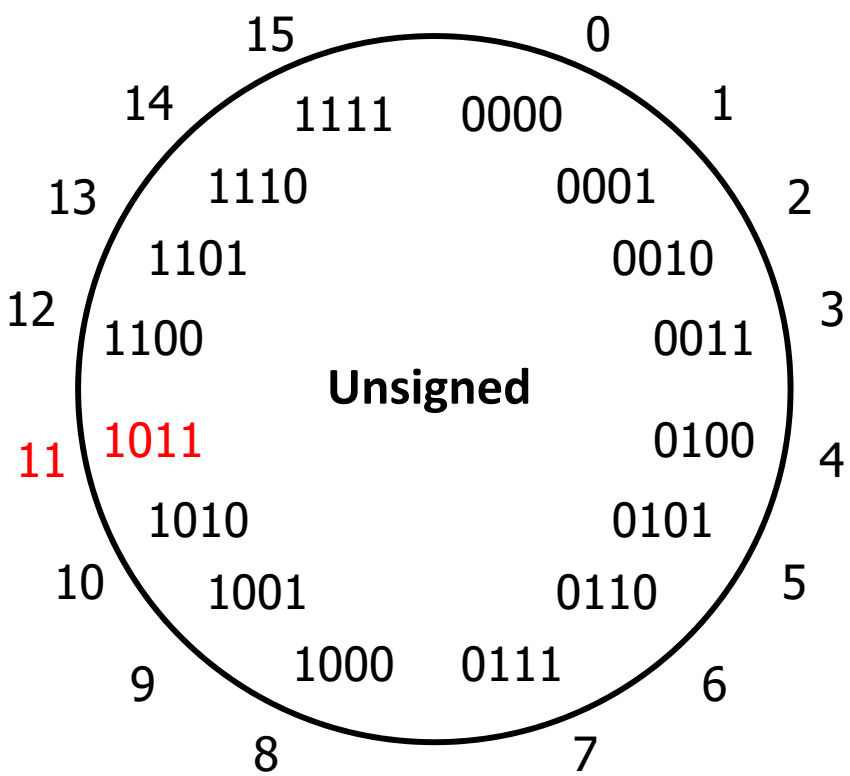
Unsigned vs. Two's Complement

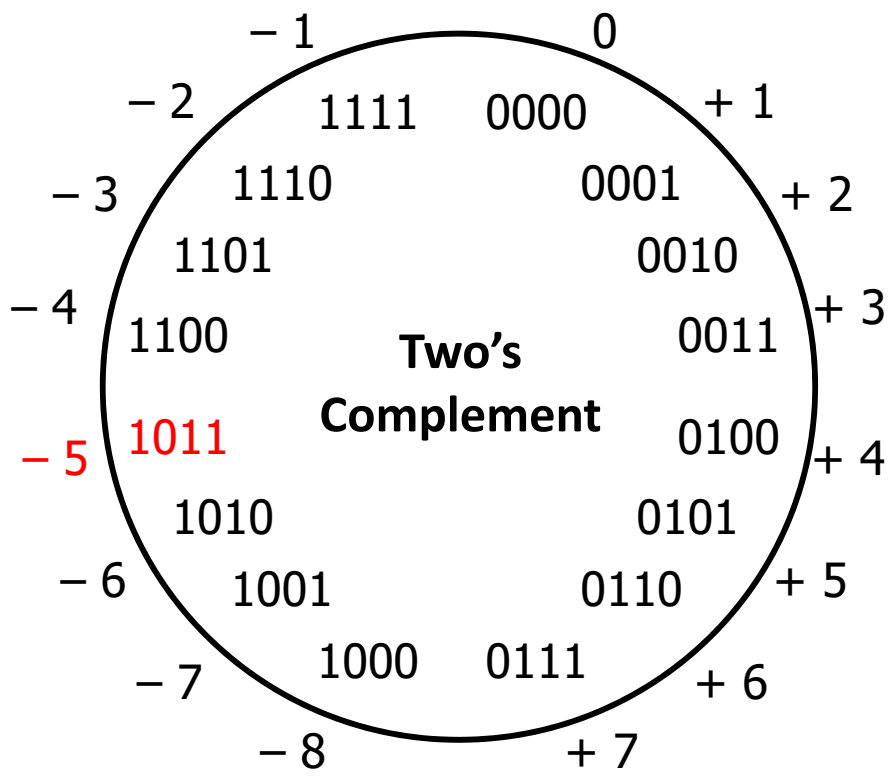
- 4-bit Example: **1011**

Unsigned: $1x2^3 + 0x2^2 + 1x2^1 + 1x2^0$ → **11**

Two's Complement: $1x(-2^3) + 0x2^2 + 1x2^1 + 1x2^0$ → **-5**

(math) difference = 16 = 2^4





Two's Complement Arithmetic

- The same addition procedure works for both unsigned and two's complement integers
  - **Simplifies hardware:** only one algorithm for addition
  - **Algorithm:** simple addition, discard the highest carry bit
    - Called modular addition: result is sum *modulo* 2^w

- **4-bit Examples:**

4	0100	-4	1100	4	0100
+3	+0011	+3	+0011	-3	+1101
=7	=0111	=-1	=1111	=1	10001
					=0001

Why Does Two's Complement Work?

- For all representable positive integers x, we want:

$$\begin{array}{r} \textit{bit representation of } x \\ +\ \textit{bit representation of } -x \\ \hline 0 \end{array}$$

(ignoring the carry-out bit)

  - What are the 8-bit negative encodings for the following?

```
  00000001        00000010        11000011
  11111111        11111110        00111101
+ ????????      + ????????      + ????????
----------      ----------      ----------
1 00000000        00000000        00000000
```

Why Does Two's Complement Work?

- For all representable positive integers x, we want:

additive inverse

$$\begin{array}{r} \textit{bit representation of } x \\ + \textit{ bit representation of } -x \\ \hline 0 \end{array}$$

(ignoring the carry-out bit)

$x + \sim x = 0xFF$ (all 1's)
$= -1$

$x + \sim x + 1 = 0$

$\boxed{\sim x + 1 = -x}$

  - What are the 8-bit negative encodings for the following?

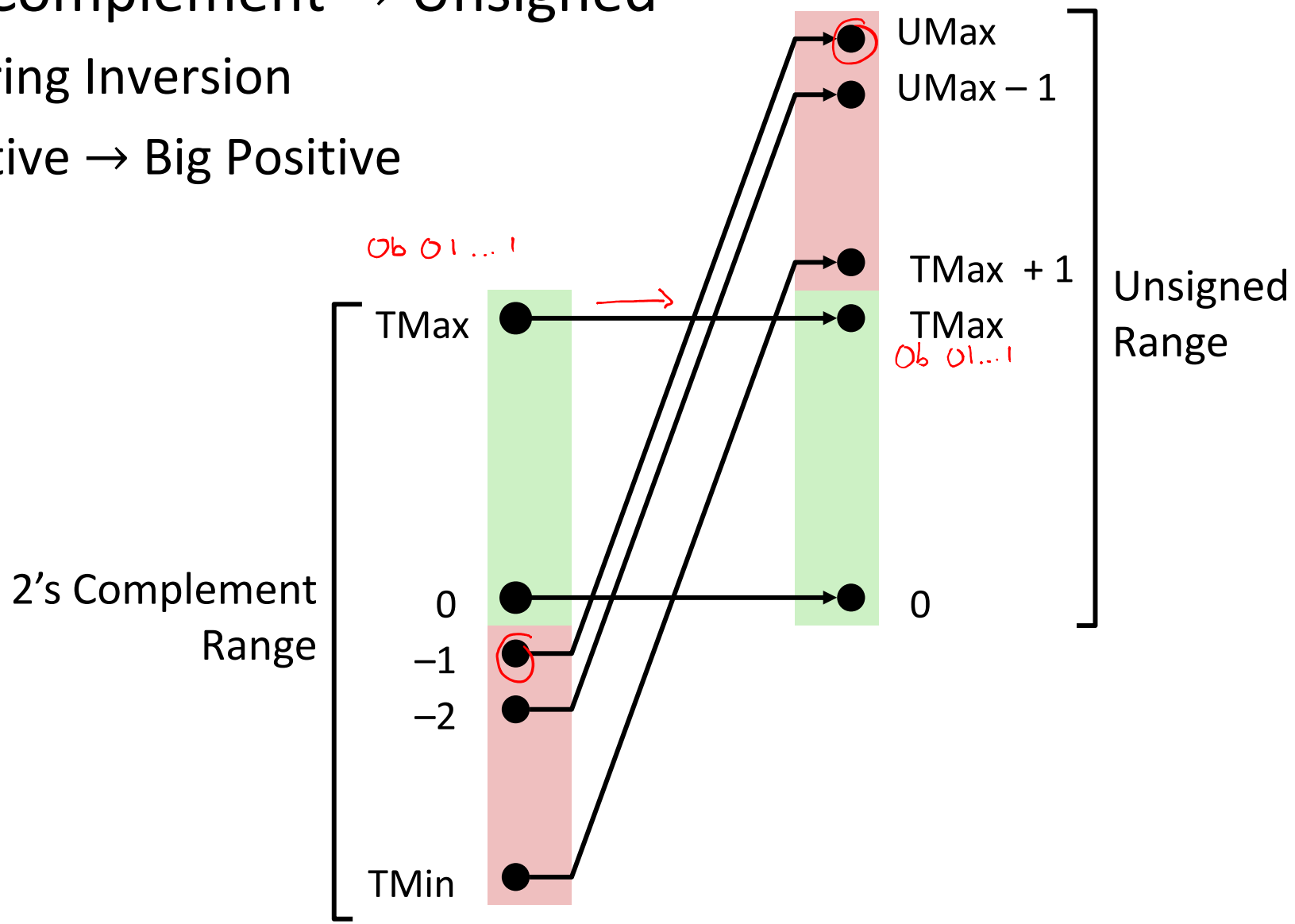
```
   00000001        00000010        11000011
+  11111111     +  11111110     +  00111101
 100000000       100000000       100000000
```

These are the bitwise complement plus 1!

`-x == ~x + 1`

Signed/Unsigned Conversion Visualized

- Two's Complement → Unsigned
  - Ordering Inversion
  - Negative → Big Positive



Values To Remember

- Unsigned Values
  - UMin = `0b00…0`
    = 0
  - UMax = `0b11…1`
    = $2^w - 1$
- Two's Complement Values
  - TMin = `0b10…0`
    = -2^{w-1}
  - TMax = `0b01…1`
    = $2^{w-1} - 1$
  - -1 = `0b11…1`
- **Example:** Values for $w = 32$

	Decimal	Hex	Binary
UMax	4,294,967,296	FF FF FF FF	11111111 11111111 11111111 11111111
TMax	2,147,483,647	7F FF FF FF	01111111 11111111 11111111 11111111
TMin	-2,147,483,648	80 00 00 00	10000000 00000000 00000000 00000000
-1	-1	FF FF FF FF	11111111 11111111 11111111 11111111
0	0	00 00 00 00	00000000 00000000 00000000 00000000

In C: Signed vs. Unsigned

- Casting
  - Bits are unchanged, just interpreted differently!
    - int tx, ty; int char
    - unsigned ux, uy; unsigned char
  - *Explicit* casting
    - tx = (int) ux;
    - uy = (unsigned) ty;
  - *Implicit* casting can occur during assignments or function calls
    - tx = ux;
    - uy = ty;
    - gcc flag -*Wsign-conversion* produces warnings for implicit casts, but -*Wall* does not

Casting Surprises

!!!

- Integer literals (constants)
  - By default, integer constants are considered *signed* integers
    - Hex constants already have an explicit binary representation
  - Use "U" (or "u") suffix to explicitly force *unsigned*
    - **Examples:** 0U, 4294967259u

- Expression Evaluation
  - When you mixed unsigned and signed in a single expression, then **signed values are implicitly cast to <u>unsigned</u>**
  - Including comparison operators <, >, ==, <=, >=

Casting Surprises

!!!

- 32-bit examples:
  - TMin = -2,147,483,648, TMax = 2,147,483,647

Left Constant	Op	Right Constant	Interpretation
0 0000 0000 0000 0000 0000 0000 0000 0000	==	**0U** 0000 0000 0000 0000 0000 0000 0000 0000	Unsigned
-1 1111 1111 1111 1111 1111 1111 1111 1111	<	**0** 0000 0000 0000 0000 0000 0000 0000 0000	Signed
-1 1111 1111 1111 1111 1111 1111 1111 1111	>	**0U** 0000 0000 0000 0000 0000 0000 0000 0000	Unsigned
2147483647 0111 1111 1111 1111 1111 1111 1111 1111	>	**-2147483648** 1000 0000 0000 0000 0000 0000 0000 0000	Signed
2147483647U 0111 1111 1111 1111 1111 1111 1111 1111	<	**-2147483648** 1000 0000 0000 0000 0000 0000 0000 0000	Unsigned
-1 1111 1111 1111 1111 1111 1111 1111 1111	>	**-2** 1111 1111 1111 1111 1111 1111 1111 1110	Signed
(unsigned) -1 1111 1111 1111 1111 1111 1111 1111 1111	>	**-2** 1111 1111 1111 1111 1111 1111 1111 1110	Unsigned
2147483647 0111 1111 1111 1111 1111 1111 1111 1111	<	**2147483648U** 1000 0000 0000 0000 0000 0000 0000 0000	Unsigned
2147483647 0111 1111 1111 1111 1111 1111 1111 1111	>	**(int) 2147483648U** 1000 0000 0000 0000 0000 0000 0000 0000	Signed

Integers

- Binary representation of integers
  - Unsigned and signed
  - Casting in C
- Consequences of finite width representations
  - Overflow, sign extension
- Shifting and arithmetic operations

Arithmetic Overflow

Bits	Unsigned	Signed
0000	0	0
0001	1	1
0010	2	2
0011	3	3
0100	4	4
0101	5	5
0110	6	6
0111	7	7
1000	8	-8
1001	9	-7
1010	10	-6
1011	11	-5
1100	12	-4
1101	13	-3
1110	14	-2
1111	15	-1

- When a calculation produces a result that can't be represented in the current encoding scheme
  - Integer range limited by fixed width
  - Can occur in both the positive and negative directions
- Computer handling of overflow
  - CPU *may be* capable of "throwing an exception" for overflow on <u>signed</u> values
  - CPU doesn't throw exception for <u>unsigned</u>
  - C and Java ignore overflow exceptions… oops!

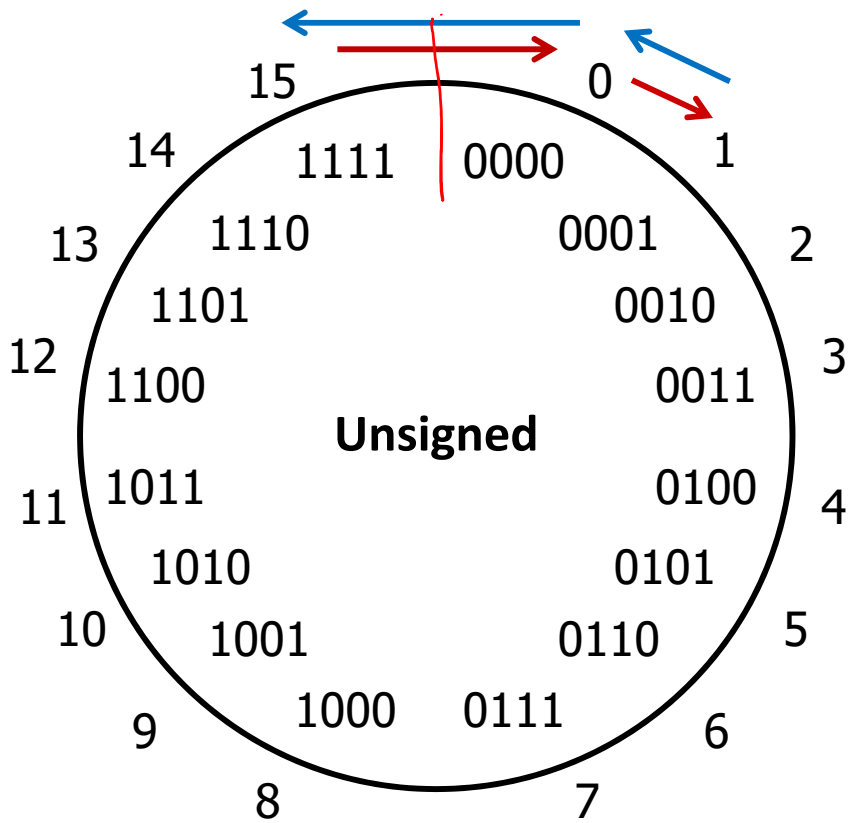
Overflow: Unsigned

- **Addition:** drop carry bit (-2^N)

```
  15        1111
+  2      + 0010
----      ------
  17        10001
  1
```

- **Subtraction:** borrow ($+2^N$)

```
   1        10001
 - 2      - 0010
----      ------
  -1        1111
  15
```



$\pm 2^N$ because of modular arithmetic

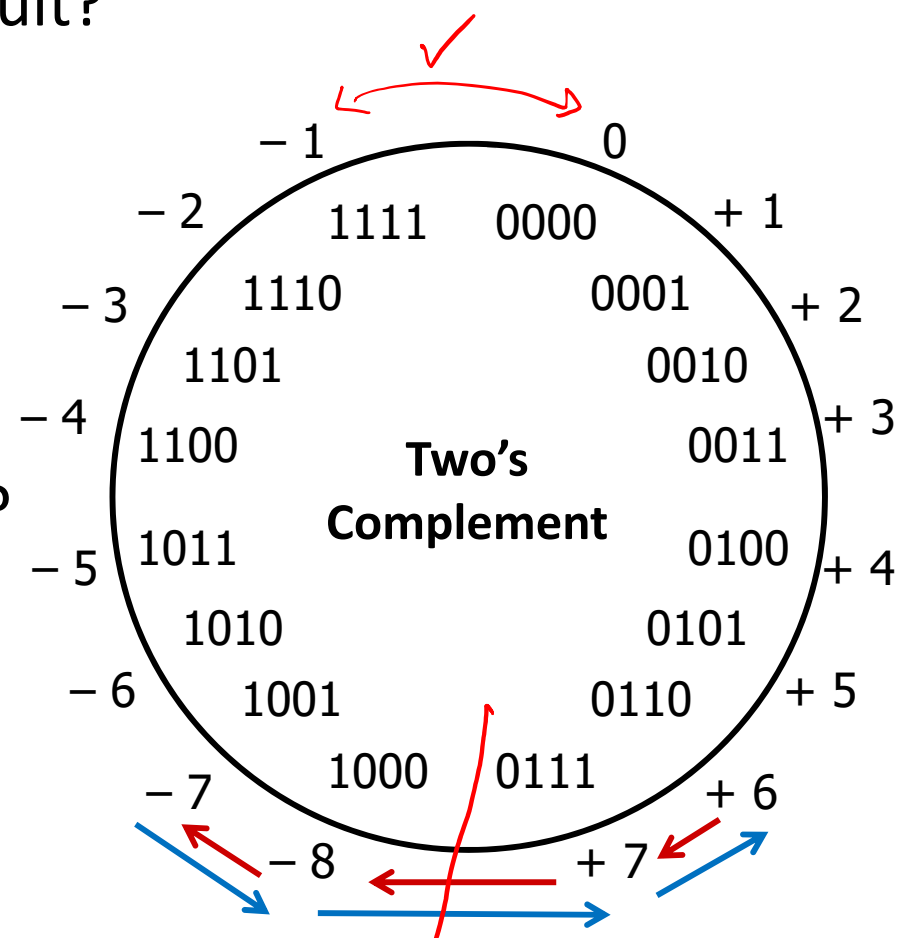
Overflow: Two's Complement

- **Addition:** (+) + (+) = (–) result?

```
   6        0110
+  3      + 0011
-------   -------
   9        1001
  -7
```

- **Subtraction:** (–) + (–) = (+)?

```
  -7        1001
-  3      - 0011
-------   -------
 -10        0110
   6
```



For signed: overflow if operands have same sign and result's sign is different

Sign Extension

- What happens if you convert a *signed* integral data type to a larger one?
  - e.g., `char` → `short` → `int` → `long`
- **4-bit → 8-bit Example:**
  - Positive Case
    - ✓ Add 0's?

4-bit:	0010	=	+2
8-bit:	00000010	=	+2

Sign Extension

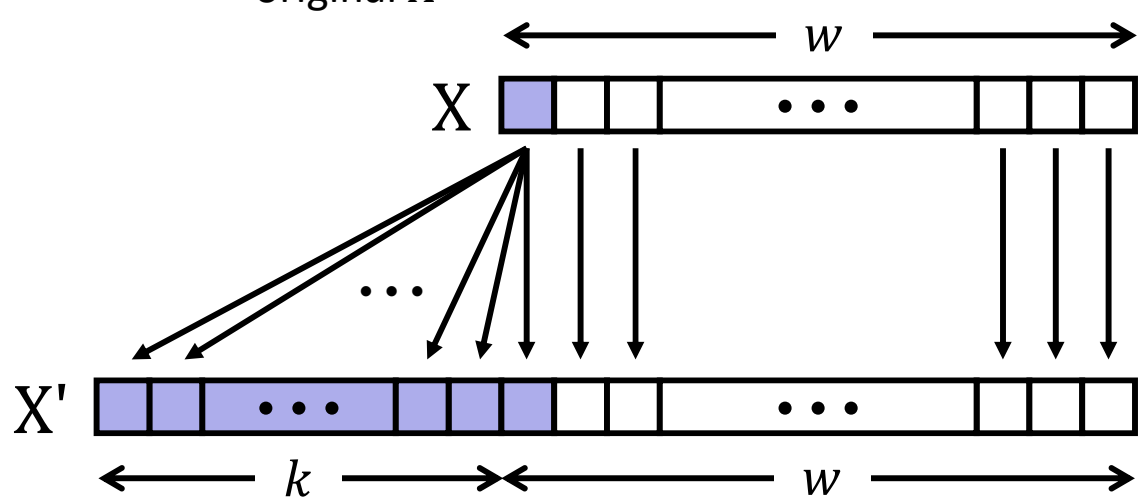
- What happens if you convert a *signed* integral data type to a larger one?
  - e.g., `char` → `short` → `int` → `long`
- **4-bit → 8-bit Example:**
  - Positive Case
    - ✓ Add 0's?
  - Negative Case
    - ✗ Add 0's?
    - ✗ Make MSB 1?
    - ✓ Add 1's?

4-bit:	0010	=	+2
8-bit:	00000010	=	+2
4-bit:	1100	=	-4
8-bit:	00001100	=	+12
	10001100	=	-116
	11111100	=	-4

00000011
+1

Sign Extension

- **Task:** Given a w-bit signed integer X, convert it to w+k-bit signed integer X′ *with the same value*
- **Rule:** Add k copies of sign bit
  - Let x_i be the i-th digit of X in binary
  - $X' = \underbrace{x_{w-1}, \ldots, x_{w-1}}_{k \text{ copies of MSB}}, \underbrace{x_{w-1}, x_{w-2}, \ldots, x_1, x_0}_{\text{original X}}$



Sign Extension Example

- Convert from smaller to larger integral data types
- C automatically performs sign extension
  - Java too

```
short int x =   12345;
int      ix = (int) x;
short int y =  -12345;
int      iy = (int) y;
```

Var	Decimal	Hex	Binary
x	12345	30 39	00110000 00111001
ix	12345	00 00 30 39	00000000 00000000 00110000 00111001
y	-12345	CF C7	11001111 11000111
iy	-12345	FF FF CF C7	11111111 11111111 11001111 11000111

Integers

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Shift Operations

- Left shift (x<<n) bit vector x by n positions
  - Throw away (drop) extra bits on left
  - Fill with 0s on right
- Right shift (x>>n) bit-vector x by n positions
  - Throw away (drop) extra bits on right
  - Logical shift (for unsigned values)
    - Fill with 0s on left
  - Arithmetic shift (for signed values)
    - Replicate most significant bit on left
    - Maintains sign of x

Shift Operations

- Left shift (x<<n)
  - Fill with 0s on right
- Right shift (x>>n)
  - Logical shift (for unsigned values)
    - Fill with 0s on left
  - Arithmetic shift (for signed values)
    - Replicate most significant bit on left
- Notes:
  - Shifts by n<0 or n≥w (bit width of x) are *undefined*
  - **In C:** behavior of >> is determined by compiler
    - In gcc / C lang, depends on data type of x (signed/unsigned)
  - **In Java:** logical shift is >>> and arithmetic shift is >>

	x	0010 0010
	x<<3	0001 0000
logical:	x>>2	0000 1000
arithmetic:	x>>2	0000 1000

	x	1010 0010
	x<<3	0001 0000
logical:	x>>2	0010 1000
arithmetic:	x>>2	1110 1000

Shifting Arithmetic?

- What are the following computing?
  - x>>n
    - 0b 0100 >> 1 = 0b 0010 ($2^2=4$, $2^1=2$)
    - 0b 0100 >> 2 = 0b 0001 ($2^0=1$)
    - Divide by 2^n
  - x<<n
    - 0b 0001 << 1 = 0b 0010 ($2^0=1$, $2^1=2$)
    - 0b 0001 << 2 = 0b 0100 ($2^2=4$)
    - Multiply by 2^n
- Shifting is faster than general multiply and divide operations

$b_i \times 2^i$

$\gg n \rightarrow b_i \times 2^{i-n}$

$\ll n \rightarrow b_i \times 2^{i+n}$

Left Shifting Arithmetic 8-bit Example

- No difference in left shift operation for unsigned and signed numbers (just manipulates bits)
  - Difference comes during interpretation: $x*2^n$?

			Signed	Unsigned
x = 25;	00011001	=	25	25
L1=x<<2;	0001100100	=	100	100
L2=x<<3;	00011001000	=	-56	200
L3=x<<4;	000110010000	=	-112	144

$25 \times 2^2 = 100$

$25 \times 2^3 = 200$

$25 \times 2^4 = 400$

signed overflow

unsigned overflow

Right Shifting Arithmetic 8-bit Examples

- **Reminder:** C operator >> does *logical* shift on unsigned values and *arithmetic* shift on signed values
  - Logical Shift: $x/2^n$?

Unsigned

```
xu = 240u;  11110000          = 240

R1u=xu>>3;  00011110000       =  30
```

$240/2^3 = 30$

```
R2u=xu>>5;  000001111 10000   =   7
```

$240/2^5 = 7.5$

rounding (down)

Right Shifting Arithmetic 8-bit Examples

- **Reminder:** C operator >> does *logical* shift on unsigned values and *arithmetic* shift on signed values
  - Arithmetic Shift: $x/2^n$?

```
xs = -16;      11110000          = -16
R1s=xu>>3;     11111110000       =  -2
R2s=xu>>5;     1111111110000     =  -1
```

$-16/2^3 = -2$

$-16/2^5 = -0.5$

rounding (down)

Peer Instruction Question

For the following expressions, find a value of `char x`, if there exists one, that makes the expression TRUE. Compare with your neighbor(s)!

- Assume we are using 8-bit arithmetic:

	Example	General
`x == (unsigned char) x` (unsigned)	x = 0	works for all x
`x >= 128U` (unsigned)	x = -1	any x < 0
`x != (x>>2)<<2`	x = 3	any x where lowest two bits are not 0b00
`x == -x` • Hint: there are two solutions	x = 0	x = 0b0...0 = 0 x = 0b01...1 = -128
`(x < 128U) && (x > 0x3F)`	x = 64	any x where upper two bits are exactly 0b01

Using Shifts and Masks

- Extract the 2nd most significant *byte* of an `int`:
  - First shift, then mask: `(x>>16) & 0xFF`

Could also mask, then shift in this case → what would new mask be?

x	00000001	00000010	00000011	00000100
x>>16	~~00000000~~	~~00000000~~	~~00000001~~	00000010
0xFF	00000000	00000000	00000000	11111111
(x>>16) & 0xFF	00000000	00000000	00000000	00000010

Using Shifts and Masks

- Extract the *sign bit* of a signed `int`:
  - First shift, then mask: `(x>>31) & 0x1`
    - Assuming arithmetic shift here, but works in either case
    - Need mask to clear 1s possibly shifted in

x	00000001 00000010 00000011 00000100
x>>31	00000000 00000000 00000000 00000000
0x1	00000000 00000000 00000000 00000001
(x>>31) & 0x1	00000000 00000000 00000000 00000000

x	10000001 00000010 00000011 00000100
x>>31	11111111 11111111 11111111 11111111
0x1	00000000 00000000 00000000 00000001
(x>>31) & 0x1	00000000 00000000 00000000 00000001

Using Shifts and Masks

- Conditionals as Boolean expressions
  - For `int x`, what does `(x<<31)>>31` do?

x=!!123	00000000 00000000 00000000 00000001
x<<31	10000000 00000000 00000000 00000000
(x<<31)>>31	11111111 11111111 11111111 11111111
!x	00000000 00000000 00000000 00000000
!x<<31	00000000 00000000 00000000 00000000
(!x<<31)>>31	00000000 00000000 00000000 00000000

  - Can use in place of conditional:
    - In C: `if(x) {a=y;} else {a=z;}` equivalent to `a=x?y:z;`
    - `a=(((x<<31)>>31)&y) | (((!x<<31)>>31)&z);`

Summary

- Sign and unsigned variables in C
  - Bit pattern remains the same, just *interpret* differently
  - Strange things can happen with our arithmetic when we convert/cast between sign and unsigned numbers
  - Type of variables affects behavior of operators (shifting, comparison)
- We can only represent so many numbers in w bits
  - When we exceed the limits, *arithmetic overflow* occurs
  - *Sign extension* tries to preserve value when expanding
- Shifting is a useful bitwise operator
  - Can be used in multiplication with constant or bit masking
  - Right shifting can be arithmetic (sign) or logical (0)