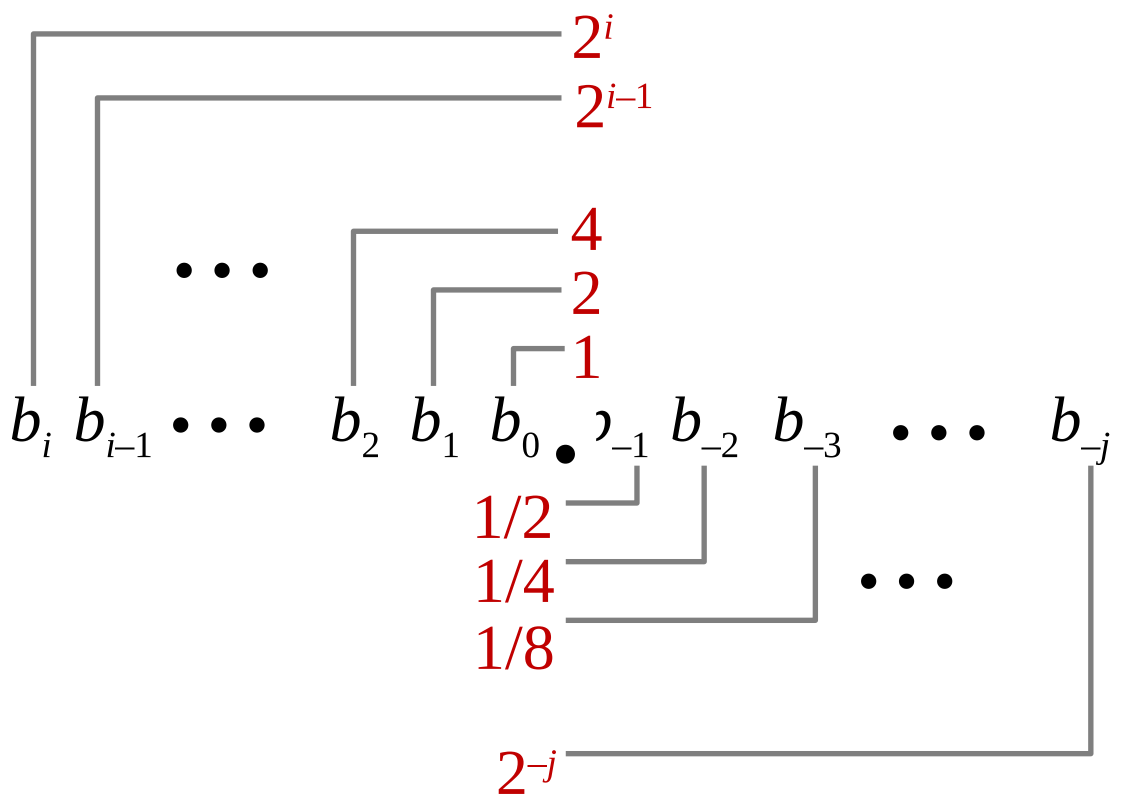
Fractional binary numbers

- What is 1011.101?

Fractional Binary Numbers



- **Representation**
  - **Bits to right of "binary point" represent fractional powers of 2**
  - **Represents rational number:** $\sum_{k=-j}^{i} b_k \times 2^k$

Fractional Binary Numbers: Examples

- **Value** **Representation**

Value	Representation
5 and 3/4	101.11_2
2 and 7/8	10.111_2
63/64	0.111111_2

- **Observations**
  - Divide by 2 by shifting right
  - Multiply by 2 by shifting left
  - Numbers of form $0.111111\ldots_2$ are just below 1.0
    - $1/2 + 1/4 + 1/8 + \ldots + 1/2^i + \ldots \rightarrow 1.0$
    - Use notation $1.0 - \varepsilon$

Representable Numbers

- **Limitation**
  - **Can only exactly represent numbers of the form $x/2^k$**
  - **Other rational numbers have repeating bit representations**

- **Value** / **Representation**

Value	Representation
1/3	$0.0101010101[01]\ldots_2$
1/5	$0.001100110011[0011]\ldots_2$
1/10	$0.0001100110011[0011]\ldots_2$

Fixed Point Representation

- **float → 32 bits; double → 64 bits**
- **We might try representing fractional binary numbers by picking a fixed place for an implied binary point**
  - **"fixed point binary numbers"**
- **Let's do that, using 8 bit floating point numbers as an example**
  - **#1: the binary point is between bits 2 and 3**

    $b_7\ b_6\ b_5 b_4\ b_3\ [.]\ b_2\ b_1\ b_0$
  - **#2: the binary point is between bits 4 and 5**

    $b_7\ b_6\ b_5\ [.]\ b_4\ b_3\ b_2\ b_1\ b_0$
  - **The position of the binary point affects the <u>range</u> and <u>precision</u>**

    range: difference between the largest and smallest representable numbers

Fixed Point Pros and Cons

- **Pros**
  - **It's simple. The same hardware that does integer arithmetic can do fixed point arithmetic**
    - **In fact, the programmer can use ints with an implicit fixed point**
      - **E.g., int balance; // number of pennies in the account**
    - **ints are just fixed point numbers with the binary point to the right of b_0**
- **Cons**
  - **There is no good way to pick where the fixed point should be**
    - **Sometimes you need range, sometimes you need precision. The more you have of one, the less of the other**

IEEE Floating Point

- **Fixing fixed point: analogous to scientific notation**
  - Not 12000000 but 1.2 x 10^7; not 0.0000012 but 1.2 x 10^-6
- **IEEE Standard 754**
  - Established in 1985 as uniform standard for floating point arithmetic
    - Before that, many idiosyncratic formats
  - Supported by all major CPUs
- **Driven by numerical concerns**
  - Nice standards for rounding, overflow, underflow
  - Hard to make fast in hardware
    - Numerical analysts predominated over hardware designers in defining standard

Floating Point Representation

- **Numerical Form:**

$$(-1)^s \; M \;\; 2^E$$

- **Sign bit s determines whether number is negative or positive**
- **Significand (mantissa) M normally a fractional value in range [1.0,2.0).**
- **Exponent E weights value by power of two**

- **Encoding**
- **MSB** `s` **is sign bit s**
- `frac` **field encodes M (but is not equal to M)**
- `exp` **field encodes E (but is not equal to E)**

s	exp	frac

Precisions

- **Single precision: 32 bits**

s	exp	frac
1	8	23

- **Double precision: 64 bits**

s	exp	frac
1	11	52

- **Extended precision: 80 bits (Intel only)**

s	exp	frac
1	15	63 or 64

Normalization and Special Values

- **"Normalized" means mantissa has form 1.xxxxx**
- 0.011 x 2^5 and 1.1 x 2^3 represent the same number, but the latter makes better use of the available bits
- Since we know the mantissa starts with a 1, don't bother to store it

- **How do we do 0? How about 1.0/0.0?**

Normalization and Special Values

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- **Special values:**

•The float value 00...0 represents zero

•If the exp == 11...1 and the mantissa == 00...0, it represents ∞

•E.g., 10.0 / 0.0 → ∞

•If the exp == 11...1 and the mantissa != 00...0, it represents NaN

•"Not a Number"

•Results from operations with undefined result

E.g., 0 * ∞

How do we do operations?

- Is representation exact?
- How are the operations carried out?

Floating Point Operations: Basic Idea

- $x +_f y = \text{Round}(x + y)$

- $x *_f y = \text{Round}(x * y)$

- **Basic idea**
  - **First compute exact result**
  - **Make it fit into desired precision**
    - **Possibly overflow if exponent too large**
    - **Possibly round to fit into `frac`**

Floating Point Multiplication

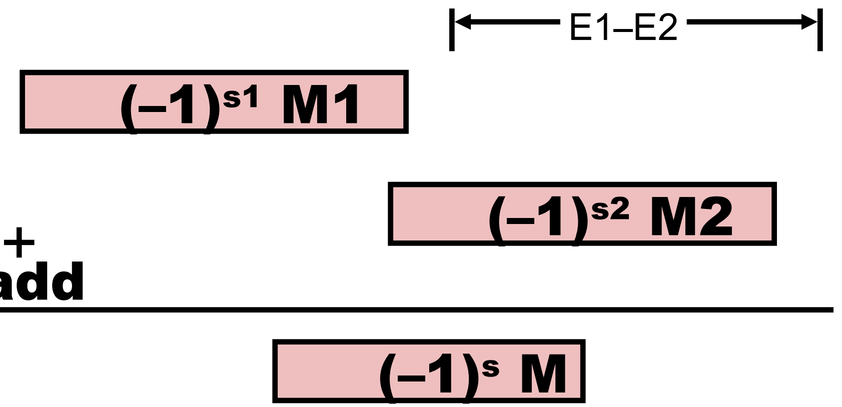
$(-1)^{s1}\ M1\ \ 2^{E1}\ \ *\ \ (-1)^{s2}\ M2\ \ 2^{E2}$

- **Exact Result:** $(-1)^{s}\ M\ \ 2^{E}$
  - **Sign s:** s1 ^ s2
  - **Significand M:** M1 * M2
  - **Exponent E:** E1 + E2
- **Fixing**
  - If M ≥ 2, shift M right, increment E
  - If E out of range, overflow
  - Round M to fit `frac` precision
- **Implementation**
  - What is hardest?

Floating Point Addition

$(-1)^{s1}\ M1\ \ 2^{E1} \quad + \quad (-1)^{s2}\ M2\ \ 2^{E2}$

Assume E1 > E2



- **Exact Result:** $(-1)^s\ M\ \ 2^E$
  - **Sign s, significand M:**
    - **Result of signed align & add**
  - **Exponent E: E1**

- **Fixing**
  - **If M ≥ 2, shift M right, increment E**
  - **if M < 1, shift M left k positions, decrement E by k**
  - **Overflow if E out of range**
  - **Round M to fit `frac` precision**

Hmm… if we round at every operation…

Mathematical Properties of FP Operations

- **Not really** associative or distributive due to rounding
- **Infinities and NaNs cause issues**
- **Overflow and infinity**

Floating Point in C

- **C Guarantees Two Levels**

  `float` **single precision**

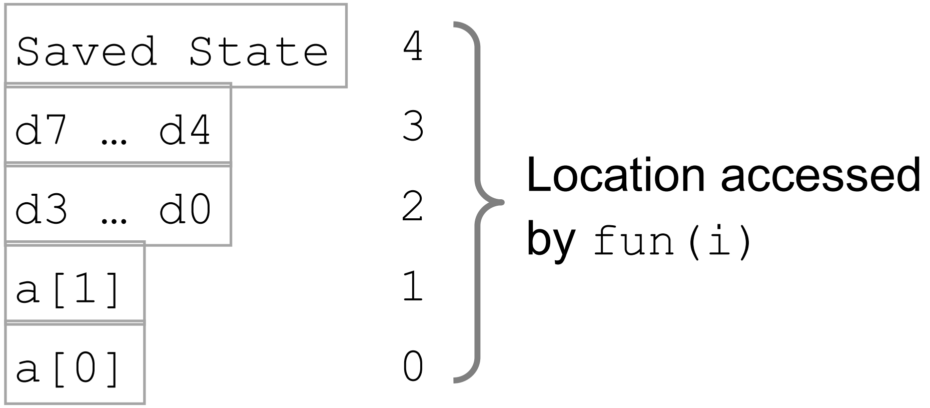
  `double` **double precision**

- **Conversions/Casting**
  - **Casting between `int`, `float`, and `double` changes bit representation**
  - `Double/float` → `int`
    - **Truncates fractional part**
    - **Like rounding toward zero**
    - **Not defined when out of range or NaN: Generally sets to TMin**
  - `int` → `double`
    - **Exact conversion, why?**
  - `int` → `float`

Memory Referencing Bug

```
double fun(int i)
{
  volatile double d[1] = {3.14};
  volatile long int a[2];
  a[i] = 1073741824; /* Possibly out of bounds */
  return d[0];
}
```

```
fun(0)  ->    3.14
fun(1)  ->    3.14
fun(2)  ->    3.1399998664856
fun(3)  ->    2.00000061035156
fun(4)  ->    3.14, then segmentation fault
```



Floating Point and the Programmer

```
#include <stdio.h>

int main(int argc, char* argv[]) {

  float f1 = 1.0;
  float f2 = 0.0;
  int i;
  for ( i=0; i<10; i++ ) {
    f2 += 1.0/10.0;
  }

  printf("0x%08x  0x%08x\n", *(int*)&f1, *(int*)&f2);
  printf("f1 = %10.8f\n", f1);
  printf("f2 = %10.8f\n\n", f2);

  f1 = 1E30;
  f2 = 1E-30;
  float f3 = f1 + f2;
  printf ("f1 == f3? %s\n", f1 == f3 ? "yes" : "no" );

  return 0;
}
```

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  int i;
  for ( i=0; i<10; i++ ) {
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  printf("0x%08x  0x%08x\n", *(int*)&f1, *(int*)&f2);
  printf("f1 = %10.8f\n", f1);
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  f1 = 1E30;
  f2 = 1E-30;
  float f3 = f1 + f2;
  printf ("f1 == f3? %s\n", f1 == f3 ? "yes" : "no" );

  return 0;
}
```

```
$ ./a.out
0x3f800000  0x3f800001
f1 = 1.000000000
f2 = 1.000000119

f1 == f3? yes
```

Summary

- As with integers, floats suffer from the fixed number of bits available to represent them
- Can get overflow/underflow, just like ints
- Some "simple fractions" have no exact representation
  - E.g., 0.1
- Can also lose precision, unlike ints
  - "Every operation gets a slightly wrong result"
- Mathematically equivalent ways of writing an expression may compute differing results
- NEVER test floating point values for equality!