Encoding Integers

- The hardware (and C) supports two flavors of integers:
  - unsigned – only the non-negatives
  - signed – both negatives and non-negatives

- There are only 2^W distinct bit patterns of W bits, so...
  - Can't represent all the integers
  - Unsigned values are 0 ... 2^W-1
  - Signed values are -2^{W-1} ... 2^{W-1}-1

Unsigned Integers

- Unsigned values are just what you expect
  - $b_7b_6b_5b_4b_3b_2b_1b_0 = b_7 2^7 + b_6 2^6 + b_5 2^5 + \ldots + b_1 2^1 + b_0 2^0$

    Interesting aside: $1+2+4+8+\ldots+2^{N-1} = 2^N - 1$

```
  00111111
 +00000001
  01000000
```

```
  63
 + 1
  64
```

- You add/subtract them using the normal "carry/borrow" rules, just in binary

- unsigned integers in C **are not the same thing as** pointers
  - **Similar: There are no negative memory addresses**
  - **Similar: Years ago sizeof(int) = sizeof(int *)**
  - **Not Similar: Today and in well written code for all time, sizeof(int) != sizeof(int *)**

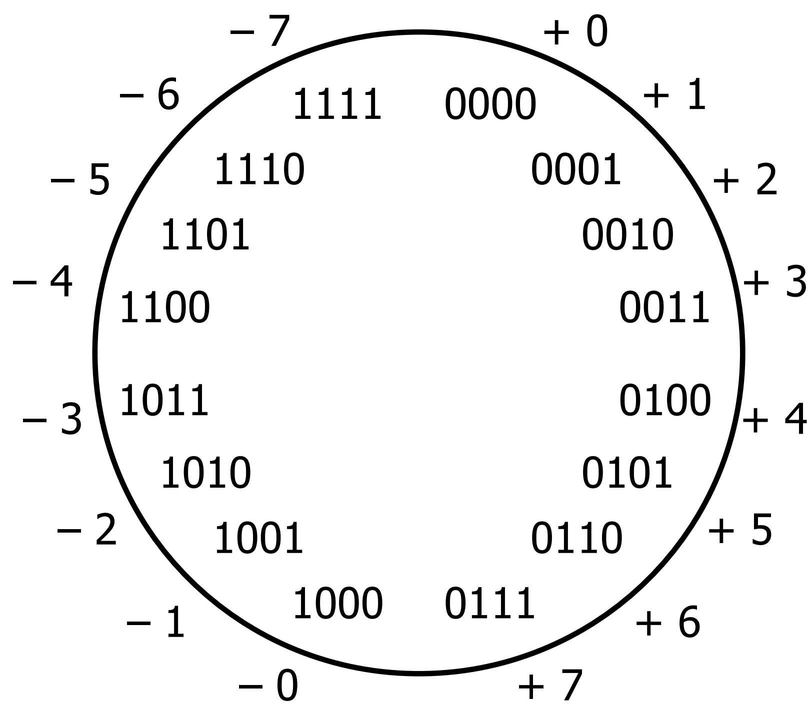
Signed Integers

- Let's do the natural thing for the positives
  - They correspond to the unsigned integers of the same value

    Example (8 bits): 0x00 = 0, 0x01 = 1, …, 0x7F = 127
- But, we need to let about half of them be negative
  - Use the high order bit to indicate something like 'negative’
  - Historically, there have been 3 flavors in use… but today there is only 1 (and for good reason).
    - Bad ideas (but were commonly used in the past!)
      - sign/magnitude
      - one’s complement
    - Good idea:
      - Two’s complement

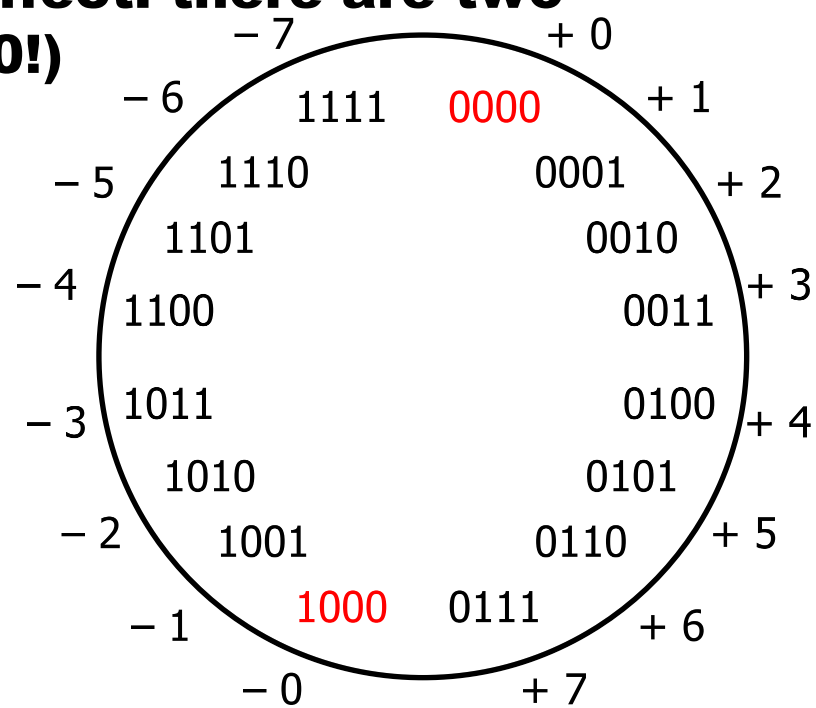
Sign-and-Magnitude Negatives

- **How should we represent -1 in binary?**
  - **<u>Possibility 1:</u> 10000001_2**
    Use the MSB for "+ or -", and the other bits to give magnitude



Sign-and-Magnitude Negatives

- **How should we represent -1 in binary?**
  - **<u>Possibility 1:</u> 10000001_2**
    Use the MSB for "+ or -", and the other bits to give magnitude
    (Unfortunate side effect: there are two representations of 0!)



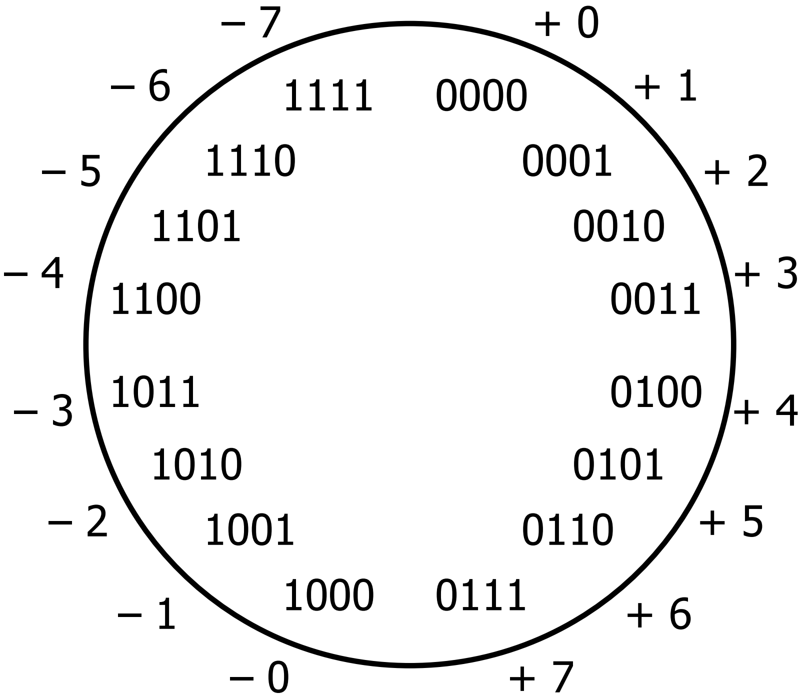
Sign-and-Magnitude Negatives

- **How should we represent -1 in binary?**
  - **<u>Possibility 1:</u> 10000001_2**

    Use the MSB for "+ or -", and the other bits to give magnitude

    Another problem: math is cumbersome

4 – 3 != 4 + (-3)

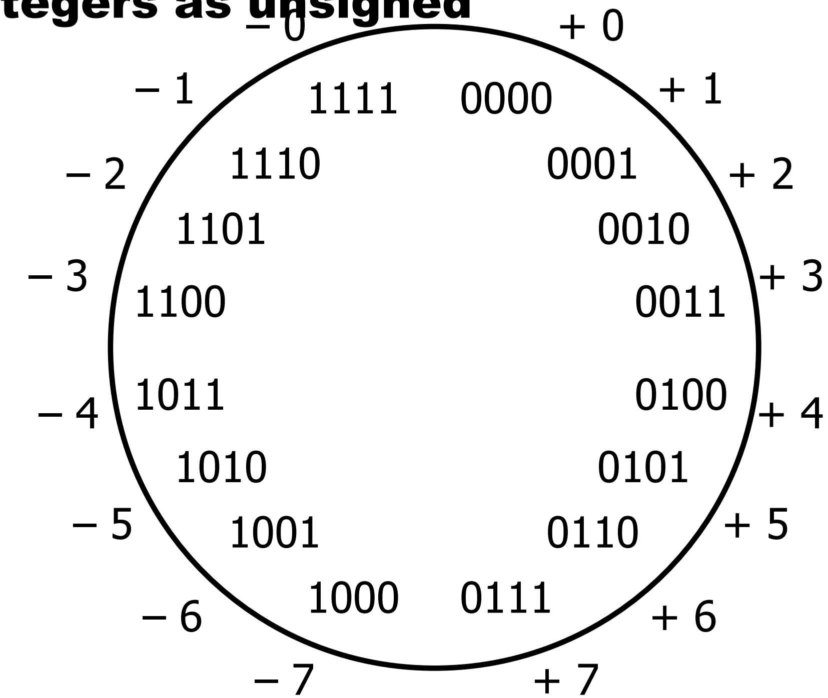


Ones' Complement Negatives

- **How should we represent -1 in binary?**
  - **<u>Possibility 2:</u> 11111110_2**

    Negative numbers: bitwise complements of positive numbers

    It would be handy if we could use the same hardware adder to add signed integers as unsigned



Ones' Complement Negatives

- **How should we represent -1 in binary?**
  - **<u>Possibility 2:</u> 11111110_2**
    Negative numbers: bitwise complements of positive numbers

Solves the arithmetic problem

Add		Invert, add, add carry		Invert and add	
4	0100	4	0100	– 4	1011
+ 3	+ 0011	– 3	+ 1100	+ 3	+ 0011
= 7	= 0111	= 1	1 0000	– 1	1110
		add carry:	+1		
			= 0001		

end-around carry

Ones' Complement Negatives

- How should we represent -1 in binary?
  - <u>Possibility 2:</u> 11111110_2
    Negative numbers: bitwise complements of positive numbers
    Use the same hardware adder to add signed integers as unsigned (but we have to keep track of the end-around carry bit)

  Why does it work?

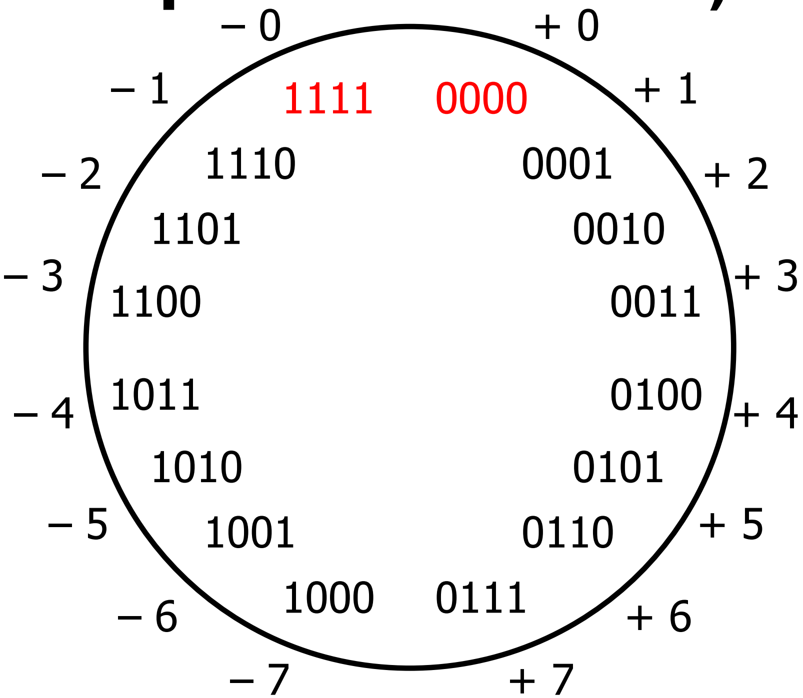
  - The ones' complement of a 4-bit positive number y
    is $1111_2 - y$
    - $0111 \equiv 7_{10}$
    - $1111_2 - 0111_2 = 1000_2 \equiv -7_{10}$
  - 1111_2 is 1 less than $10000_2 = 2^4 - 1$

Ones' Complement Negatives

- **How should we represent -1 in binary?**
  - **Possibility 2:** **11111110_2**

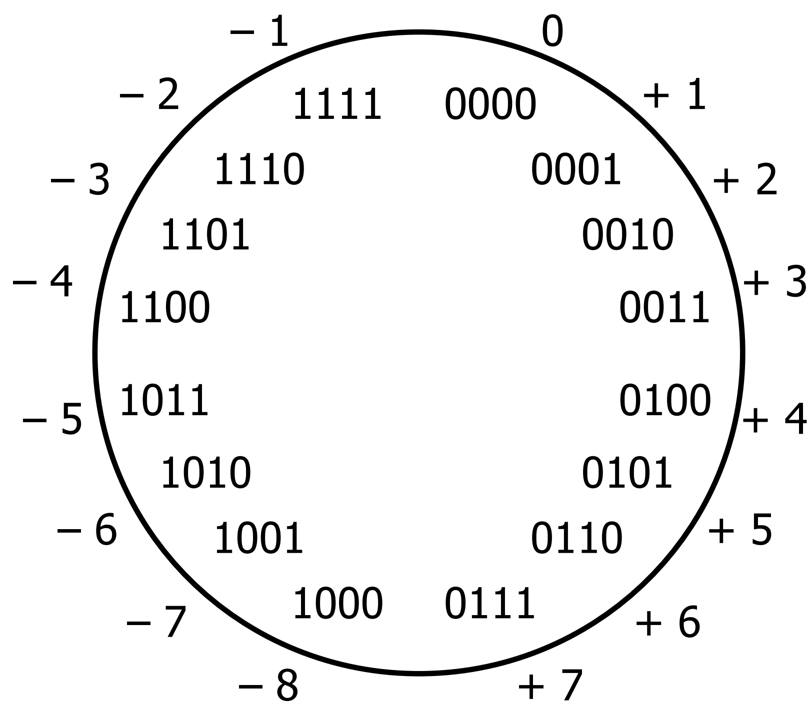
    Negative numbers: bitwise complements of positive numbers

    (But there are still two representations of 0!)



Two's Complement Negatives

- **How should we represent -1 in binary?**
  - **<u>Possibility 3:</u> 11111111_2**
    Bitwise complement plus one
    (Only one zero)



Two's Complement Negatives

- **How should we represent -1 in binary?**
  - **<u>Possibility 3:</u> 11111111_2**
    Bitwise complement plus one
    (Only one zero)
  - **Simplifies arithmetic**
    Use the same hardware adder to add signed integers as unsigned (simple addition; discard

Add		Invert and add		Invert and add	
4	0100	4	0100	– 4	1100
+ 3	+ 0011	– 3	+ 1101	+ 3	+ 0011
= 7	= 0111	= 1	1 0001	– 1	1111
		drop carry	= 0001		

Two's Complement Negatives

- How should we represent -1 in binary?
  - Two's complement: Bitwise complement plus one

Why does it work?

- Recall: The ones' complement of a b-bit positive number y
  is $(2^b – 1) – y$
- Two's complement adds one to the bitwise complement,
  thus, -y is $2^b – y$ (or -x == (~x + 1))
  - $–y$ and $2^b – y$ are equal mod 2^b
    (have the same remainder when divided by 2^b)
  - Ignoring carries is equivalent to doing arithmetic mod 2^b

Two's Complement Negatives

- **How should we represent -1 in binary?**
  - **Two's complement: Bitwise complement plus one**

What should the 8-bit representation of -1 be?

```
  00000001
+????????      (want whichever bit string gives right result)
  00000000

  00000010      00000011
+????????     +????????
  00000000      00000000
```

Unsigned & Signed Numeric Values

X	Unsigned	Signed
0000	0	0
0001	1	1
0010	2	2
0011	3	3
0100	4	4
0101	5	5
0110	6	6
0111	7	7
1000	8	–8
1001	9	–7
1010	10	–6
1011	11	–5
1100	12	–4
1101	13	–3
1110	14	–2
1111	15	–1

- Both signed and unsigned integers have limits
- If you compute a number that is too big, you wrap: 6 + 4 = ? 15U + 2U = ?
- If you compute a number that is too small, you wrap: -7 – 3 = ? 0U – 2U = ?
- Answers are only correct mod 2^b

- The CPU may be capable of "throwing an exception" for overflow on signed values
- It won't for unsigned
- But C and Java just cruise along silently when overflow occurs...

Mapping Signed ↔ Unsigned

Bits	Signed	Unsigned
0000	0	0
0001	1	1
0010	2	2
0011	3	3
0100	4	4
0101	5	5
0110	6	6
0111	7	7
1000	-8	8
1001	-7	9
1010	-6	10
1011	-5	11
1100	-4	12
1101	-3	13
1110	-2	14
1111	-1	15

=

+16

Numeric Ranges

- **Unsigned Values**
  - **UMin = 0**
    000…0
  - **UMax = $2^w - 1$**
    111…1

Two's Complement Values

TMin = -2^{w-1}
100…0

TMax = $2^{w-1} - 1$
011…1

Other Values

Minus 1
111…1 0xFFFFFFFF (32 bits)

Values for W = 16

	Decimal	Hex	Binary
UMax	65535	FF FF	11111111 11111111
TMax	32767	7F FF	01111111 11111111
TMin	-32768	80 00	10000000 00000000
-1	-1	FF FF	11111111 11111111
0	0	00 00	00000000 00000000

Values for Different Word Sizes

	W			
	8	16	32	64
UMax	255	65,535	4,294,967,295	18,446,744,073,709,551,615
TMax	127	32,767	2,147,483,647	9,223,372,036,854,775,807
TMin	-128	-32,768	-2,147,483,648	-9,223,372,036,854,775,808

- **Observations**
  - |TMin | = TMax + 1
  - Asymmetric range
  - UMax = 2 * TMax + 1

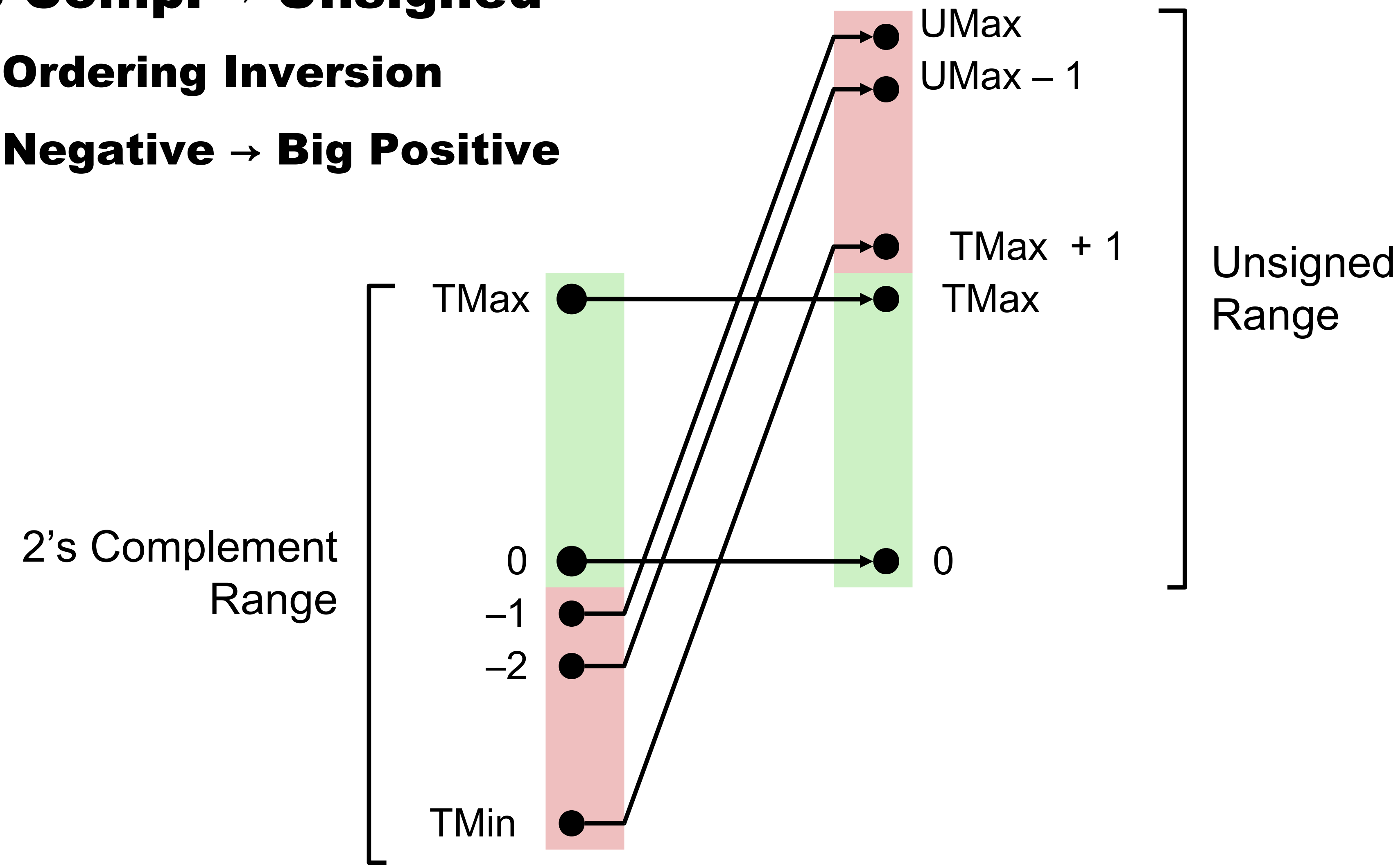
- **C Programming**
  - #include <limits.h>
  - Declares constants, e.g.,
  - ULONG_MAX
  - LONG_MAX
  - LONG_MIN
  - Values platform specific

Conversion Visualized

2's Comp. → Unsigned

Ordering Inversion

Negative → Big Positive



Signed vs. Unsigned in C

- **Constants**
  - By default are considered to be signed integers
  - Unsigned if have "U" as suffix
    - `0U, 4294967259U`
  - Size can be typed too 1234567890123456ULL
- **Casting**
  - `int tx, ty;`
  - `unsigned ux, uy;`
  - Explicit casting between signed & unsigned same as U2T and T2U
    - `tx = (int) ux;`
    - `uy = (unsigned) ty;`
  - Implicit casting also occurs via assignments and procedure calls
    - `tx = ux;`
    - `uy = ty;`

Casting Surprises

Expression Evaluation

- **If mix unsigned and signed in single expression,**
  signed values implicitly cast to unsigned
- **Including comparison operations <, >, ==, <=, >=**
- **Examples for W = 32: TMIN = -2,147,483,648 TMAX = 2,147,483,647**

Constant$_1$	Constant$_2$	Relation	Evaluation
0	0U	==	unsigned
-1	0	<	signed
-1	0U	>	unsigned
2147483647	-2147483647-1	>	signed
2147483647U	-2147483647-1	<	unsigned
-1	-2	>	signed
(unsigned)-1	-2	>	unsigned
2147483647	2147483648U	<	unsigned
2147483647	(int) 2147483648U	>	signed

General advice on types

- Be as explicit as possible

```
typedef unsigned int uint32_t;
uint32_t  i; for(i = 0; i < n; i++) { ... }
```

- Use modern C dialect features / use the type system to catch errors at compile time:

```
// fast and loose
#define my_constant 1234
// better
#define my_constant 1234U
// generally (but not always) best
const unsigned int my_constant = 1234;
```

- Use opaque types as much as possible

```
struct my_type; struct my_type *allocate_object_of_my_type();
```

- C compilers have a lot of legacy cruft in this area. Much can go wrong...

  e.g. is unsigned long long x:4; a 4 bit field of a 64 bit type? or a 32 bit one?

Shift Operations

- **Left shift:** `x << y`
  - **Shift bit-vector `x` left by `y` positic**
    - **Throw away extra bits on lef**
    - **Fill with 0s on right**
  - **Multiply by 2**y**
- **Right shift:** `x >> y`
  - **Shift bit-vector `x` right by `y` positions**
    - **Throw away extra bits on rig**
  - **Logical shift (for unsigned)**
    - **Fill with 0s on left**
  - **Arithmetic shift (for signed)**
    - **Replicate most significant bi**
    - **Maintain sign of x**
  - **Divide by 2**y**
  - **correct truncation (towards 0) requires some care with signed numbers**

Argument x	01100010
<< 3	00010000
Logical >> 2	00011000
Arithmetic >> 2	00011000

Argument x	10100010
<< 3	00010000
Logical >> 2	00101000
Arithmetic >> 2	11101000

What if $y < 0$ or $y \geq$ word_size?

Using Shifts and Masks

Extract 2nd most significant byte of an integer

First shift: x >> (2 * 8)

Then mask: (x >> 16) & 0xFF

x	01100001 01100010 01100011 01100100
x >> 16	00000000 00000000 01100001 01100010
(x >> 16) & 0xFF	00000000 00000000 00000000 11111111 00000000 00000000 00000000 01100010

Extracting the sign bit

(x >> 31) & 1 - need the "& 1" to clear out all other bits except LSB

Conditionals as Boolean expressions (assuming x is 0 or 1 here)

if (x) a=y else a=z; which is the same as a = x ? y : z;

Sign Extension

Task:

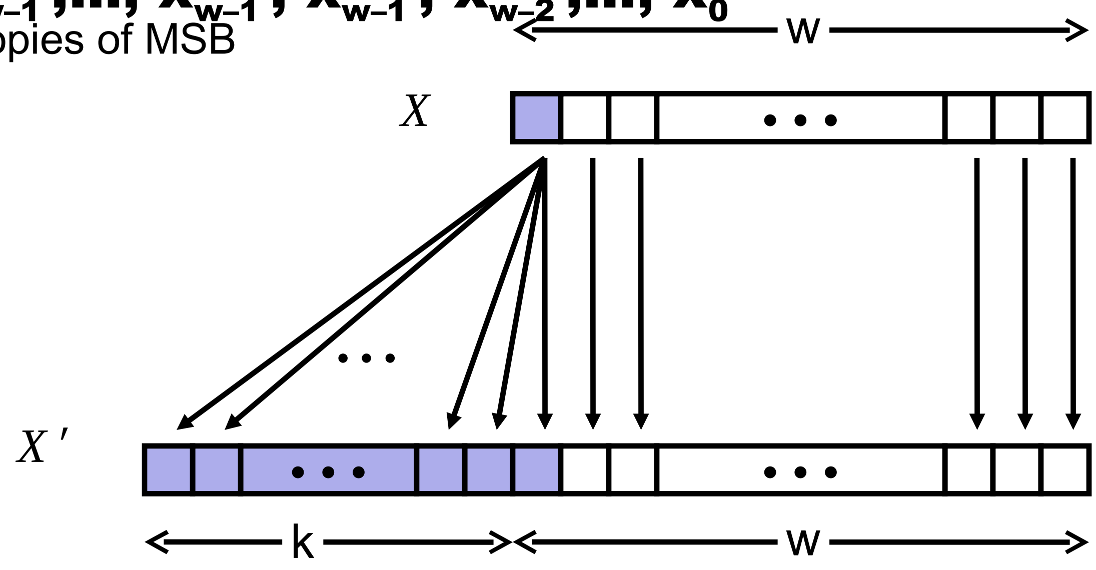
Given w-bit signed integer x

Convert it to w+k-bit integer with same value

Rule:

Make k copies of sign bit:

$$X' = \underbrace{x_{w-1},\ldots, x_{w-1}}_{\text{k copies of MSB}}, x_{w-1}, x_{w-2}, \ldots, x_0$$



Sign Extension Example

```
short int x =  12345;
int       ix = (int) x;
short int y = -12345;
int       iy = (int) y;
```

	Decimal	Hex	Binary
x	12345	30 39	00110000 01101101
ix	12345	00 00 30 39	00000000 00000000 00110000 01101101
y	-12345	CF C7	11001111 11000111
iy	-12345	FF FF CF C7	11111111 11111111 11001111 11000111

- **Converting from smaller to larger integer data type**
- **C automatically performs sign extension**
- **You might have to if converting a bizarre data type to a native one (e.g. PMC counters are sometimes 48 bits)**