Today's Topics

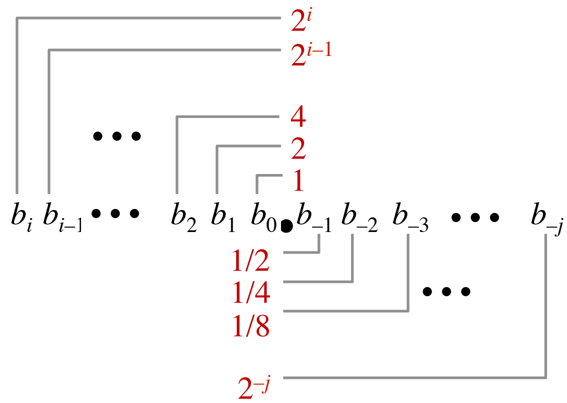
- Floating Point Numbers

- Background: Fractional binary numbers
- IEEE floating point standard: Definition
- Example and properties
- Rounding, addition, multiplication
- Floating point in C
- Summary

Fractional binary numbers

- What is 1011.101?

Fractional Binary Numbers



- **Representation**
  - Bits to right of "binary point" represent fractional powers of 2
  - Represents rational number: $\sum_{k=-j}^{i} b_k \cdot 2^k$

Fractional Binary Numbers: Examples

- **Value** **Representation**
  - 5 and 3/4 $\texttt{101.11}_2$
  - 2 and 7/8 $\texttt{10.111}_2$
  - 63/64 $\texttt{0.111111}_2$

- **Observations**
  - Divide by 2 by shifting right
  - Multiply by 2 by shifting left
  - Numbers of the form $\texttt{0.111111}\ldots_2$ are just below 1.0
    - $1/2 + 1/4 + 1/8 + \ldots + 1/2^i + \ldots \rightarrow 1.0$
    - Use notation $1.0 - \varepsilon$

Representable Numbers

- **Limitation**
  - Can only exactly represent numbers of the form $x/2^k$
  - Other rational numbers have repeating bit representations

- **Value** **Representation**
  - 1/3 `0.0101010101[01]`…$_2$
  - 1/5 `0.001100110011[0011]`…$_2$
  - 1/10 `0.0001100110011[0011]`…$_2$

Fixed Point Representation

- **float → 32 bits; double → 64 bits**
- **We might try representing fractional binary numbers by picking a fixed place for an implied binary point**
  - "fixed point binary numbers"
- **Let's do that, using 8 bit floating point numbers as an example**
  - #1: the binary point is between bits 2 and 3
    $b_7\ b_6\ b_5\ b_4\ b_3$ [.] $b_2\ b_1\ b_0$
  - #2: the binary point is between bits 4 and 5
    $b_7\ b_6\ b_5$ [.] $b_4\ b_3\ b_2\ b_1\ b_0$
  - The position of the binary point affects the range and precision
    - range: difference between largest and smallest numbers possible
    - precision: smallest possible difference between any two numbers

Fixed Point Pros and Cons

- **Pros**
  - It's simple. The same hardware that does integer arithmetic can do fixed point arithmetic
    - In fact, the programmer can use ints with an implicit fixed point
      - E.g., int balance; // number of pennies in the account
    - ints are just fixed point numbers with the binary point to the right of b_0
- **Cons**
  - There is no good way to pick where the fixed point should be
    - Sometimes you need range, sometimes you need precision
    - The more you have of one, the less of the other

What else could we do?

IEEE Floating Point

- **Fixing fixed point: analogous to scientific notation**
  - Not 12000000 but 1.2 x 10^7; not 0.0000012 but 1.2 x 10^{-6}
- **IEEE Standard 754**
  - Established in 1985 as uniform standard for floating point arithmetic
    - Before that, many idiosyncratic formats
  - Supported by all major CPUs
- **Driven by numerical concerns**
  - Nice standards for rounding, overflow, underflow
  - Hard to make fast in hardware
    - Numerical analysts predominated over hardware designers in defining standard

Floating Point Representation

- **Numerical Form:**

$$(-1)^{s} * M * 2^{E}$$

  - Sign bit ***s*** determines whether number is negative or positive
  - Significand (mantissa) ***M*** normally a fractional value in range [1.0,2.0).
  - Exponent ***E*** weights value by power of two

- **Encoding**
  - MSB **`s`** is sign bit ***s***
  - **`frac`** field encodes ***M*** (but is *not equal* to M)
  - **`exp`** field encodes ***E*** (but is *not equal* to E)

s	exp	frac

Precisions

- **Single precision: 32 bits**

s	exp	frac
1	8	23

- **Double precision: 64 bits**

s	exp	frac
1	11	52

- **Extended precision: 80 bits (Intel only)**

s	exp	frac
1	15	63 or 64

Normalization and Special Values

- **"Normalized" means mantissa has form 1.xxxxx**
  - 0.011 x 2^5 and 1.1 x 2^3 represent the same number, but the latter makes better use of the available bits
  - Since we know the mantissa starts with a 1, don't bother to store it

- **How do we represent 0.0? How about 1.0/0.0?**

Normalization and Special Values

- "Normalized" means mantissa has form 1.xxxxx
  - 0.011 x 2^5 and 1.1 x 2^3 represent the same number, but the latter makes better use of the available bits
  - Since we know the mantissa starts with a 1, don't bother to store it
- Special values:
  - The float value 00...0 represents zero
  - If the exp == 11...1 and the mantissa == 00...0, it represents ∞
  - E.g., 1.0/0.0 = −1.0/−0.0 = +∞, 1.0/−0.0 = −1.0/0.0 = −∞
- If the exp == 11...1 and the mantissa != 00...0, it represents NaN
  - "Not a Number"
  - Results from operations with undefined result
    - E.g., sqrt(–1), ∞ − ∞, ∞ * 0

Normalized Values

- **Condition: `exp` ≠ `000…0` and `exp` ≠ `111…1`**
- **Exponent coded as *biased* value: *exp* = *E + Bias***
  - `exp` is an unsigned value ranging from 1 to 2^e-2
    - Allows negative values for E (= exp – Bias)
  - *Bias* = 2^{e-1} - 1, where e is number of exponent bits (bits in exp)
    - Single precision: 127 (*exp*: 1…254, *E*: -126…127)
    - Double precision: 1023 (*exp*: 1…2046, *E*: -1022…1023)
- **Significand coded with implied leading 1: *M* = `1.xxx…x`$_2$**
  - `xxx…x`: bits of `frac`
  - Minimum when `000…0` (*M* = 1.0)
  - Maximum when `111…1` (*M* = 2.0 – ε)
  - Get extra leading bit for "free"

Normalized Encoding Example

- Value: `Float F = 12345.0;`
  - $12345_{10} = 11000000111001_2$
    $= 1.1000000111001_2 \times 2^{13}$

- Significand
  M = `1.`$\underline{\texttt{1000000111001}}_2$
  `frac=` $\underline{\texttt{1000000111001}}\texttt{0000000000}_2$

- Exponent
  E = 13
  Bias = 127
  `exp` = 140 = $\texttt{10001100}_2$

- Result:

0	10001100	10000001110010000000000
`s`	`exp`	`frac`

How do we do operations?

- Is representation exact?
- How are the operations carried out?

Floating Point Operations: Basic Idea

- `x +f y = Round(x + y)` ($x +_f y$)
- `x *f y = Round(x * y)` ($x *_f y$)
- **Basic idea**
  - First compute exact result
  - Make it fit into desired precision
    - Possibly overflow if exponent too large
    - Possibly round to fit into frac

Floating Point Multiplication

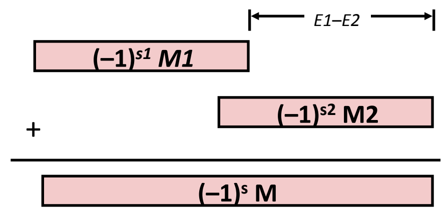
$$(-1)^{s1}\ M1\ \ 2^{E1}\quad *\quad (-1)^{s2}\ M2\ \ 2^{E2}$$

- **Exact Result:** $(-1)^s\ M\ \ 2^E$
  - Sign s: s1 ^ s2 // xor of s1 and s2
  - Significand M: M1 * M2
  - Exponent E: E1 + E2
- **Fixing**
  - If M ≥ 2, shift M right, increment E
  - If E out of range, overflow
  - Round M to fit frac precision

Floating Point Addition

$(-1)^{s1}\ M1\ \ 2^{E1}\ \ +\ \ (-1)^{s2}\ M2\ \ 2^{E2}$ *Assume E1 > E2*

- **Exact Result: $(-1)^{s}\ M\ \ 2^{E}$**
  - Sign s, significand M:
    - Result of signed align & add
  - Exponent E: E1



- **Fixing**
  - If M ≥ 2, shift M right, increment E
  - if M < 1, shift M left k positions, decrement E by k
  - Overflow if E out of range
  - Round M to fit frac precision

Hmm… if we round at every operation…

Mathematical Properties of FP Operations

- Not really associative or distributive due to rounding
- Infinities and NaNs cause issues
- Overflow and infinity

Floating Point in C

- **C Guarantees Two Levels**
  - `float` single precision
  - `double` double precision
- **Conversions/Casting**
  - Casting between `int`, `float`, and `double` changes bit representation
  - `Double/float` → `int`
    - Truncates fractional part
    - Like rounding toward zero
    - Not defined when out of range or NaN: generally sets to TMin
  - `int` → `double`
    - Exact conversion, as long as int has ≤ 53-bit word size
  - `int` → `float`
    - Will round according to rounding mode

Memory Referencing Bug

```
double fun(int i)
{
  volatile double d[1] = {3.14};
  volatile long int a[2];
  a[i] = 1073741824; /* Possibly out of bounds */
  return d[0];
}
```

```
fun(0)  ->    3.14
fun(1)  ->    3.14
fun(2)  ->    3.1399998664856
fun(3)  ->    2.00000061035156
fun(4)  ->    3.14, then segmentation fault
```

Explanation:

Saved State	4
d7 … d4	3
d3 … d0	2
a[1]	1
a[0]	0

Location accessed by `fun(i)`

Representing 3.14 as a Double FP Number

- **1073741824 = 0100 0000 0000 0000 0000 0000 0000 0000**
- **3.14 = 11.0010 0011 1101 0111 0000 1010 000…**
- **$(-1)^s\ M\ 2^E$**
  - S = 0 encoded as 0
  - M = 1.1001 0001 1110 1011 1000 0101 000…. (leading 1 left out)
  - E = 1 encoded as 1024 (with bias)

s	exp (11)	frac (first 20 bits)
0	**100 0000 0000**	**1001 0001 1110 1011 1000**

frac (the other 32 bits)
0101 0000 …

Memory Referencing Bug (Revisited)

```
double fun(int i)
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  volatile long int a[2];
  a[i] = 1073741824; /* Possibly out of bounds */
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fun(0)  ->    3.14
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fun(3)  ->    2.00000061035156
fun(4)  ->    3.14, then segmentation fault
```

Saved State		4
d7 … d4	0100 0000 0000 1001 0001 1110 1011 1000	3
d3 … d0	0101 0000 …	2
a[1]		1
a[0]		0

Location accessed by `fun(i)`

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```

Saved State		4
d7 … d4	0100 0000 0000 1001 0001 1110 1011 1000	3
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```
fun(0)  ->     3.14
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fun(4)  ->     3.14, then segmentation fault
```

Saved State		4
d7 … d4	0100 0000 0000 0000 0000 0000 0000 0000	3
d3 … d0	0101 0000 …	2
a[1]		1
a[0]		0

Location accessed by `fun(i)`

Floating Point and the Programmer

```
#include <stdio.h>

int main(int argc, char* argv[]) {

  float f1 = 1.0;
  float f2 = 0.0;
  int i;
  for ( i=0; i<10; i++ ) {
    f2 += 1.0/10.0;
  }

  printf("0x%08x  0x%08x\n", *(int*)&f1, *(int*)&f2);
  printf("f1 = %10.8f\n", f1);
  printf("f2 = %10.8f\n\n", f2);

  f1 = 1E30;
  f2 = 1E-30;
  float f3 = f1 + f2;
  printf ("f1 == f3? %s\n", f1 == f3 ? "yes" : "no" );

  return 0;
}
```

```
$ ./a.out
0x3f800000  0x3f800001
f1 = 1.000000000
f2 = 1.000000119

f1 == f3? yes
```

Summary

- **As with integers, floats suffer from the fixed number of bits available to represent them**
  - Can get overflow/underflow, just like ints
  - Some "simple fractions" have no exact representation
    - E.g., 0.1
  - Can also lose precision, unlike ints
    - "Every operation gets a slightly wrong result"
- **Mathematically equivalent ways of writing an expression may compute different results**
  - Violates associativity/distributivity
- **NEVER test floating point values for equality!**

Additional details

- **Denormalized values – to get finer precision near zero**
- **Tiny floating point example**
- **Distribution of representable values**
- **Rounding**

Denormalized Values

- **Condition: `exp` = 000…0**

- **Exponent value: E = exp – $Bias$ + 1 (instead of E = exp – Bias)**
- **Significand coded with implied leading 0: M = 0 . `xxx…x`$_2$**
    - `xxx…x`: bits of `frac`
- **Cases**
    - `exp` = 000…0, `frac` = 000…0
        - Represents value 0
        - Note distinct values: +0 and –0 (why?)
    - `exp` = 000…0, `frac` ≠ 000…0
        - Numbers very close to 0.0
        - Lose precision as get smaller
        - Equispaced

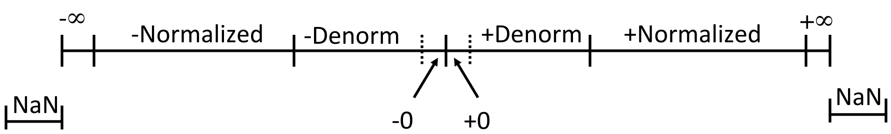
Special Values

- **Condition: `exp` = 111…1**

- **Case: `exp` = 111…1, `frac` = 000…0**
    - Represents value ∞ (infinity)
    - Operation that overflows
    - Both positive and negative
    - E.g., 1.0/0.0 = −1.0/−0.0 = +∞, 1.0/−0.0 = −1.0/0.0 = −∞

- **Case: `exp` = 111…1, `frac` ≠ 000…0**
    - Not-a-Number (NaN)
    - Represents case when no numeric value can be determined
    - E.g., sqrt(–1), ∞ − ∞, ∞ * 0

Visualization: Floating Point Encodings



Tiny Floating Point Example

s	exp	frac
1	4	3

- **8-bit Floating Point Representation**
  - the sign bit is in the most significant bit.
  - the next four bits are the exponent, with a bias of 7.
  - the last three bits are the **`frac`**

- **Same general form as IEEE Format**
  - normalized, denormalized
  - representation of 0, NaN, infinity

Dynamic Range (Positive Only)

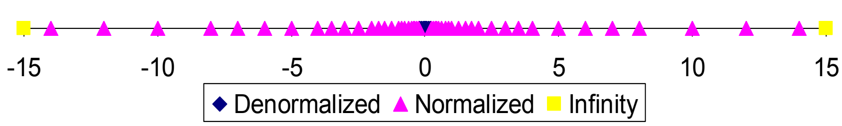
	s	exp	frac	*E*	Value	
Denormalized numbers	0	0000	000	-6	0	
	0	0000	001	-6	1/8*1/64 = 1/512	closest to zero
	0	0000	010	-6	2/8*1/64 = 2/512	
	…					
	0	0000	110	-6	6/8*1/64 = 6/512	
	0	0000	111	-6	7/8*1/64 = 7/512	largest denorm
Normalized numbers	0	0001	000	-6	8/8*1/64 = 8/512	smallest norm
	0	0001	001	-6	9/8*1/64 = 9/512	
	…					
	0	0110	110	-1	14/8*1/2 = 14/16	
	0	0110	111	-1	15/8*1/2 = 15/16	closest to 1 below
	0	0111	000	0	8/8*1 = 1	
	0	0111	001	0	9/8*1 = 9/8	closest to 1 above
	0	0111	010	0	10/8*1 = 10/8	
	…					
	0	1110	110	7	14/8*128 = 224	
	0	1110	111	7	15/8*128 = 240	largest norm
	0	1111	000	n/a	inf	

Distribution of Values

- **6-bit IEEE-like format**
  - e = 3 exponent bits
  - f = 2 fraction bits
  - Bias is $2^{3-1}-1 = 3$

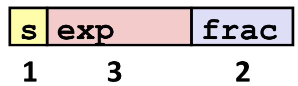
s	exp	frac
1	3	2

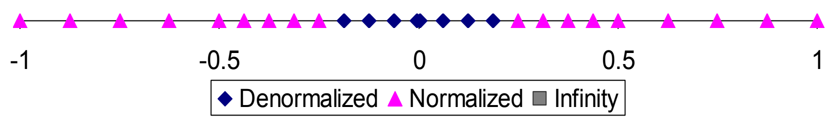
- **Notice how the distribution gets denser toward zero.**



Distribution of Values (close-up view)

- **6-bit IEEE-like format**
  - e = 3 exponent bits
  - f = 2 fraction bits
  - Bias is 3





Interesting Numbers `{single,double}`

Description	exp	frac	Numeric Value
Zero	00…00	00…00	0.0
Smallest Pos. Denorm.	00…00	00…01	$2^{-\{23,52\}} * 2^{-\{126,1022\}}$
▪ Single ≈ $1.4 * 10^{-45}$			
▪ Double ≈ $4.9 * 10^{-324}$			
Largest Denormalized	00…00	11…11	$(1.0 - \varepsilon) * 2^{-\{126,1022\}}$
▪ Single ≈ $1.18 * 10^{-38}$			
▪ Double ≈ $2.2 * 10^{-308}$			
Smallest Pos. Norm.	00…01	00…00	$1.0 * 2^{-\{126,1022\}}$
▪ Just larger than largest denormalized			
One	01…11	00…00	1.0
Largest Normalized	11…10	11…11	$(2.0 - \varepsilon) * 2^{\{127,1023\}}$
▪ Single ≈ $3.4 * 10^{38}$			
▪ Double ≈ $1.8 * 10^{308}$			

Special Properties of Encoding

- **Floating point zero (0^+) exactly the same bits as integer zero**
  - All bits = 0
- **Can (Almost) Use Unsigned Integer Comparison**
  - Must first compare sign bits
  - Must consider $0^- = 0^+ = 0$
  - NaNs problematic
    - Will be greater than any other values
    - What should comparison yield?
  - Otherwise OK
    - Denorm vs. normalized
    - Normalized vs. infinity

Rounding

- **Rounding Modes (illustrate with $ rounding)**

	$1.40	**$1.60**	**$1.50**	**$2.50**	**–$1.50**
▪ Towards zero	$1	$1	$1	$2	–$1
▪ Round down (-∞)	$1	$1	$1	$2	–$2
▪ Round up (+∞)	$2	$2	$2	$3	–$1
▪ Nearest (default)	$1	$2	$2	$2	–$2

- **What are the advantages of the modes?**

Closer Look at Round-To-Nearest

- **Default Rounding Mode**
  - Hard to get any other kind without dropping into assembly
  - All others are statistically biased
    - Sum of set of positive numbers will consistently be over- or under-estimated
- **Applying to Other Decimal Places / Bit Positions**
  - When exactly halfway between two possible values
    - Round so that least significant digit is even
  - E.g., round to nearest hundredth

1.2349999	1.23	(Less than half way)
1.2350001	1.24	(Greater than half way)
1.2350000	1.24	(Half way—round up)
1.2450000	1.24	(Half way—round down)

Rounding Binary Numbers

- **Binary Fractional Numbers**
  - "Half way" when bits to right of rounding position = $100\ldots_2$
- **Examples**
  - Round to nearest 1/4 (2 bits right of binary point)

Value	Binary	Rounded	Action	Rounded Value
2 3/32	10.00011_2	10.00_2	(<1/2—down)	2
2 3/16	10.00110_2	10.01_2	(>1/2—up)	2 1/4
2 7/8	10.11100_2	11.00_2	(1/2—up)	3
2 5/8	10.10100_2	10.10_2	(1/2—down)	2 1/2