# CSE 332 Winter 2026
# Lecture 9: AVL Trees pt. 2

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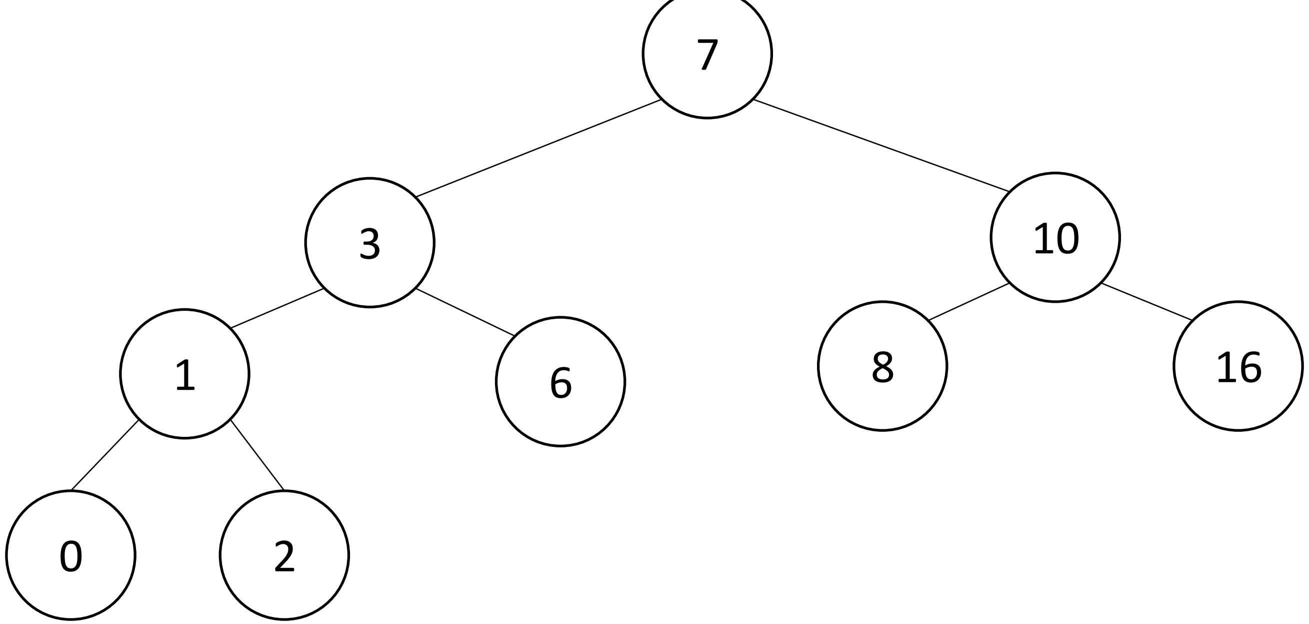
# Dictionary (Map) ADT

- Contents:
  - Sets of key+value pairs
  - Keys must be comparable
- Operations:
  - insert(key, value)
    - Adds the (key,value) pair into the dictionary
    - If the key already has a value, overwrite the old value
      - Consequence: Keys cannot be repeated
  - find(key)
    - Returns the value associated with the given key
  - delete(key)
    - Remove the key (and its associated value)

# Naïve attempts

| Data Structure | Time to insert | Time to find | Time to delete |
|---|---|---|---|
| Unsorted Array | $\Theta(n)$ | $\Theta(n)$ | $\Theta(n)$ |
| Unsorted Linked List | $\Theta(n)$ | $\Theta(n)$ | $\Theta(n)$ |
| Sorted Array | $\Theta(n)$ | $\Theta(\log n)$ | $\Theta(n)$ |
| Sorted Linked List | $\Theta(n)$ | $\Theta(n)$ | $\Theta(n)$ |
| Heap | $\Theta(n)$ | $\Theta(n)$ | $\Theta(n)$ |
| Binary Search Tree | $\Theta(n)$ | $\Theta(n)$ | $\Theta(n)$ |
| AVL Tree | $\Theta(\log n)$ | $\Theta(\log n)$ | $\Theta(\log n)$ |

# Binary Search Tree




- Binary Tree
  - Definition:
    - Tree where each node has at most 2 children
- Order Property
  - All keys in the left subtree are smaller than the root
  - All keys in the right subtree are larger than the root
  - Consequence: cannot have repeated values

# Improving the worst case

- How can we get a better worst case running time?
  - Add rules about the shape of our BST
- AVL Tree
  - A BST with some shape rules
    - Algorithms need to change to accommodate those

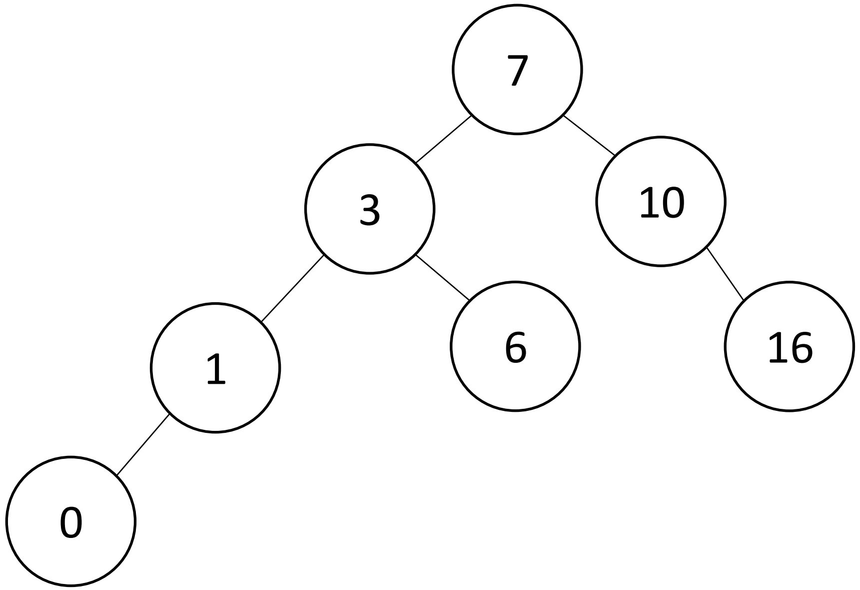
# AVL Tree

- A Binary Search tree that maintains that the left and right subtrees of every node have heights that differ by at most one.
  - height of left subtree and height of right subtree off by at most 1
  - Not too weak (ensures trees are short)
  - Not too strong (works for any number of nodes)

- Idea of AVL Tree:
  - When you insert/delete nodes, if tree is “out of balance” then modify the tree
  - Modification = “rotation”

# Find Operation (Same as BST)

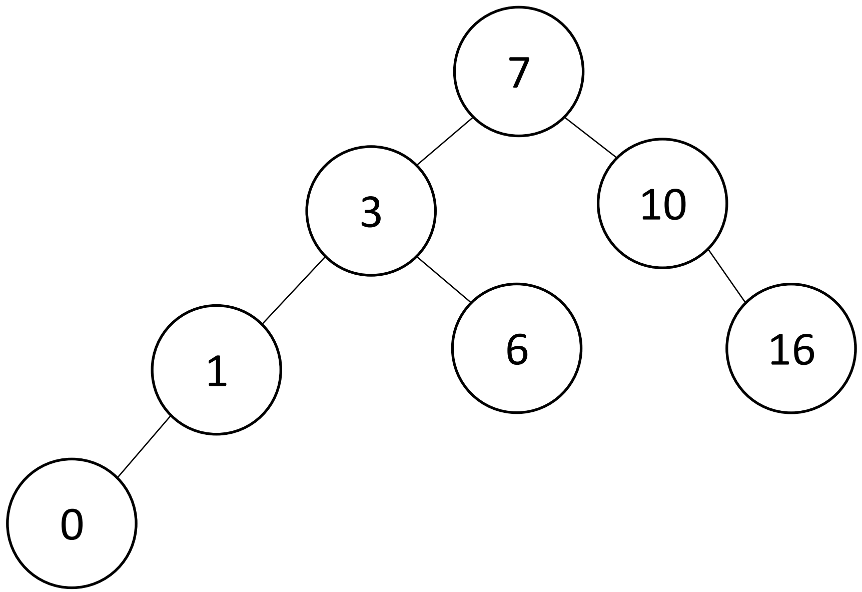
```
find(key, root){
        if (root == Null){
                return Null;
        {
        if (key == root.key){
                return root.value;
        }
        if (key < root.key){
                return find(key, root.left);
        }
        if (key > root.key){
                return find(key, root.right);
        }
        return Null;
}
```



# Inserting into an AVL Tree

- Starts out the same way as BST:
  - “Find” where the new node should go
  - Put it in the right place (it will be a leaf)
- Next check the balance
  - If the tree is still balanced, you’re done!
  - Otherwise we need to do rotations

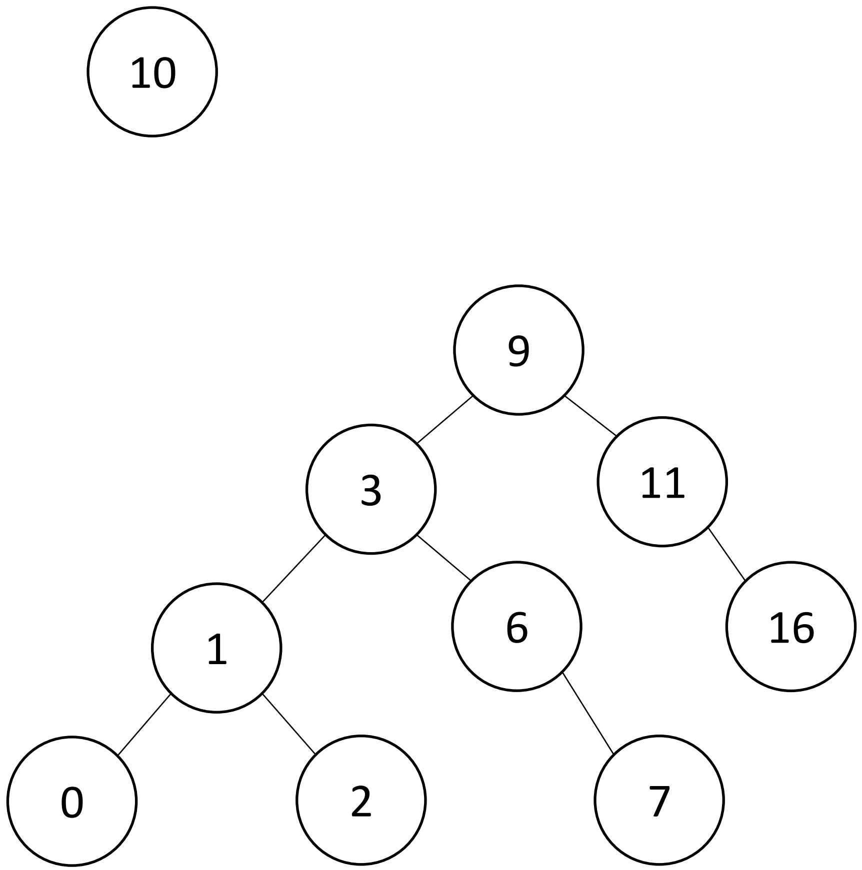
# Insert Operation (for BST)




```
insert(key, value, root){
        root = insertHelper(key, value, root);
}
insertHelper(key, value, root){
        if(root == null)
                return new Node(key, value);
        if (root.key < key)
                root.right = insertHelper(key, value, root.right);
        else
                root.left = insertHelper(key, value, root.left);
        return root;
}
```

Note: Insert happens only at the leaves!

# Insert Example

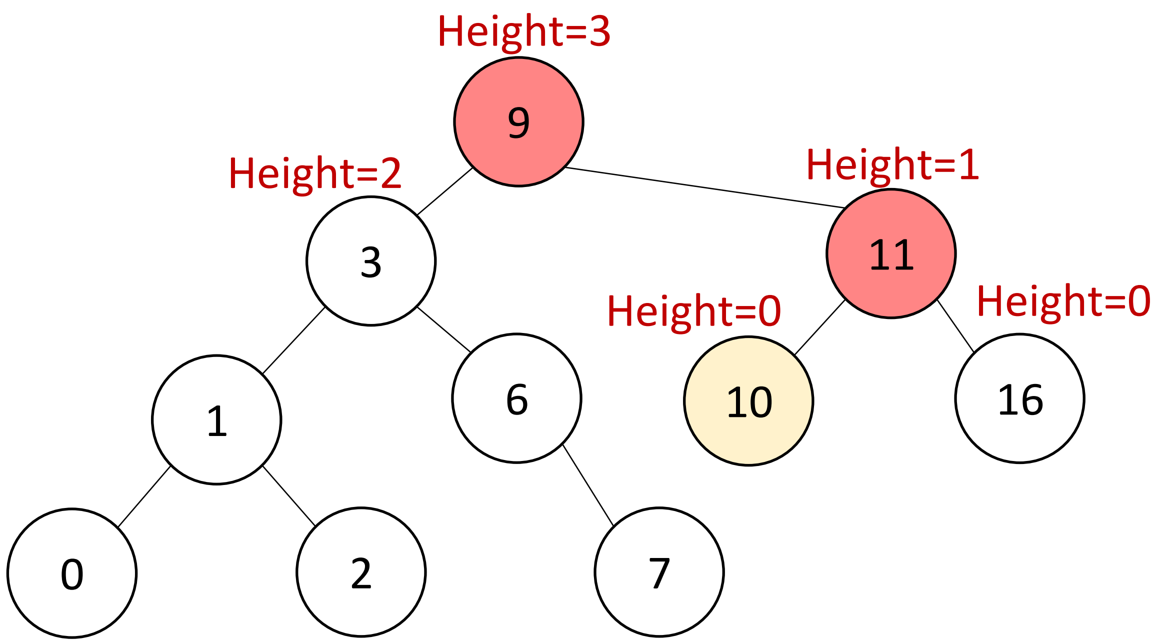


# Insert Example

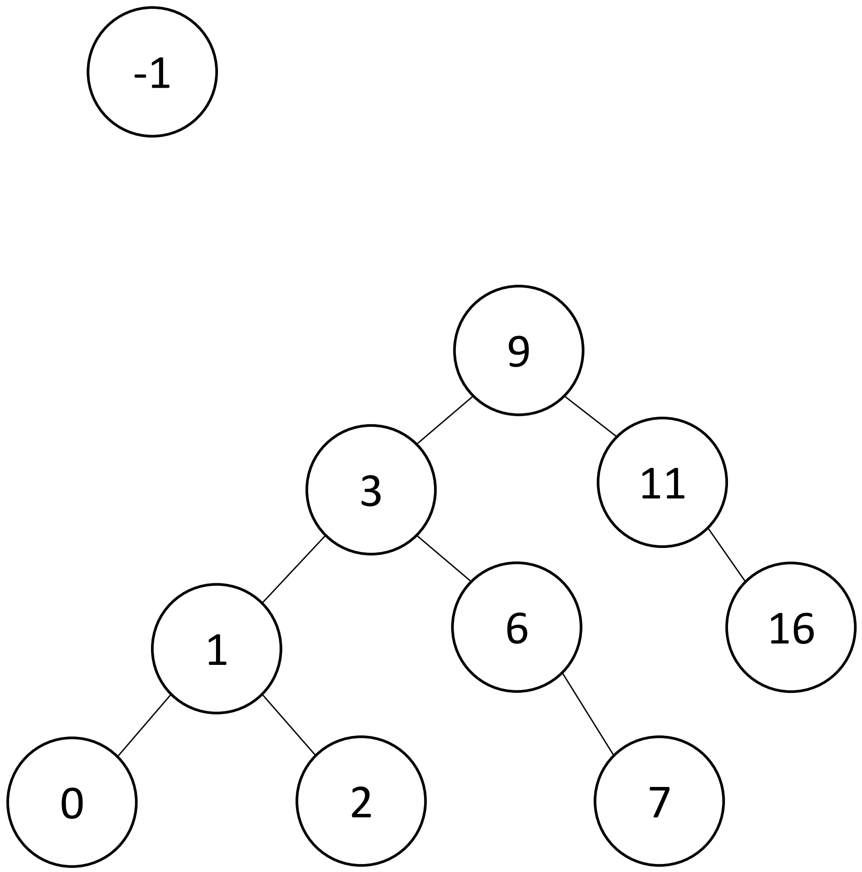
Is the tree still balanced?

To confirm we only need to check nodes in the path from root to the new node

Why? We assume the tree was balanced before the insert, so unchanged subtrees cannot be unbalanced.



# Insert Example



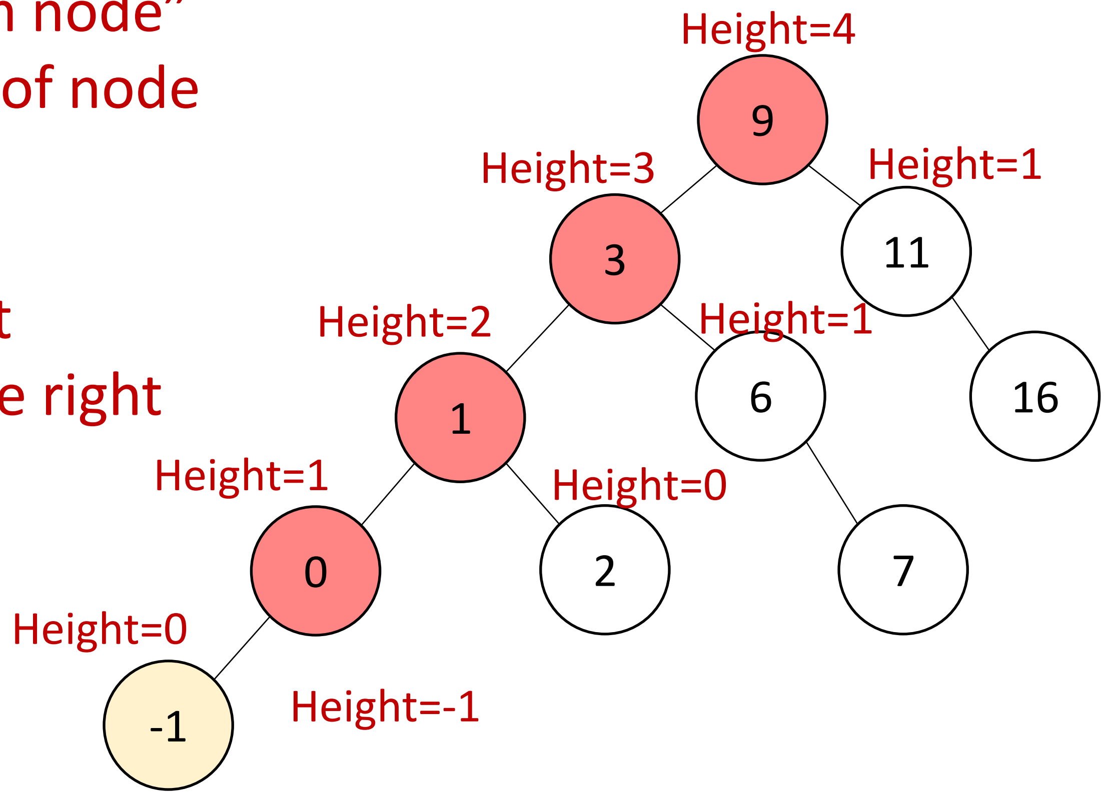
# Insert Example

Not Balanced!

Node 9 is the "problem node"

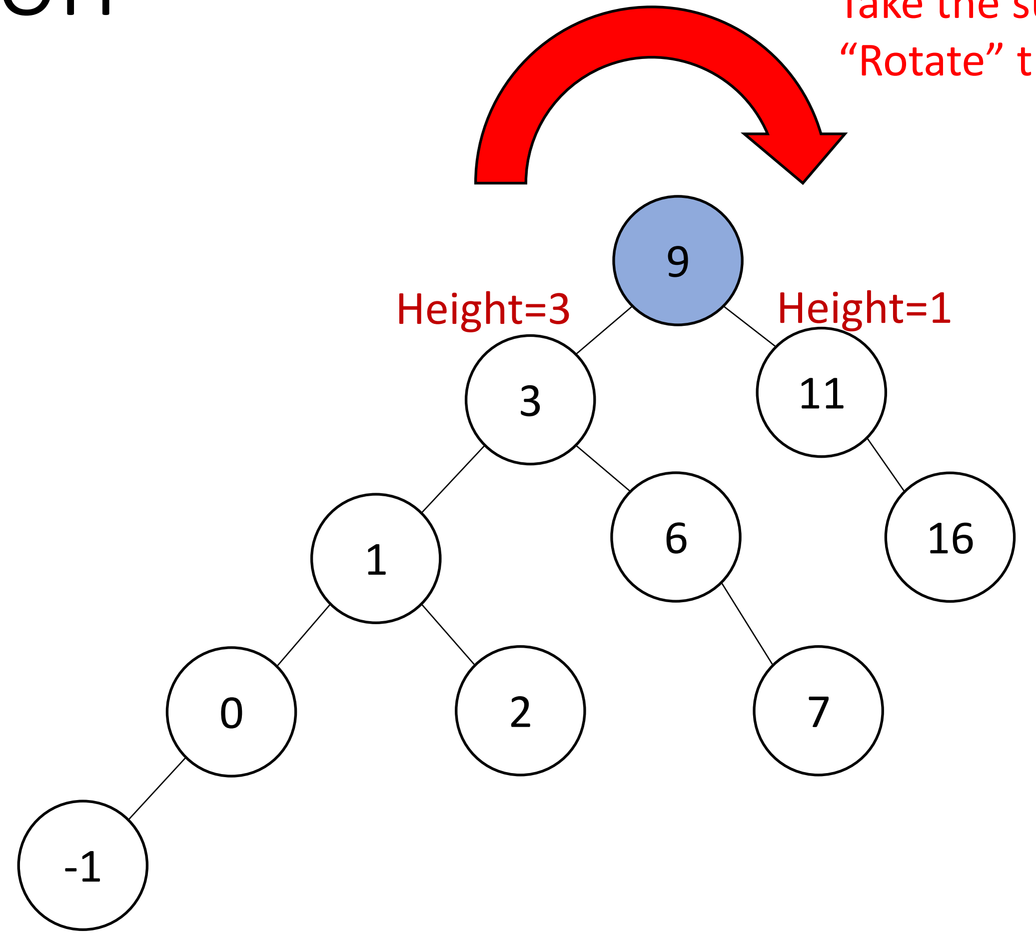
Left and right children of node 9 are different by 2.

Idea: "shorten" the left subtree, "lengthen" the right

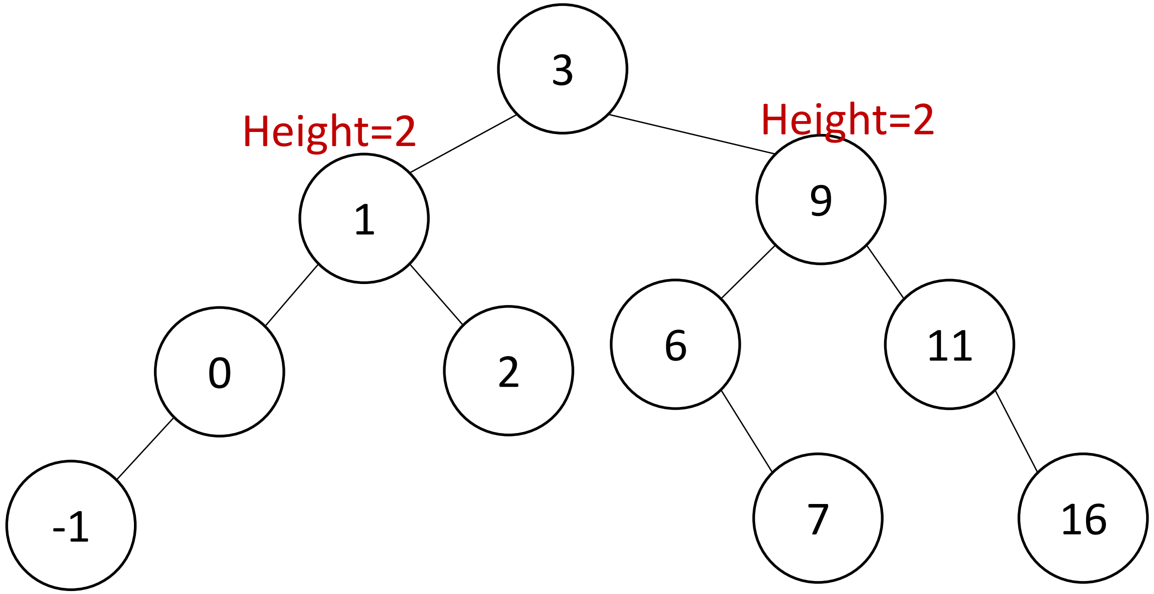


# Right Rotation

Solution:
Take the subtree starting with the problem node,
“Rotate” that tree to the right

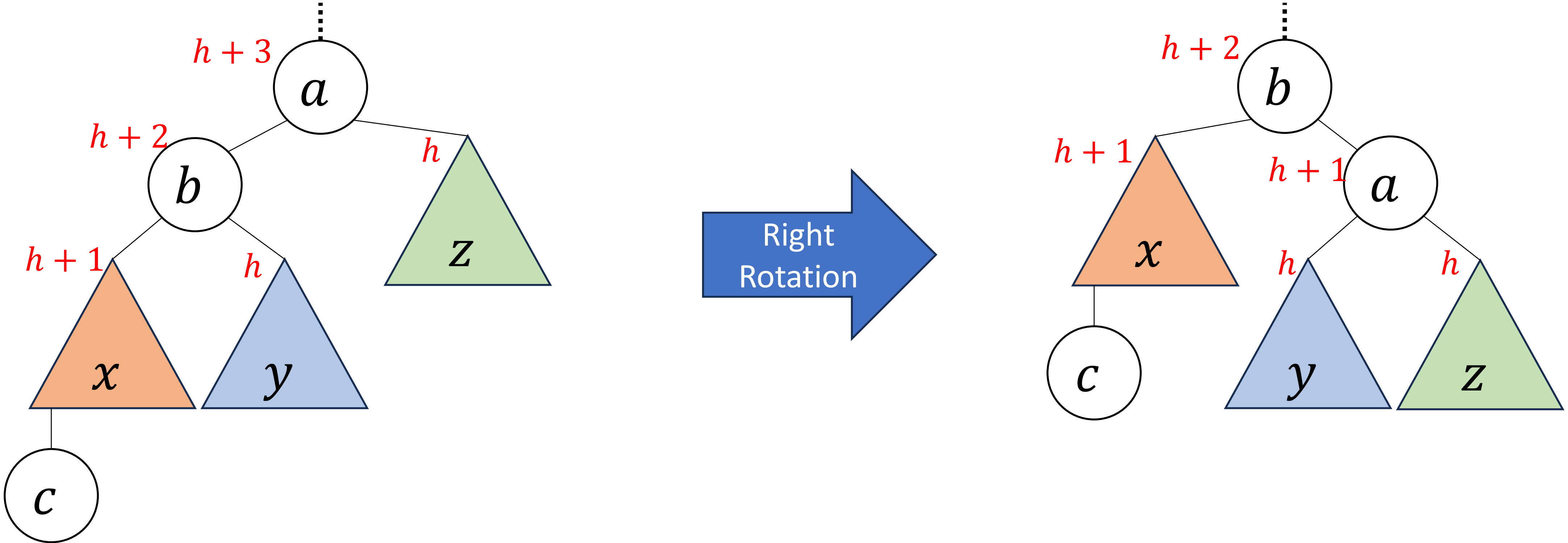


# Balanced!



# Right Rotation

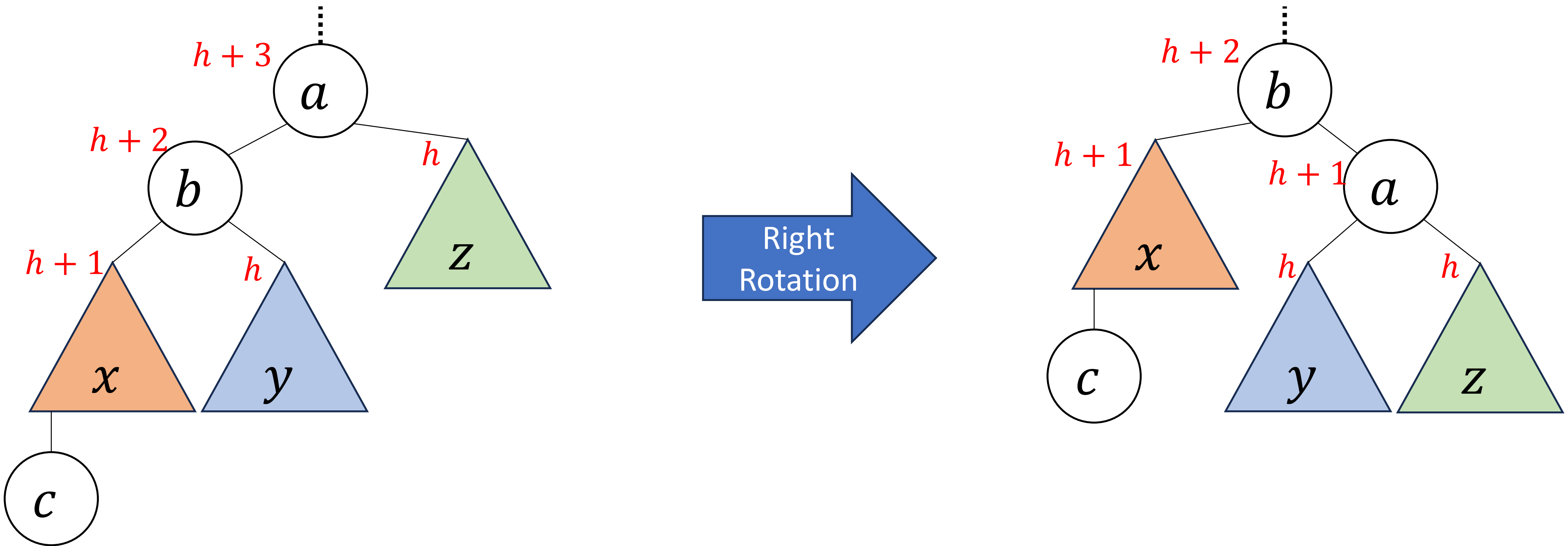
- We just inserted $c$, node $a$ is the deepest “problem” node
- Make the left child the new root
- Make the old root the right child of the new
- Make the new root’s right subtree the old root’s left subtree



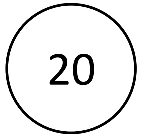
# Right Rotation - Implementation

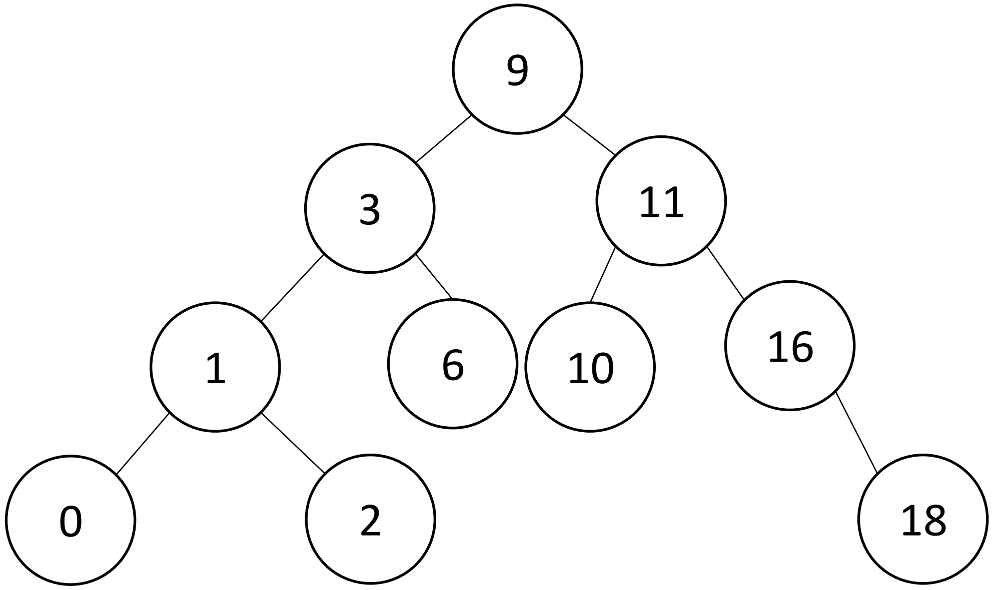
```
b=a.left
a.left=b.right
b.right=a
return b
```

Running time: $\Theta(1)$



# Insert Example

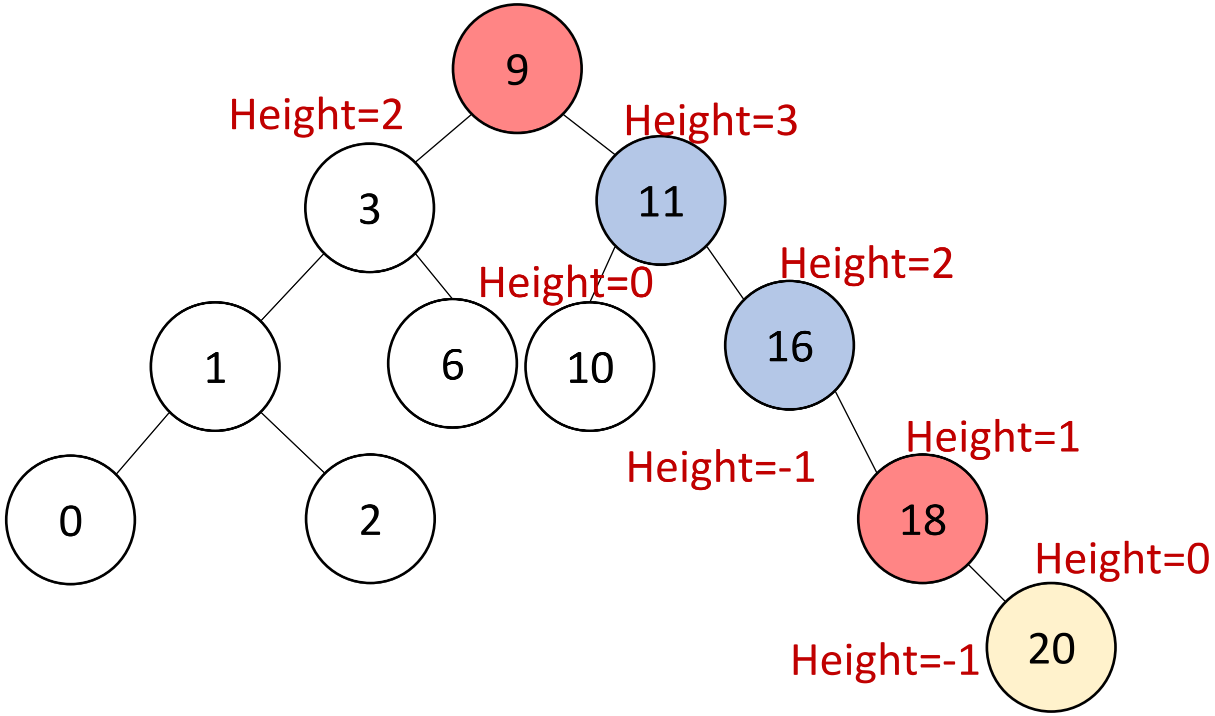


# Not Balanced! Multiple Problem Nodes

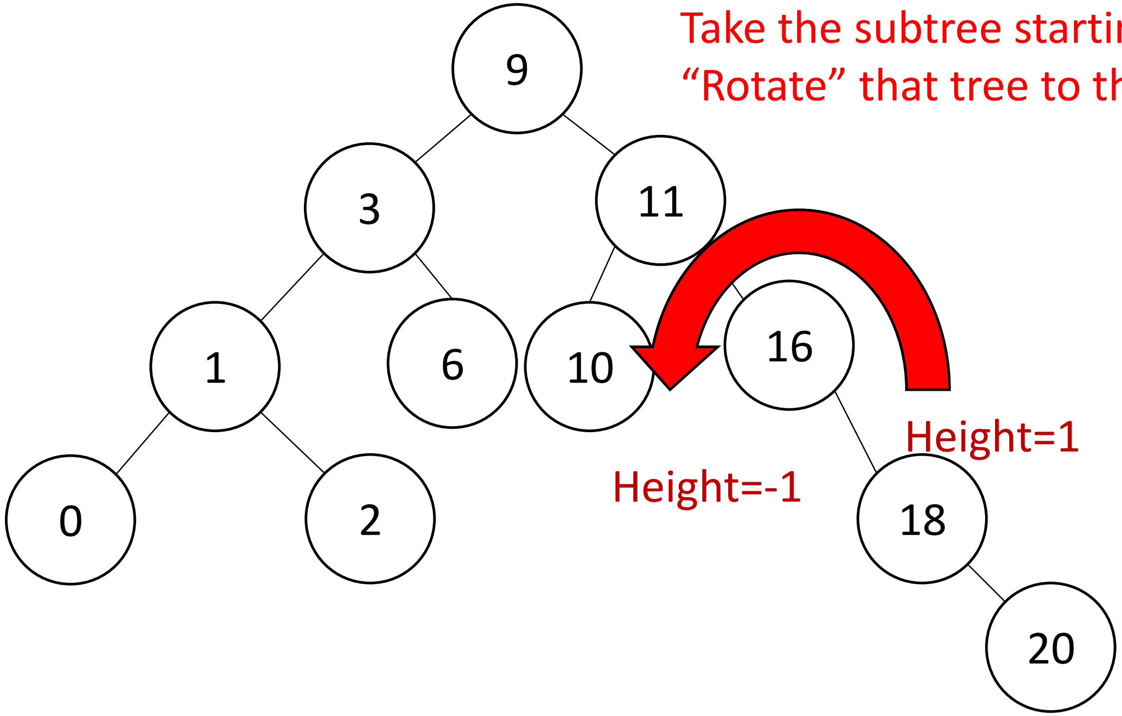
Here, nodes 11 and 16 are problem nodes.

There may be multiple places where we can do a rotation to rebalance the tree, but the *deepest* problem node *always works*!

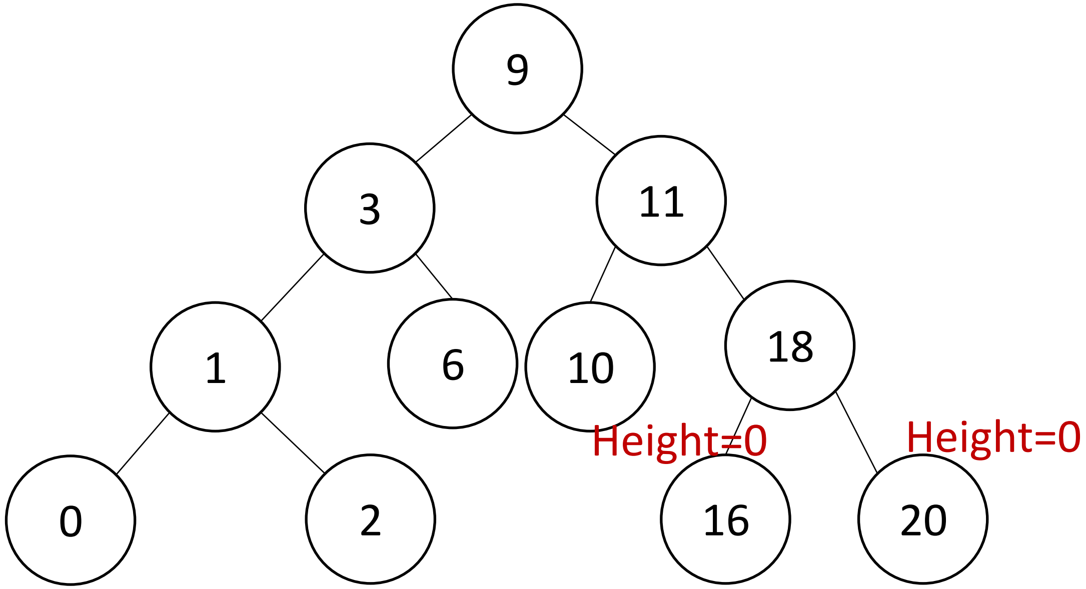


# Left Rotation

Solution:
Take the subtree starting with the problem node,
“Rotate” that tree to the left

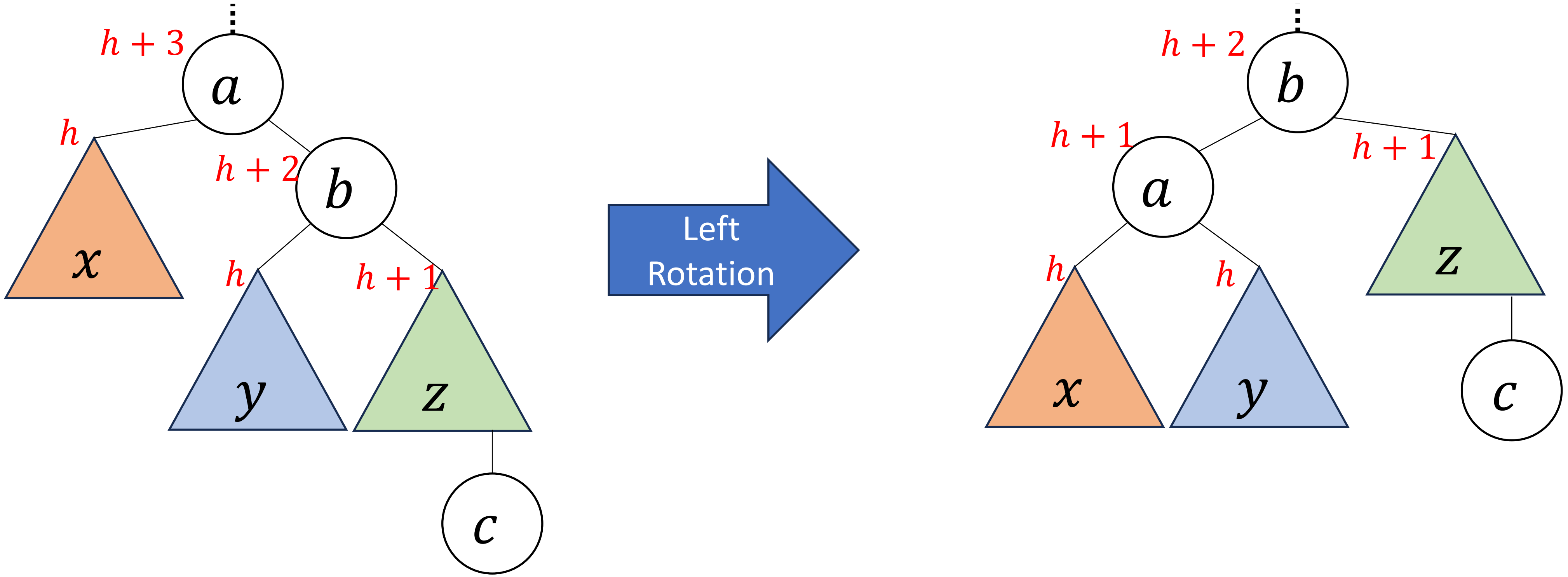


# Balanced!



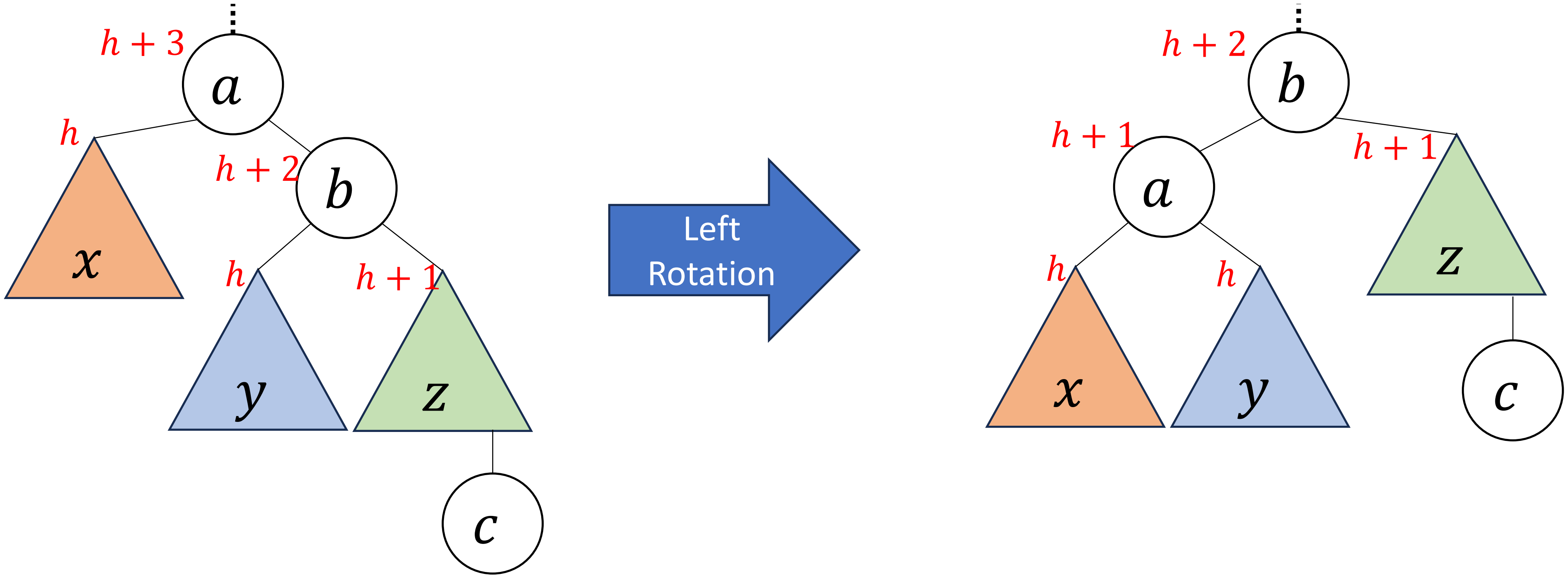
# Left Rotation

- We just inserted $c$, node $a$ is the deepest "problem" node
- Make the right child the new root
- Make the old root the left child of the new
- Make the new root's left subtree the old root's right subtree



# Left Rotation (Implementation)

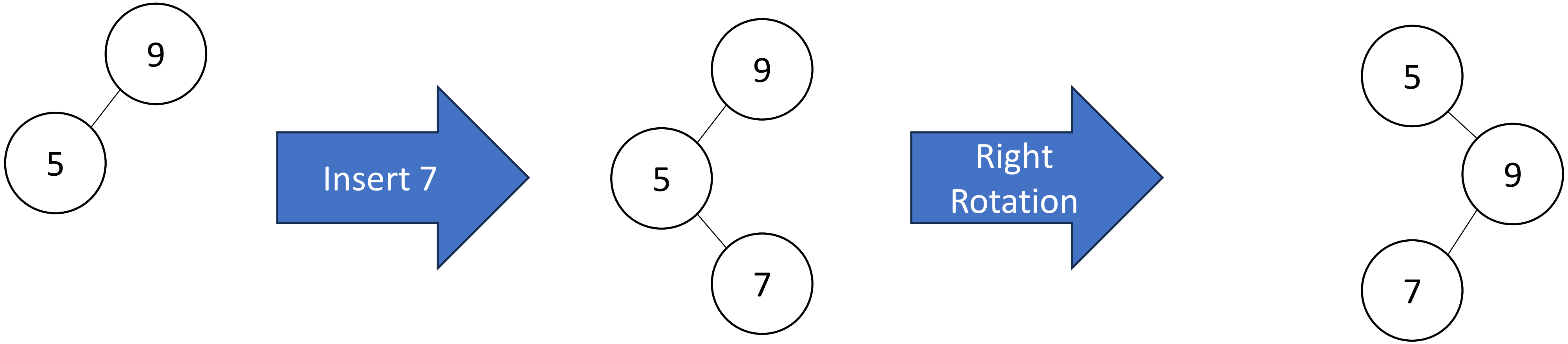
```
b=a.right
a.right=b.left
b.left=a
return b
```



# Insertion Story So Far

- After insertion, update the heights of the node's ancestors
- Check for unbalance
- If unbalanced then at the deepest unbalanced root:
  - If the left subtree was deeper then rotate right
  - If the right subtree was deeper then rotate left

This is incomplete!
There are some cases
where this doesn't work!



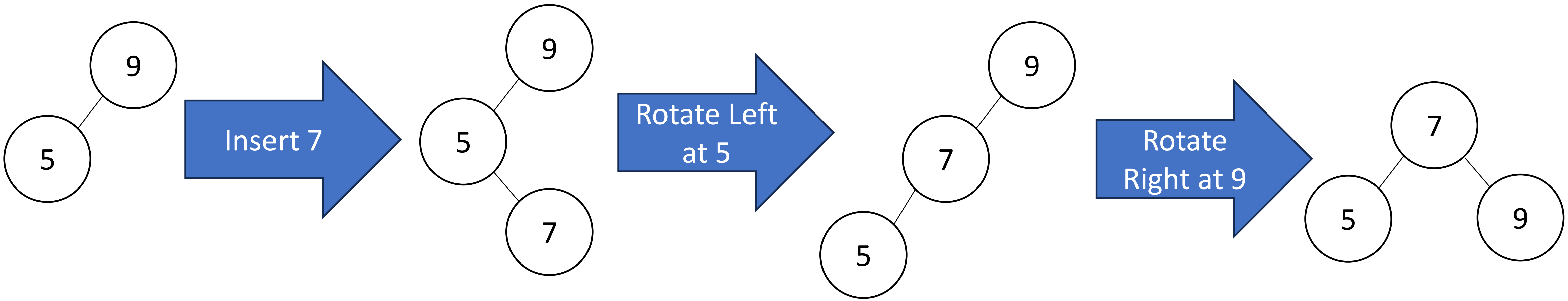
# Insertion Story So Far

- After insertion, update the heights of the node's ancestors
- Check for unbalance
- If unbalanced then at the deepest unbalanced root:
  - Case LL: If we inserted in the **left** subtree of the **left** child then rotate right
  - Case RR: If we inserted in the **right** subtree of the **right** child then rotate left
  - Case LR: If we inserted into the **right** subtree of the **left** child then ???
  - Case RL: If we inserted into the **left** subtree of the **right** child then ???

Cases LR and RL require 2 rotations!

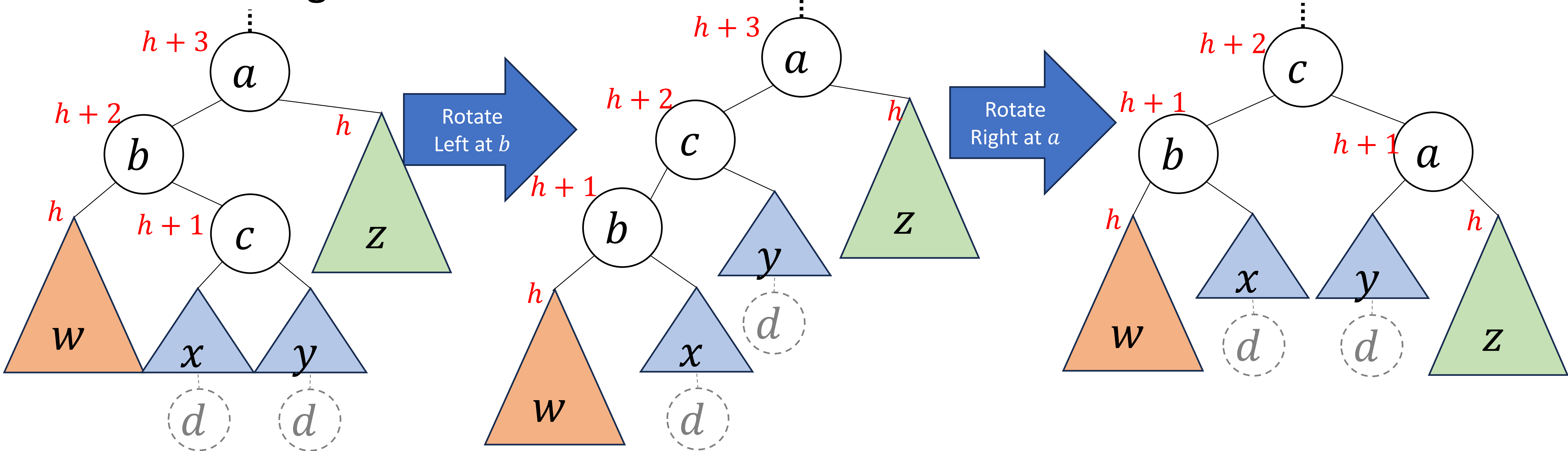
# Case LR

- From deepest problem node:
    - Rotate left at the left child
    - Rotate right at the problem node



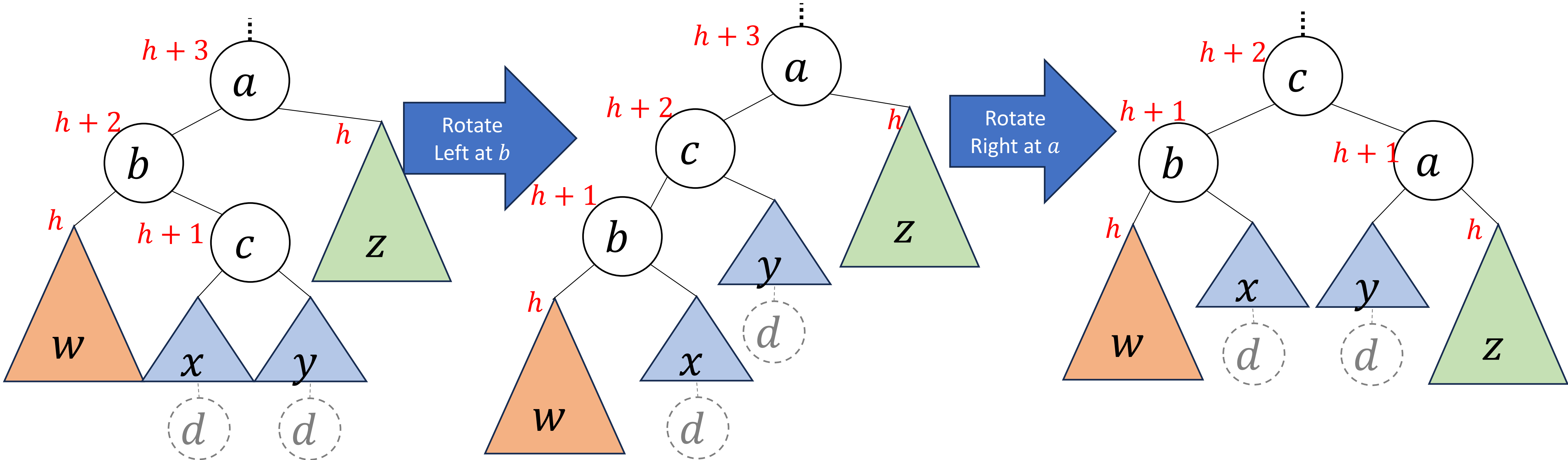
# Case LR in General

- We just inserted $d$, node $a$ is the deepest "problem" node
- Imbalance caused by inserting in the left child's right subtree
- Rotate left at the left child
- Rotate right at the unbalanced node



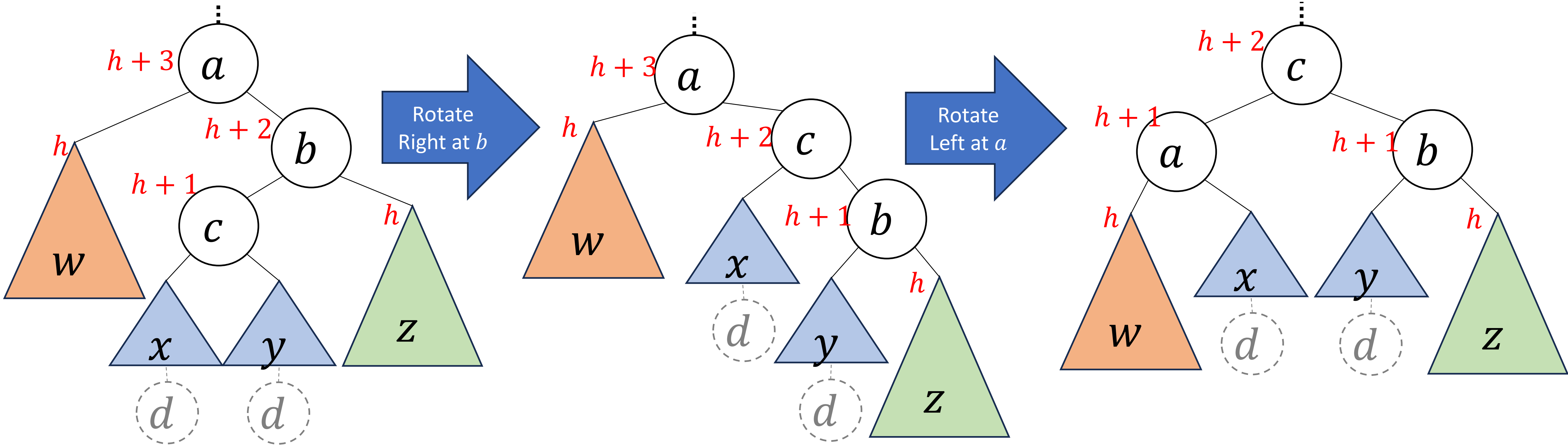
# Case LR Implementation

```
b=a.left
c=b.right
b.right=c.left
a.left=c.right
c.right=a
c.left=b
return c
```



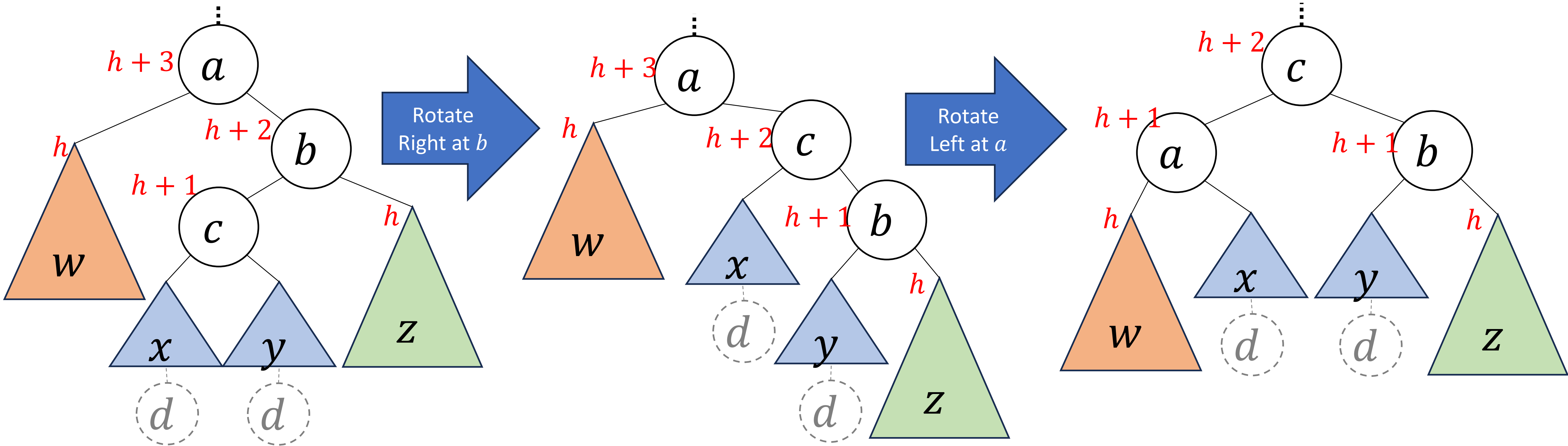
# Case RL in General

- We just inserted $d$, node $a$ is the deepest “problem” node
- Imbalance caused by inserting in the right child’s left subtree
- Rotate right at the right child
- Rotate left at the unbalanced node



# Case RL Implementation

```
b=a.right
c=b.left
b.left=c.right
a.right=c.left
c.left=a
c.right=b
return c
```



# Insert Summary

- After a BST insertion, update the heights of the node’s ancestors
- From leaf to root, check if each node is balanced
- If a node is unbalanced then at the deepest unbalanced node:
  - Case LL: If we inserted in the **left** subtree of the **left** child then: rotate right
  - Case RR: If we inserted in the **right** subtree of the **right** child then: rotate left
  - Case LR: If we inserted into the **right** subtree of the **left** child then: rotate left at the left child and then rotate right at the root
  - Case RL: If we inserted into the **left** subtree of the **right** child then: rotate right at the right child and then rotate left at the root
- Done after either reaching the root or applying **one** of the above cases

# Insert Operation (for AVL)

```
insert(key, value, root){
        root = insertHelper(key, value, root);
}
insertHelper(key, value, root){
        if(root == null)
                return new Node(key, value);
        if (root.key < key)
                root.right = insertHelper(key, value, root.right);
        else
                root.left = insertHelper(key, value, root.left);
        if(isUnbalanced(root))
                root=rotate(root);
        return root;
}
```

# Delete Summary

- Tldr: same cases, reverse direction of rotation, may need to repeat with ancestors
- After a BST deletion, update the heights of the node’s ancestors
- From leaf to root, check if each node is unbalanced
- If a node is unbalanced then at the deepest unbalanced node:
  - Case LL: If we deleted in the **left** subtree of the **left** child then: rotate left
  - Case RR: If we deleted in the **right** subtree of the **right** child then: rotate right
  - Case LR: If we deleted into the **right** subtree of the **left** child then: rotate right at the left child and then rotate left at the root
  - Case RL: If we deleted into the **left** subtree of the **right** child then: rotate left at the right child and then rotate right at the root
- Continue checking until reach the root

# Why is this $\Theta(\log n)$ time?

- We get poor running times when $height \approx n$
- Let $M(h)$ be the minimum count of nodes in an AVL tree of height $h$
- An AVL tree of height $h$ must have one subtree of height $h - 1$
- This means the other subtree has height at least $h - 2$
- $M(h) = M(h - 1) + M(h - 2) + 1$

# Comparing to Fibonacci Sequence

- $M(h) = M(h-1) + M(h-2) + 1$
- $F(n) = F(n-1) + F(n-2)$
  - Fibonacci Sequence
- So $M(h) > F(h)$
- For large values of $h$, $F(h) \approx \phi^h$
  - $\phi$ being the golden ratio. $\phi > 1.6$
- This means that an AVL tree of height $h$ has at least $\phi^h$ nodes
- So a tree of $n$ nodes has height at most $\log_\phi(n)$
  - We need $n \leq \phi^h$, so $\log_\phi n \leq h$
- All operations run in time $O(\log n)$
  - The maximum number of nodes for height $h$ is $2^{n+1} - 1$, so they are also $\Omega(\log n)$