

# CSE 332 Winter 2026

## Lecture 4: Algorithm Analysis and Priority Queues

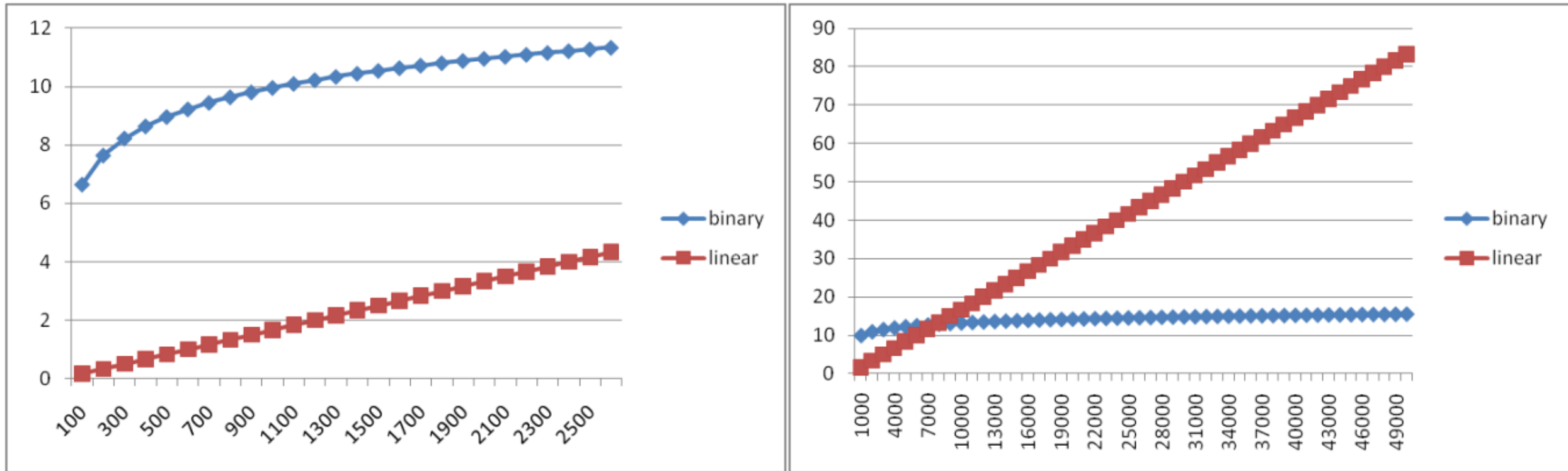
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<http://www.cs.uw.edu/332>

# Goals for Algorithm Analysis

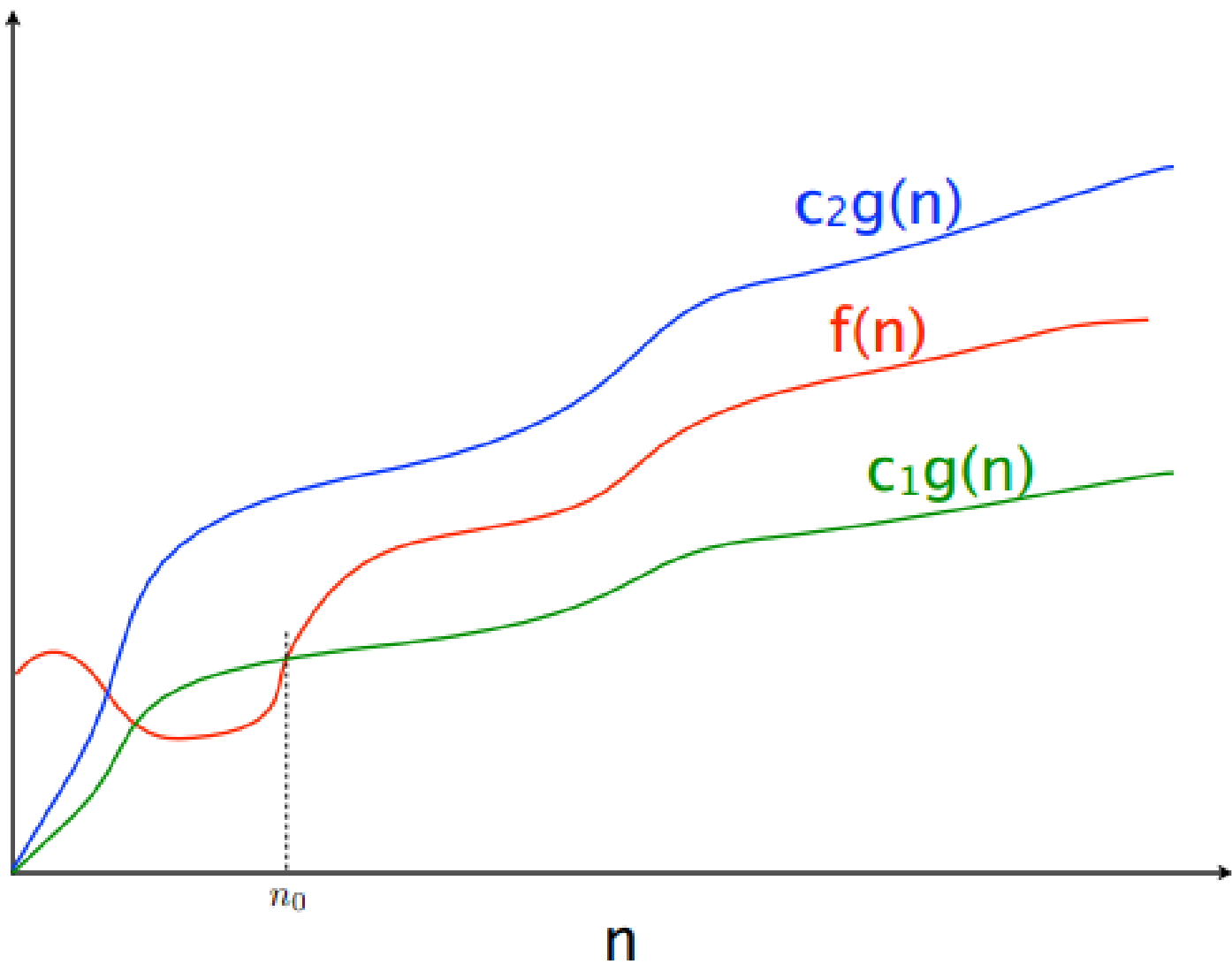
- Identify a *function* which maps the algorithm's input size to a measure of resources used
  - Domain of the function: **sizes** of the input
    - Number of characters in a string, number of items in a list, number of pixels in an image
  - Codomain of the function: **counts** of resources used
    - Number of times the algorithm adds two numbers together, number times the algorithm does a  $>$  or  $<$  comparison, maximum number of bytes of memory the algorithm uses at any time
- Important note: Make sure you know the “units” of your domain and codomain!
  - Domain = inputs to the function
  - Codomain = outputs to the function

# Comparing



# Comparing Running Times

- Suppose I have these algorithms, all of which have the same input/output behavior:
  - Algorithm A's worst case running time is  $10n + 900$
  - Algorithm B's worst case running time is  $100n - 50$
  - Algorithm C's worst case running time is  $\frac{n^2}{100}$
- Which algorithm is best?



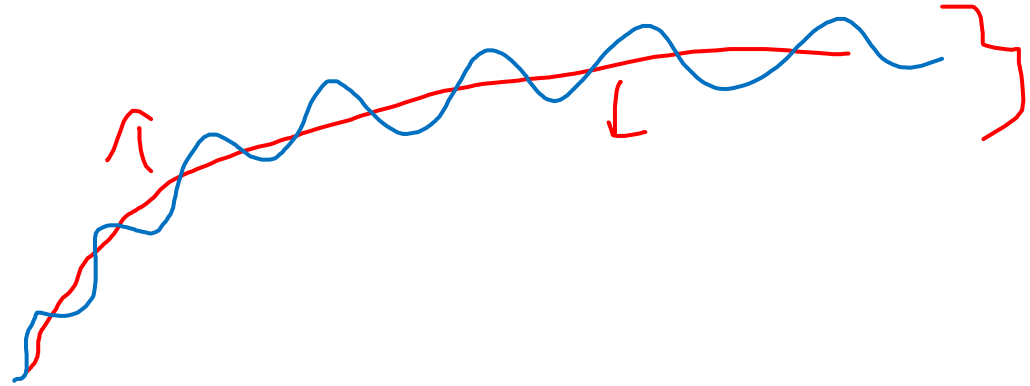
$\in$

$$f(n) = O(g(n))$$

$$f(n) = \Theta(g(n))$$

$$f(n) = \Omega(g(n))$$

# Asymptotic Notation



- $O(g(n))$ 
  - The **set of functions** with asymptotic behavior less than or equal to  $g(n)$
  - **Upper-bounded** by a constant times  $g$  for large enough values  $n$
  - $f \in O(g(n)) \equiv \exists c > 0. \exists n_0 > 0. \forall n \geq n_0. f(n) \leq c \cdot g(n)$
- $\Omega(g(n))$ 
  - the **set of functions** with asymptotic behavior greater than or equal to  $g(n)$
  - **Lower-bounded** by a constant times  $g$  for large enough values  $n$
  - $f \in \Omega(g(n)) \equiv \exists c > 0. \exists n_0 > 0. \forall n \geq n_0. f(n) \geq c \cdot g(n)$
- $\Theta(g(n))$ 
  - “**Tightly**” within constant of  $g$  for large  $n$
  - $\Omega(g(n)) \cap O(g(n))$

# Asymptotic Notation Example

- Show:  $10n + 100 \in O(n^2)$ 
  - **Technique:** find values  $c > 0$  and  $n_0 > 0$  such that  $\forall n > n_0. 10n + 100 \leq c \cdot n^2$
  - **Scratch work:**
    - $10n + 100 \leq f'(n) \wedge f'(n) \leq cn^2$
    - $10n + 100 \leq 10n + 100n$  whenever  $n \geq 1$
    - $10n + 100n = 110n$
    - Sufficient to show that  $110n \leq cn^2$ 
      - Choose  $c = 110, n_0 = 1$
  -

# Asymptotic Notation Example

- Show:  $10n + 100 \in O(n^2)$ 
  - **Technique:** find values  $c > 0$  and  $n_0 > 0$  such that  $\forall n \geq n_0. 10n + 100 \leq c \cdot n^2$
  - **Proof:**
    - Let  $c = 110$  and  $n_0 = 1$
    - **Goal:**  $10n + 100 \leq 110n^2$  for all  $n \geq 1$
    - We can first observe that  $100 \leq 100n$
    - So we know  $10n + 100 \leq 110n$
    - And also  $110n \leq 110n^2$
    - Therefore  $c$  and  $n_0$  exist



# Asymptotic Notation Example

- Show:  $13n^2 - 50n \in \Omega(n^2)$ 
  - **Technique:** find values  $c > 0$  and  $n_0 > 0$  such that  $\forall n \geq n_0. 13n^2 - 50n \geq c \cdot n^2$
  - **Scratch work:**
    - $13n^2 - 50n \geq \dots \geq cn^2$
    - $13n - 50 \geq 13n - n \geq cn$  provided  $n \geq 50$
    - $13n - 50 \geq (13 - 1)n \geq cn$
    - $13n - 50 \geq 12n \geq cn$
    - We can choose any  $c \leq 12$
    - $n_0 = 50$

# Asymptotic Notation Example

- Show:  $13n^2 - 50n \in \Omega(n^2)$ 
  - **Technique:** find values  $c > 0$  and  $n_0 > 0$  such that  $\forall n \geq n_0. 13n^2 - 50n \geq c \cdot n^2$
  - **Proof:**
    - Let  $c = 12$  and  $n_0 = 50$
    - $13n^2 - 50n \geq 12n^2$
    - $13n - 50 \geq 12n$
    - $n - 50 \geq 0$
    - $n \geq 50$
    - Which is true for our choice of  $n_0$

# Asymptotic Notation Example

- Show:  $n^2 \notin O(n)$ 
  - There is no choice of a  $c$  and  $n_0$  pair to make the inequality work for all  $n > n_0$ 
    - No matter you choose as your choice of  $c$ , we can find an arbitrarily large value of  $n$  that will violate the inequality
  - Assume towards contradiction that there exists a  $c$  and  $n_0$  such that  $n^2 \leq cn$
  - $n^2 \leq cn$
  - $\equiv n \leq c$
  - So this is only true for choices of  $n$  below the choice of  $c$ , so no  $n_0$  can exist
  - Contradiction!

# Asymptotic Notation Example

Proof by  
Contradiction!

- To Show:  $n^2 \notin O(n)$ 
  - **Technique: Contradiction**
  - **Proof:** Assume  $n^2 \in O(n)$ . Then  $\exists c, n_0 > 0$  s.t.  $\forall n > n_0, n^2 \leq cn$   
Let us derive constant  $c$ . For all  $n > n_0 > 0$ , we know:  
 $cn \geq n^2,$   
 $c \geq n.$

Since  $c$  is lower bounded by  $n$ ,  $c$  cannot be a constant and make this True.

Contradiction. Therefore  $n^2 \notin O(n)$ .

# Gaining Intuition

- When doing asymptotic analysis of functions:
  - If multiple expressions are added together, ignore all but the “biggest”
    - If  $f(n)$  grows asymptotically faster than  $g(n)$ , then  $\underline{f(n)} + \underline{g(n)} \in \Theta(f(n))$
  - Ignore all multiplicative constants
    - $f(n) + \underline{c} \in \Theta(f(n))$  for any constant  $c \in \mathbb{R}$
  - Ignore bases of logarithms
  - Do NOT ignore:
    - Non-multiplicative and non-additive constants (e.g. in exponents, bases of exponents)
    - Logarithms themselves
- Examples:
  - $4n + 5$
  - $0.5n \log n + 2n + 7$
  - $n^3 + 2^n + 3n$
  - $n \log(10n^2)$

$$\begin{array}{cc} n^2 & 2^n \\ \hline n^3 & 3^n \end{array}$$

# More Examples

- Is each of the following True or False?
  - $4 + 3n \in O(n)$
  - $n + 2 \log n \in O(\log n)$
  - $\log n + 2 \in O(1)$
  - $n^{50} \in O(1.1^n)$
  - $3^n \in \Theta(2^n)$

# Common Categories

- $O(1)$  “constant”
- $O(\log n)$  “logarithmic”
- $O(n)$  “linear”
- $O(n \log n)$  “log-linear”
- $O(n^2)$  “quadratic”
- $O(n^3)$  “cubic”
- $O(n^k)$  “polynomial”
- $O(k^n)$  “exponential”

# ADT: Queue

- What is it?
  - A “First In First Out” (FIFO) collection of items
- What Operations do we need?
  - Enqueue
    - Add a new item to the queue
  - Dequeue
    - Remove the “oldest” item from the queue
  - Is\_empty
    - Indicate whether or not there are items still on the queue



# ADT: Priority Queue

- What is it?
  - A collection of items and their “priorities”
  - Allows quick access/removal to the “top priority” thing
- What Operations do we need?
  - insert(item, priority)
    - Add a new item to the PQ with indicated priority
    - Usually, smaller priority value means more important
  - extract
    - Remove and return the “top priority” item from the queue
  - Is\_empty
    - Indicate whether or not there are items still on the queue
- Note: the “priority” value can be any type/class so long as it’s comparable (i.e. you can use “<” or “compareTo” with it)

# Priority Queue, example

```
PriorityQueue PQ = new PriorityQueue();
```

```
PQ.insert(5,5)
```

```
PQ.insert(6,6)
```

```
PQ.insert(1,1)
```

```
PQ.insert(3,3)
```

```
PQ.insert(8,8)
```

```
Print(PQ.extract) // 1
```

```
Print(PQ.extract) // 3
```

```
Print(PQ.extract) // 5
```

```
Print(PQ.extract) // 6
```

```
Print(PQ.extract) // 8
```

# Priority Queue, example

```
PriorityQueue PQ = new PriorityQueue();
```

```
PQ.insert(5,5) // has 5
```

```
PQ.insert(6,6) // has 5, 6
```

```
PQ.insert(1,1) // 1, 5, 6
```

```
Print(PQ.extract) // 1, has: 5, 6
```

```
PQ.insert(3,3) // 3, 5, 6
```

```
Print(PQ.extract) // 3, has: 5, 6
```

```
Print(PQ.extract) // 5, has: 6
```

```
PQ.insert(8,8) // 6, 8
```

```
Print(PQ.extract) // 6
```

```
Print(PQ.extract) // 8
```

Applications?

# Thinking through implementations

Data Structure	Worst case time to insert	Worst case time to extract
Unsorted Array		
Unsorted Linked List		
Sorted Array		
Sorted Linked List		
Binary Search Tree		

For simplicity, Assume we know the maximum size of the PQ in advance (otherwise we'd do an amortized analysis, but get the same answers...)

# Thinking through implementations

Data Structure	Worst case time to insert	Worst case time to extract
Unsorted Array	$\Theta(1)$	$\Theta(n)$
Unsorted Linked List	$\Theta(1)$	$\Theta(n)$
Sorted Array	$\Theta(n)$	$\Theta(1)$
Sorted Linked List	$\Theta(n)$	$\Theta(1)$
Binary Search Tree	$\Theta(n)$	$\Theta(n)$

For simplicity, Assume we know the maximum size of the PQ in advance (otherwise we'd do an amortized analysis, but get the same answers...)

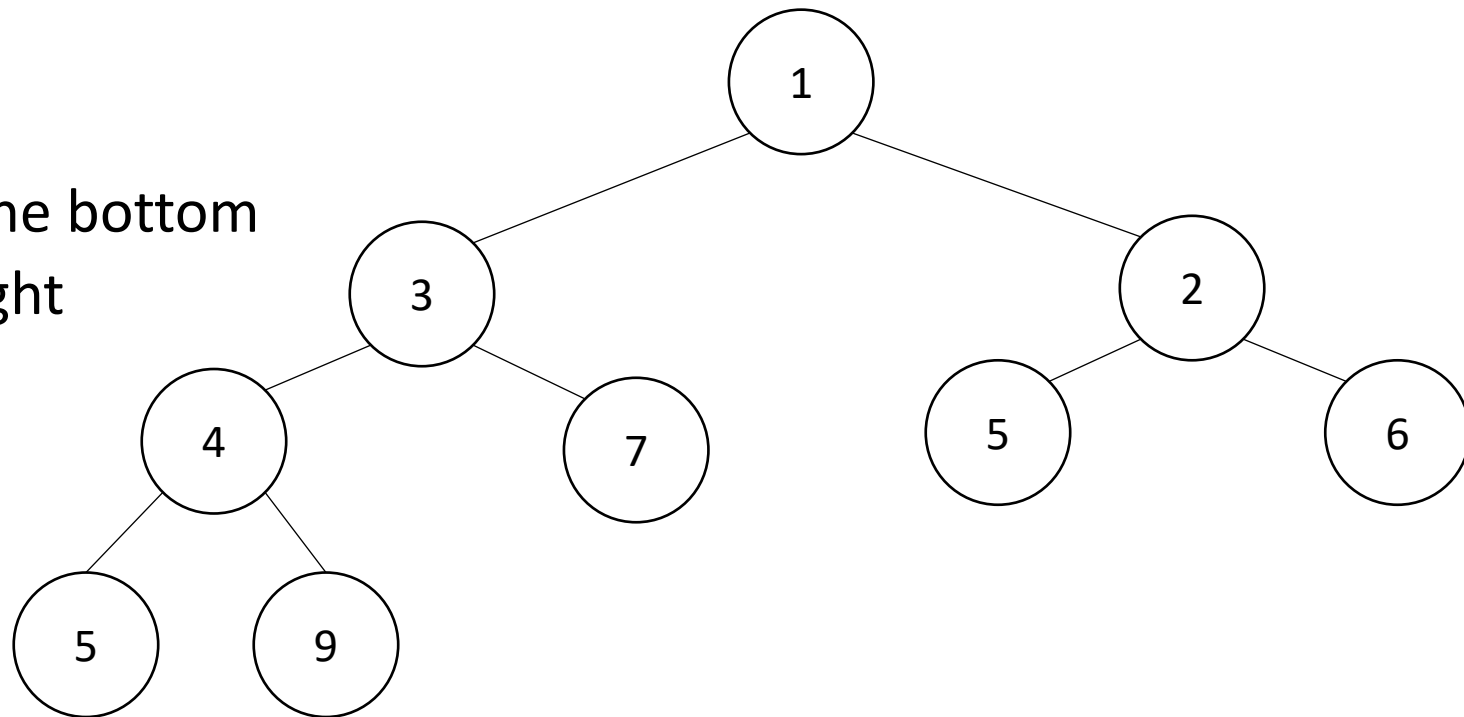
# Thinking through implementations

Data Structure	Worst case time to insert	Worst case time to extract
Unsorted Array	$\Theta(1)$	$\Theta(n)$
Unsorted Linked List	$\Theta(1)$	$\Theta(n)$
Sorted Array	$\Theta(n)$	$\Theta(1)$
Sorted Linked List	$\Theta(n)$	$\Theta(1)$
Binary Search Tree	$\Theta(n)$	$\Theta(n)$
Binary Heap	$\Theta(\log n)$	$\Theta(\log n)$

For simplicity, Assume we know the maximum size of the PQ in advance (otherwise we'd do an amortized analysis, but get the same answers...)

# Trees for Heaps

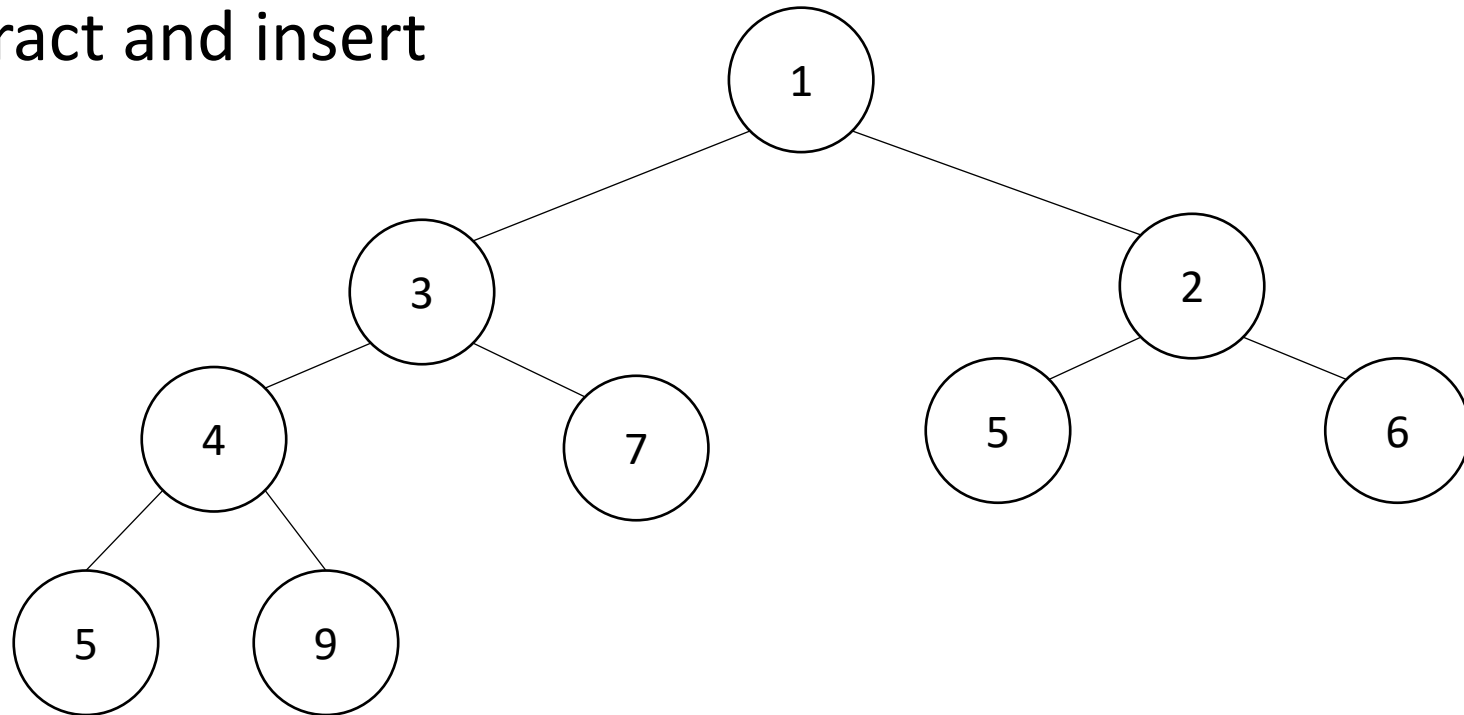
- Binary Trees:
  - The branching factor is 2
  - Every node has  $\leq 2$  children
- Complete Tree:
  - All “layers” are full, except the bottom
  - Bottom layer filled left-to-right





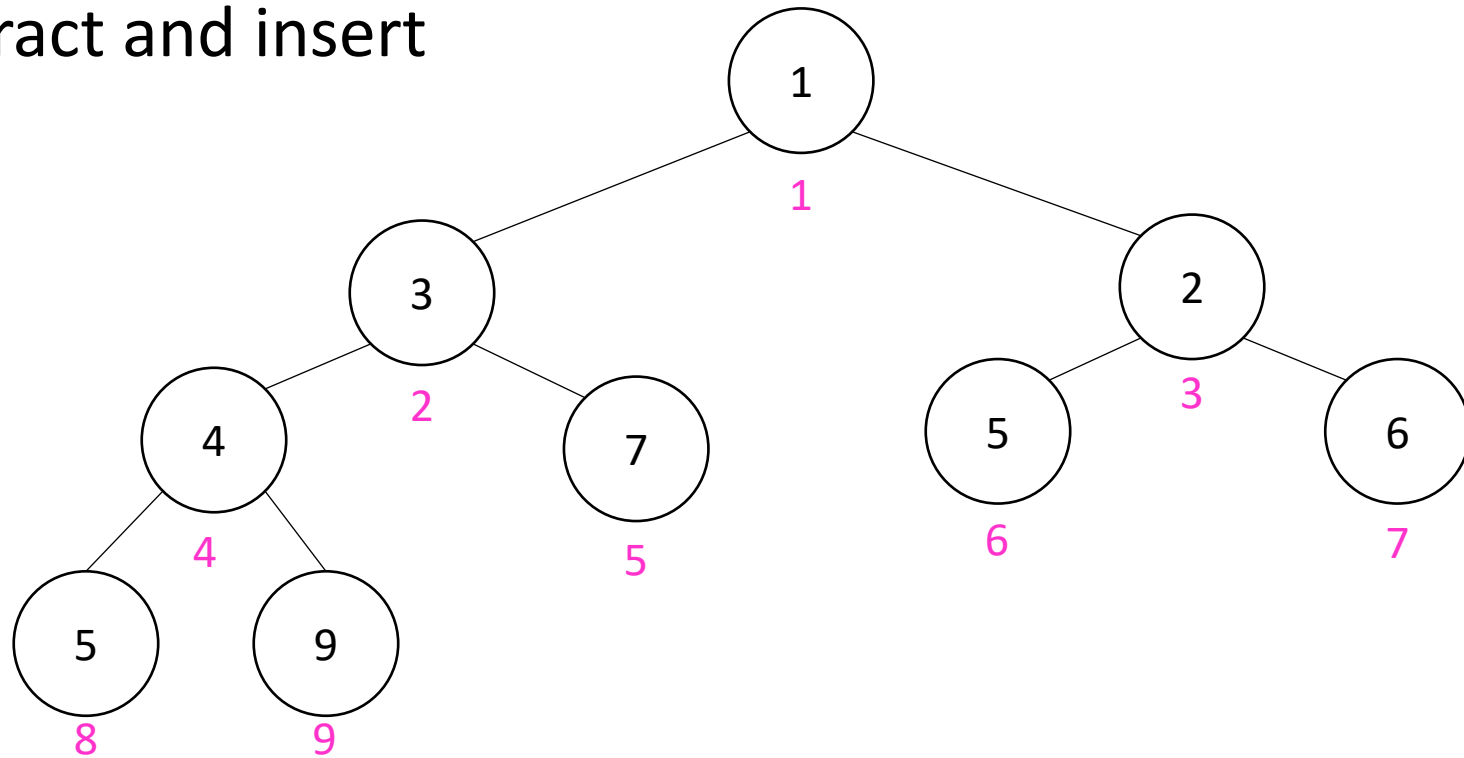
# Heap – Priority Queue Data Structure

- Idea: We need to keep some ordering, but it doesn't need to be entirely sorted
- $\Theta(\log n)$  worst case for extract and insert



# Heap – Priority Queue Data Structure

- Idea: We need to keep some ordering, but it doesn't need to be entirely sorted
- $\Theta(\log n)$  worst case for extract and insert



# Challenge!

- What is the maximum number of total nodes in a binary tree of height  $h$ ?
  - $2^{h+1} - 1$
  - $\Theta(2^h)$
- If I have  $n$  nodes in a binary tree, what is its minimum height?
  - $\Theta(\log n)$
- **Heap Idea:**
  - If  $n$  values are inserted into a complete tree, the height will be roughly  $\log n$
  - Ensure each insert and extract requires just one “trip” from root to leaf

# (Min) Heap Data Structure

- Keep items in a complete binary tree
- Maintain the “(Min) Heap Property” of the tree
  - Every node's priority is  $\leq$  its children's priority
  - Max Heap Property: every node's priority is  $\geq$  its children
- Where is the min?
- How do I insert?
- How do I extract?
- How to do it in Java?

